Financial Flexibility under Non-Exclusive Lending

Yunzhi Hu and Felipe Varas*

July 31, 2026

Abstract

This paper studies optimal financial flexibility in debt issuance. A borrower raises funds for an initial investment and may need financing after observing a private liquidity shock. Under non-exclusive lending, new lenders price only their claims, so additional borrowing dilutes existing debt and leads to excessive leverage. The optimal simple debt contract is an endogenous debt limit. It captures the intertemporal commitment–flexibility tradeoff: more borrowing today finances investment but increases dilution incentives, requiring tighter limits on future borrowing. Richer clauses, including performance-sensitive debt and contingent prepayment provisions, restore the exclusive-lending benchmark by compensating existing lenders when new financing is raised.

Keywords: financial flexibility, debt dilution, lending non-exclusivity, covenant, optimal delegation

*Hu: Kenan-Flagler Business School, UNC Chapel Hill; Yunzhi_Hu@kenan-flagler.unc.edu. Varas: Naveen Jindal School of Management, University of Texas at Dallas; felipe.varas@utdallas.edu. We are grateful to Jesse Davis, Alessandro Dovis, Paolo Fulghieri, Brett Green, Zhiguo He, Ron Kaniel, Narayana Kocherlakota, Adriano Rampini, Jacob Sagi, Fenghua Song (discussant), Andy Skrzypacz, Joao Thereze, Vivian Yue, Neng Wang, Pavel Zryumov and seminar participants from UNC, WashU Olin Finance Conference, 14th Advances in Macro-Finance Conference, and Rochester Simon for useful comments.

1 Introduction

Financial flexibility is the ability of a firm to obtain financing when it needs it. According to the presidential address by Graham (2022), financial flexibility is the most important factor in debt and capital structure decisions. In 2022, 87% of large firms and 78% of small firms ranked it as the top consideration for debt policy. Firms value this flexibility because future financing needs are difficult to predict: new investment opportunities may arise, cash flows may deteriorate, or financial distress may force the firm back to capital markets.

However, preserving the ability to issue debt in the future can come at a cost.[1] In debt issuance, a firm that can freely borrow in the future may dilute the claims of existing creditors. A recent literature in dynamic capital structure argues that this force induces firms to persistently increase leverage even when doing so is suboptimal, thereby eroding the value from borrowing (Admati et al., 2018; DeMarzo and He, 2021). Debt contracts must therefore balance future borrowing flexibility against the protection of existing creditors.

We develop a three-date model in which, at the initial date, a borrower needs to borrow a fixed amount to finance an investment project. The project generates cash flows at the final date contingent on the borrower's effort, with higher debt reducing the borrower's incentive to exert effort. At the interim date, the borrower privately observes a liquidity shock, modeled as a shock to her discount factor, which creates incentives to further borrow. Crucially, subsequent borrowing is non-exclusive: the borrower is not obligated to return to the initial lenders and can instead borrow from new lenders. Because the borrower does not internalize how new debt dilutes initial lenders' claims, this naturally leads to over-borrowing.

Under simple debt contracts, the financing problem is equivalent to delegation without transfers. The principal is the borrower at the initial date, who chooses the initial debt and the set of future debt choices before the liquidity shock is known. The agent is the same borrower at the interim date, who chooses total debt after privately observing the shock. The initial borrower maximizes total firm value, whereas the interim borrower maximizes continuation value net of the legacy debt claim. Initial lenders can restrict future debt choices, but cannot use state-contingent transfers to compensate themselves for dilution.

The delegation formulation allows us to characterize the optimal financing contract. Initial debt plays two roles: it raises funds for investment and determines the value of the legacy claim that the borrower disregards when choosing additional debt. Initial borrowing, therefore, affects both the incentive to dilute existing creditors and the amount of future borrowing discretion that should be

[1] A large literature studies the costs of flexibility that come from excess cash or discretion over internal resources, including the free-cash-flow problem in Jensen (1986) and related contracting problems such as Hart and Tirole (1988). Our focus is different: we study flexibility in debt issuance.

provided. For a given level of initial debt, we construct a Lagrange multiplier that globally verifies the optimality of a limit on total debt. We then characterize the joint choice of initial debt and the limit. The financing constraint always binds, and a primitive curvature condition on the debt-price schedule ensures that the joint solution is globally unique.

The solution contains both a within-period and an intertemporal commitment–flexibility trade-off. For a given initial debt level, borrowers with severe liquidity shocks hit the cap and under-borrow relative to the efficient benchmark, while borrowers with mild shocks remain unconstrained and over-borrow. Across dates, greater financing needs require more initial debt, which strengthens the incentive to dilute existing creditors and therefore requires a tighter limit on future borrowing.

We compare this result with exclusive lending as the benchmark. When the borrower must return to the initial lender for future financing, creditor dilution is eliminated because the lender internalizes the effect of new borrowing. If the initial investment is small, private information about the future shock by itself does not distort refinancing relative to the observable-state benchmark. When financing needs are larger, however, the financing constraint binds and refinancing is distorted even under exclusive lending. The distortion takes a different form from the debt limit under non-exclusive lending: borrowers with mild liquidity shocks will receive a flat debt allocation because limited liability at $t = 1$ prevents the borrower from receiving a negative transfer. This benchmark is useful because it identifies the allocation that richer non-exclusive contracts can restore.

Finally, we study contractual provisions that compensate existing lenders when the borrower raises new financing. Performance-sensitive debt adjusts existing lenders' claims, while contingent prepayment provisions, including debt sweeps, redirect part of the proceeds from new borrowing to them. Both mechanisms make the borrower internalize the dilution cost of new debt and can implement the exclusive-lending benchmark under non-exclusive lending. We also discuss why these contracts may be difficult to implement in practice. They require state-contingent adjustments or transfers that may be hard to monitor, verify, and enforce. In those cases, simple borrowing covenants emerge as a more practical way to protect creditors, because a total debt limit is easier to monitor than richer contingent compensation.

Related Literature

Our paper is most closely related to the mechanism-design literature on commitment and flexibility, rooted in the theory of delegation pioneered by Holmström (1984). This literature studies how a principal chooses a delegation set when an agent values flexibility to respond to future private information but also faces incentive problems that make commitment valuable. We build on the Lagrangian approach developed by Amador, Werning, and Angeletos (2006) to study consumption–savings problems with hyperbolic preferences and that Amador and Bagwell (2013) extends to a general class of delegation problems. A recent strand of this literature studies the commitment–flexibility tradeoff in the context of fiscal policy, examining how privately observed spending needs, present bias, limited enforcement, and political turnover shape fiscal rules and transitions between responsible and irresponsible regimes (Halac and Yared, 2014, 2018, 2022, 2025).

Relative to the existing delegation literature, the main difference is that the disagreement between the principal and the agent is generated by the initial financing choice rather than fixed exogenously. In our model, the gap between the initial and interim borrower's objectives is exactly the value of legacy debt. Initial debt therefore determines the incentive to dilute existing creditors, while the debt limit determines the discretion available to the interim borrower. Because initial debt must also satisfy the financing constraint, the initial claim and the delegation set must be determined jointly. We adapt the saddle-point approach in this literature by first establishing the optimal debt limit for a given initial claim and then solving for initial debt. This yields the intertemporal financing tradeoff that more initial borrowing must be accompanied by a tighter limit on future debt.

Our paper is also related to the literature on contracting under non-exclusivity. In these models, a contracting party cannot prevent its counterparty from trading with others, so bilateral contracts must remain incentive-compatible in the presence of outside contracts or side trades. In moral-hazard models, non-exclusive contracts weaken effort incentives when additional contracts dilute existing lenders' claims (Kahn and Mookherjee, 1998; Bisin and Guaitoli, 2004; Bizer and DeMarzo, 1992). In adverse-selection models, non-exclusive trade complicates screening privately informed sellers, particularly when sellers can issue securities to multiple buyers (Attar et al., 2011, 2014, 2020; Asriyan and Vanasco, 2024). Other work asks whether institutions can recover some of the protection of exclusivity. Bisin and Rampini (2006) show that bankruptcy protects existing creditors even when borrowers can trade with other lenders. Unlike much of this literature, which takes non-exclusivity to mean that outside contracts cannot be restricted, we allow the initial contract to limit the borrower's aggregate outside borrowing without requiring the borrower to return to the initial lender. Our problem is to determine the optimal limit on outside borrowing and, with richer contracts, the optimal compensation to existing lenders when such borrowing occurs. More broadly, creditor dilution is a contracting externality: borrowing from new lenders lowers initial lenders' payoffs, as in Segal (1999), who studies contracts in which one agent's trade affects other agents' payoffs.

Within corporate finance, our paper is most closely related to work on financing when borrowers cannot commit to future borrowing. Fama and Miller (1972) and Black and Scholes (1973) first highlighted that borrowers can increase leverage by issuing new debt without internalizing the resulting dilution of outstanding debt. Subsequent work studies how this debt-dilution problem affects borrower incentives and financing choices (Bizer and DeMarzo, 1992). More recent work studies the dynamic consequences of this commitment problem, including how the inability to commit to future financing choices shapes leverage dynamics and the value of commitment (Admati et al., 2018; DeMarzo, 2019; DeMarzo and He, 2021; DeMarzo et al., 2023).

Covenants are a standard contractual response to the risk that later financing dilutes existing creditors (Smith and Warner, 1979). In a non-exclusive lending setting, Attar et al. (2019) study how covenants restrict side trading and reduce lender competition. Related to our paper, Donaldson et al. (2025) study anti-dilution covenants under non-exclusive borrowing and the tradeoff between creditor protection and valuable later financing. Their mechanism instead relies on ex post creditor control: once project quality is known, creditors can accelerate or waive, thereby managing changes in debt priority. By contrast, we formulate optimal contracting under non-exclusive lending as a delegation problem that jointly determines initial debt and future borrowing discretion under private liquidity information. The endogenous financing constraint links these choices: more initial debt expands current financing but requires a tighter debt limit to preserve future flexibility. We also show that state-contingent compensation can implement the exclusive-lending allocation without exclusivity.

2 Model

We consider a three-date financing problem with both moral hazard and private information. A borrower raises funds for an initial investment at $t = 0$, learns a privately observed discount-factor shock at $t = 1$, and privately chooses effort at $t = 2$.

2.1 Borrower, Investment, and Discount-Factor Shock

There are three dates, $t = 0, 1, 2$, and a risk-neutral borrower. The borrower has a project that requires a fixed investment of $I > 0$ at $t = 0$ but has no resources to finance it, hence the need to borrow. The project generates no cash flow at $t = 1$ and a final cash flow Y at $t = 2$ if successful (zero otherwise).[2] At the beginning of $t = 2$, the borrower privately chooses an effort level $q \in [0, 1]$, where q corresponds to the success probability. The effort incurs a private cost $c(q)$. We assume that $c(0) = c'(0) = 0$, that $c(q)$ is twice continuously differentiable, and that $c''(q) > 0$ for every $q \in (0, 1]$. We also assume $c'(1) > Y$. These conditions imply that effort is uniquely determined

[2] Under this binary cash flow framework, there is no fundamental distinction between different securities, such as equity and debt. In addition, it implies that seniority is irrelevant within the class of debt contracts. We choose to model final cash flows in this parsimonious manner to eliminate the complicated security design problem.

and interior whenever $x_1 < Y$. At $x_1 = Y$, the borrower chooses $q(Y) = 0$; however, this level of debt is never chosen.

The borrower does not discount cash flows between $t = 0$ and $t = 1$. At $t = 1$, she receives a discount factor shock and discounts cash flows at $t = 2$ at θ. The realization of θ is her private information. We assume that θ is drawn from a set $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq (0, 1]$, with a continuously differentiable density function $g(\theta) > 0$ and cumulative density function $G(\theta)$. Throughout the paper, we make the following standard assumption, which leads to the result that $g(\theta)/G(\theta)$ is decreasing.

Assumption 1. *The cumulative density function $G(\theta)$ is log-concave.*

2.2 Lenders and Contract

There exists a competitive pool of dispersed lenders. Throughout the paper, lenders are risk-neutral and do not discount future cash flows. Given that lenders are competitive, the borrower captures all surplus.

The contract specifies how the cash flow at $t = 2$ is split among the borrower, the initial lenders at $t = 0$, and any new lenders at $t = 1$. At $t = 0$, initial lenders provide an upfront transfer τ_0 to finance the investment I and receive a claim x_0 on the final cash flow. At $t = 1$, the borrower returns to the market and reports her shock as $\hat{\theta}$. New lenders provide an interim transfer $\tau_1(\hat{\theta})$ in exchange for the additional claim $x_1(\hat{\theta}) - x_0$, so that total promised repayment becomes $x_1(\hat{\theta})$. Note that at $t = 1$, x_0 is a legacy claim that can be diluted by additional borrowing. We interpret these claims as debt contracts and refer to x_1 as the face value of total debt. For a given realization of the discount-factor shock θ, report $\hat{\theta}$, and effort q, the borrower's payoff is

$$\tau_0 - I + \tau_1(\hat{\theta}) + \theta \left[q \left(Y - x_1(\hat{\theta}) \right) - c(q) \right].$$

Since the borrower has no initial wealth and is protected by limited liability, the contract must satisfy the following financing and feasibility constraints:

$$\tau_0 \geq I, \qquad \tau_1(\hat{\theta}) \geq 0, \qquad 0 \leq x_0 \leq x_1(\hat{\theta}) \leq Y \quad \text{for all } \hat{\theta} \in \Theta. \tag{1}$$

For the rest of the paper, we refer to the constraint $\tau_0 \geq I$ as the *financing constraint.*

Under non-exclusive lending, the lenders who provide funds at $t = 0$ need not be the same lenders who provide funds at $t = 1$. This separation is natural when lenders are dispersed, because it may be difficult to assemble the original lending group again at the refinancing date. Even when the initial lenders are concentrated, for example in a bank relationship, regulatory capital constraints may limit their ability to expand exposure to a given borrower or sector at $t = 1$. Moreover, we show later that, if given the choice, the borrower prefers to issue debt to new lenders rather than to existing lenders at the refinancing date.

Figure 1 summarizes the sequence of events on three dates.

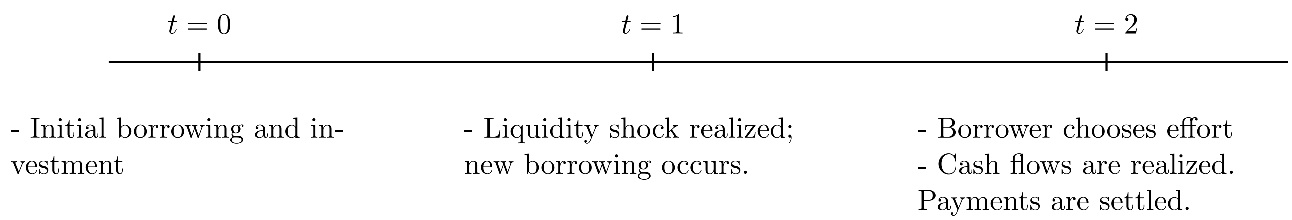

Figure 1: Timeline of events

2.3 Effort Choice and Valuations

We now turn to the borrower's problem at $t = 2$. The key tradeoff is moral hazard: higher debt weakens the borrower's incentive to exert effort.[3] Given a face value of debt x_1, the borrower chooses effort q to maximize her payoff

$$V_2(x_1) = \max_{q \in [0,1]} q(Y - x_1) - c(q), \tag{2}$$

where $V_2(x_1)$ is the value of the borrower's residual claim net of effort cost. Because there is no outsider equity, $\theta V_2(x_1)$ is also the firm's equity value at $t = 1$. Let $q(x_1)$ denote the optimal effort choice. The following lemma describes properties of $V_2(x_1)$ and $q(x_1)$ used throughout the paper.

Lemma 1. *Let $q(x_1)$ be the solution to (2) for each x_1.*

- *$\forall x_1 \in [0, Y]$, $V_2(x_1)$ is decreasing and convex with derivative $V_2'(x_1) = -q(x_1)$.*
- *$\forall x_1 \in [0, Y]$, $q(x_1) = (c')^{-1}(Y - x_1)$.*
- *$\forall x_1 \in [0, Y]$, $q(x_1)$ is strictly decreasing. For $x_1 \in [0, Y)$, its derivative is $q'(x_1) = -\frac{1}{c''(q(x_1))} < 0$.*
- *If $c(q)$ is three times continuously differentiable, then, for $x_1 \in (0, Y)$,*

$$q''(x_1) = -\frac{c'''(q(x_1))}{c''(q(x_1))} \big(q'(x_1)\big)^2.$$

[3] The solution without a moral hazard problem is trivial: the borrower always pledges all the cash flows at $t = 0$. An alternative to introducing moral hazard is to assume bankruptcy costs.

The condition $c'(0) = 0$ implies that $q(x_1) > 0$ and $V_2(x_1) > 0$ whenever $x_1 < Y$, while $q(Y) = V_2(Y) = 0$. We impose two assumptions. Assumption 2 states that the maximum expected repayment $x_1 q(x_1)$ is sufficient to cover the initial investment I. Assumption 3 and Lemma 1 imply that $q(x_1)$ is concave. This assumption will guarantee that the objective function of several of the programs we later consider is concave.

Assumption 2. *The cost function $c(q)$ satisfies*

$$I^{\max} := \max_{x_1 \in [0,Y]} \; x_1 q(x_1) \geq I. \tag{3}$$

Assumption 3. *$c(q)$ is three times continuously differentiable and satisfies $c'''(q) \geq 0$.*

For the rest of this paper, we define

$$W_1(x_1, \theta) := \underbrace{x_1 q(x_1)}_{\text{debt}} + \underbrace{\theta V_2(x_1)}_{\text{equity}}, \tag{4}$$

as the (ex-post) $t = 1$ firm value for a given θ. Let $x_1^*(\theta)$ be the level of debt that maximizes this,

$$x_1^*(\theta) = \arg \max_{x_1 \in [0,Y]} \; W_1(x_1, \theta). \tag{5}$$

We refer to $x_1^*(\theta)$ as the *ex post-efficient* debt level. It maximizes total surplus when moral hazard is the only friction and therefore captures the basic trade-off between providing liquidity and preserving effort incentives. Clearly, the efficient debt level decreases with θ: borrowers with a higher discount factor borrow less total debt.

Note that the ex post-efficient allocation corresponds to the solution after knowing θ. If the borrower chooses at $t = 0$ without knowing θ, the optimal choice is $x_1^*(\mathbb{E}[\theta])$, because $W_1(x_1, \theta)$ is linear in θ. For the remainder of this paper, we refer to $x_1^*(\mathbb{E}[\theta])$ as the *ex-ante optimal* debt.

The efficient debt choice is strictly below Y. Indeed, $W_1(0, \theta) > W_1(Y, \theta) = 0$, so $x_1^*(\theta) < Y$ for every $\theta \in \Theta$. The same conclusion holds when x_1^* is evaluated at zero: Assumption 2 guarantees a debt level in $(0, Y)$ with strictly positive proceeds, whereas $x_1 = 0$ and $x_1 = Y$ yield zero proceeds.

Linear Example. *Throughout our analysis, we study a specific example where the cost function $c(q)$ is quadratic. Under this specification, the optimal effort $q(x_1)$ is linear in x_1, and the borrower's payoff at $t = 2$ is quadratic. For the rest of this paper, we will refer to this case as the linear example.*

If $c(q) = \frac{1}{2}cq^2$ where $c > Y$, then

$$\begin{aligned} q(x_1) &= \frac{Y - x_1}{c} \\ V_2(x_1) &= \frac{1}{2}\frac{(Y - x_1)^2}{c}. \end{aligned} \tag{6}$$

Assumption 2 is satisfied only if

$$\frac{Y^2}{4c} \geq I.$$

Moreover, we have

$$\begin{aligned} W_1(x_1, \theta) &= x_1\frac{Y - x_1}{c} + \frac{\theta}{2}\frac{(Y - x_1)^2}{c} \\ x_1^*(\theta) &= \frac{1 - \theta}{2 - \theta}Y. \end{aligned}$$

3 Non-Exclusive Lending with Simple Debt Contracts

Under non-exclusive lending, initial lenders anticipate dilution from new debt issued at $t = 1$ and, therefore, seek to limit future issuance to protect their claims. We consider the contract $\{\tau_0, x_0, x_1(\hat{\theta})\}$ offered to initial lenders, where total debt is limited by $x_1(\hat{\theta})$, a function of the borrower's reported type $\hat{\theta}$ at $t = 1$. By doing so, we restrict the mechanism to those where transfers cannot be made to the initial lenders and adjustments cannot be applied to x_0 conditional on the reported type $\hat{\theta}$ at $t = 1$. We examine such mechanisms later in Section 5.

Since the borrower faces a liquidity shock, it is without loss of generality to assume she issues new debt up to the debt limit $x_1(\hat{\theta})$, so $\Delta x_1(\hat{\theta}) \coloneqq x_1(\hat{\theta}) - x_0$ is the additional debt issued at $t = 1$.

3.1 Reduction to Delegation Problem

We first show that the problem can be equivalently formulated as a delegation problem. Because the borrower offers the contract, new lenders' participation constraint must bind, implying $\tau_1 = (x_1 - x_0)q(x_1)$, where $x_1 \geq x_0$ is required to satisfy the non-negativity constraint $\tau_1 \geq 0$. At $t = 1$, the borrower's continuation payoff for any given (θ, x_1) is:

$$V_1(x_1, x_0, \theta) = \underbrace{(x_1 - x_0)q(x_1)}_{\text{new debt issuance}} + \theta V_2(x_1) = W_1(x_1, \theta) - x_0 q(x_1). \tag{7}$$

Meanwhile, the participation constraint for initial lenders must also bind, implying

$$\tau_0 = x_0\mathbb{E}[q(x_1(\theta))],$$

and the financing constraint becomes

$$x_0 \mathbb{E}\left[q(x_1(\theta))\right] \geq I. \tag{8}$$

Therefore, the borrower's expected payoff at $t = 0$ is given by:

$$\tau_0 - I + \mathbb{E}\left[V_1(x_1(\theta), x_0, \theta)\right] = \mathbb{E}\left[W_1(x_1, \theta)\right] - I,$$

so that the objective is to maximize $\mathbb{E}\left[W_1(x_1, \theta)\right]$. The borrower has incentives to report its type truthfully only if the following incentive compatibility constraint is satisfied

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} \; V_1(x_1(\hat{\theta}), x_0, \theta). \tag{9}$$

By the envelope theorem, this can be rewritten as

$$\begin{aligned} & V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) d\tilde{\theta} \quad \forall \theta \in \Theta \\ & x_1(\theta) \text{ is non-increasing.} \end{aligned} \tag{10}$$

The optimal mechanism at $t = 0$ solves the following problem:

$$\begin{aligned} & \max_{x_0, x_1(\theta) \geq x_0} \; \mathbb{E}\Big[W_1(x_1, \theta)\Big] \\ & \text{subject to (8) and (10).} \end{aligned} \tag{11}$$

This problem relates to a delegation setup, where the principal is the borrower at time $t = 0$, and the agent is the borrower at time $t = 1$. The principal delegates the choice of total debt to the agent, who has private information about the refinancing need captured by θ. The principal seeks to maximize firm value (net investment I), while the agent maximizes $V_1(x_1, x_0, \theta)$, which, as shown by equation (7), is the firm value net of initial debt at $t = 1$. This conflict relates to the insight of Aguiar et al. (2019), which shows that equilibrium debt issuance decisions can be characterized as the solution to a planner's problem that ignores the payoff to existing creditors.

Why does our mechanism-design setting resemble a delegation problem (Amador et al., 2006; Amador and Bagwell, 2013)? Intuitively, any transfer from the principal ($t = 0$ borrower) to the agent ($t = 1$ borrower) must be mediated through new lenders. Because lenders break even, no effective transfers can be made to motivate the agent's incentives. Instead, the principal can only restrict the set of debt choices available to the agent.

Two features distinguish this problem from standard delegation. First, the financing constraint

(8) couples the initial choice x_0 to the continuation allocation and, as we show below, always binds at the optimum. Second, the disagreement is not a primitive preference parameter. It is created by the initial debt chosen by the principal:

$$W_1(x_1, \theta) - V_1(x_1, x_0, \theta) = x_0 q(x_1),$$

so the difference between the principal's and the agent's payoff exactly equals the value of the initial debt.

Thus, x_0 simultaneously relaxes the financing constraint and increases the wedge between the two objective functions. This is the source of the intertemporal commitment-flexibility tradeoff.

The optimization problem in (11) is not jointly convex in x_0 and $x_1(\theta)$, so neither pointwise maximization nor the standard one-dimensional delegation argument determines the joint optimum. We proceed in two steps. In Subsection 3.2, we fix x_0 and construct a Lagrange multiplier that globally verifies the optimality of a debt limit. In Subsection 3.3, we endogenize x_0, prove that the financing constraint binds, and provide a primitive curvature condition under which the joint solution is unique.

3.2 Constrained Debt Limits

Throughout this subsection, we take the initial debt level x_0 as given. Let us define the value function

$$\begin{gathered} W_0(x_0) = \max_{x_1(\theta) \geq x_0} \mathbb{E}\Big[W_1(x_1, \theta)\Big] \\ \text{subject to (8) and (10).} \end{gathered} \tag{12}$$

Before proceeding, it is helpful to consider a local version of the incentive compatibility constraint, which applies at any point θ where $x_1(\theta)$ is differentiable. Specifically, the first-order condition of (9) implies

$$\frac{\partial V_1(x_1(\theta), x_0, \theta)}{\partial x_1} \cdot \frac{dx_1(\theta)}{d\theta} = 0. \tag{13}$$

This condition implies that a necessary condition for truthful reporting is either: (i) the borrower receives the unconstrained optimal debt level $x_1(\theta)$ that maximizes her continuation payoff $V_1(x_1(\hat{\theta}), x_0, \theta)$, or (ii) the allowed debt level $x_1(\theta)$ is locally insensitive to the type reported, so the agent does not benefit from locally misreporting the liquidity shock. Following the literature, we call the first case *flexible financing* and define

$$x_1^f(\theta, x_0) = \arg \max_{x_1 \in [x_0, Y]} V_1(x_1, x_0, \theta) \tag{14}$$

with shorthand $x_1^f(\theta)$ when no confusion arises. Any feasible x_0 is strictly below Y: otherwise, $x_1 = Y$ and the financing constraint cannot hold because $I > 0$ and $q(Y) = 0$. Moreover, for any feasible $x_0 < Y$,

$$V_1(x_0, x_0, \theta) = \theta V_2(x_0) > V_1(Y, x_0, \theta) = 0.$$

Thus, $x_1^f(\theta, x_0) < Y$, and the associated effort choice is interior. Its first-order condition is:[4]

$$(x_1^f(\theta) - x_0)q'(x_1^f(\theta)) + (1 - \theta)q(x_1^f(\theta)) = 0. \tag{15}$$

The next lemma summarizes properties of the flexible-debt solution.

Lemma 2. *The flexible-debt solution $x_1^f(\theta, x_0)$ satisfies the following properties:*

1. *$x_1^f(\theta, x_0)$ is nonincreasing in θ and nondecreasing in x_0.*
2. *$x_1^f(\theta, x_0) \geq x_1^*(\theta)$.*
3. *$x_1^f(\theta, x_0) \geq x_0$. If $\bar{\theta} = 1$, then $x_1^f(\bar{\theta}, x_0) = x_0$.*

The first result in Lemma 2 holds because higher values of θ reduce the benefits of debt issuance, while higher initial x_0 strengthens dilution incentives. The second result is driven entirely by these debt dilution incentives so that flexible borrowing tends to be excessive. The third result reflects the same underlying mechanism as the leverage-ratchet effect (Admati et al., 2018). When $\bar{\theta} = 1$, the most patient borrower has no need to borrow.

To simplify the exposition, we assume:

Assumption 4. $\bar{\theta} = 1$.

Under Assumption 4, Lemma 2 establishes that the most patient borrower does not issue any debt at $t = 1$, so $x_1^f(\bar{\theta}) = x_0$. This assumption eliminates the need to consider limits that constrain all types, reducing the number of cases to analyze.[5] Moreover, since Lemma 2 establishes that $x_1^f(\theta, x_0)$ is nonincreasing in θ, for any debt limit between x_0 and $x_1^f(\underline{\theta})$, there exists a threshold type $\theta_L \in \Theta$ such that $x_1^f(\theta_L)$ is exactly this limit. In equilibrium, some borrowers hit the debt limit, allowing us to specify the limit equivalently as a threshold type θ_L.

We now solve (12) and establish conditions under which the optimal mechanism takes the form of a debt limit

$$x_1^L(\theta) := \min\{x_1^f(\theta), x_1^f(\theta_L)\}.$$

[4] The function $V_1(x_1, x_0, \theta)$ is concave in x_1 under Assumption 3.

[5] When $\bar{\theta} < 1$, we would need to consider additional cases to allow for debt limits of the form $0 < x_1^L < x_1^f(\bar{\theta})$. This corresponds to the degenerate delegation sets in Amador et al. (2018). The analysis in the appendix considers this more general case and allows for $\bar{\theta} < 1$.

By construction, this debt policy satisfies the local incentive compatibility conditions (13) and is also globally incentive compatible (i.e., satisfies (10)). Figure 2 plots the debt policy (solid red line) against the flexible debt level $x_1^f(\theta)$ (dashed red line) and the efficient debt level $x_1^*(\theta)$ (solid green line). Several patterns emerge. First, by Lemma 2, $x_1^f(\theta)$ is nonincreasing in θ, so the debt limit binds at low θ but becomes slack as θ rises. Second, Lemma 2 also gives $x_1^f(\theta) \geq x_1^*(\theta)$, since the former maximizes firm value net of initial debt, while the latter maximizes total firm value. As a result, the debt limit leads to under-borrowing at low θ and over-borrowing at high θ.

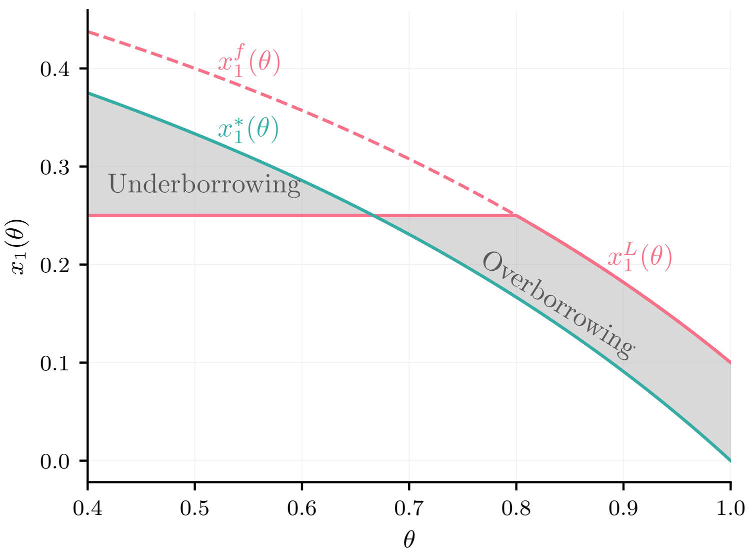

Figure 2: Debt limit and efficient debt level for $\theta \sim U[0.4, 1]$, $Y = 1$, $c(q) = \frac{1}{2}q^2$, $x_0 = 0.1$, and $\theta_L = 0.8$.

The optimal debt limit can thus be found by solving:

$$\begin{aligned} &\max_{\theta_L \in \Theta} \ \mathbb{E}\left[W_1\left(x_1^f(\theta_L), \theta\right)\mathbf{1}_{\theta \leq \theta_L} + W_1\left(x_1^f(\theta), \theta\right)\mathbf{1}_{\theta > \theta_L}\right] \\ &\text{subject to } x_0 \mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] \geq I. \end{aligned} \tag{16}$$

We formulate the Lagrangian. Let λ be the multiplier and define $\delta = \frac{1}{\lambda+1}$. Since q is decreasing and Lemma 2 establishes that $x_1^f(\theta, x_0)$ is nonincreasing in θ, it is easily established that $\mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right]$ increases in θ_L. This means that we only need to consider two candidates for the solution of problem (16): either (i) the constraint is slack, and θ_L is chosen to maximize the objective function; or (ii) the constraint is binding and θ_L is determined by the financing constraint. The following proposition characterizes the solution to problem (16).

Proposition 1 (Optimal Debt Limit). *Let θ_L^{bc} be the lowest type that satisfies the financing constraint:*

$$\theta_L^{\text{bc}} := \min\left\{\theta_L \in \Theta :\ x_0 \mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] \geq I\right\}.$$

Let θ_L^{uc} be the optimal threshold in the absence of the financing constraint.

- *If $x_0 > x_1^*\big(\mathbb{E}[\theta]\big)$, then $\theta_L^{\text{uc}} = \bar{\theta} = 1$.*
- *If $x_0 \leq x_1^*\big(\mathbb{E}[\theta]\big)$, then θ_L^{uc} solves*

$$x_0 \cdot \frac{q'\big(x_1^f(\theta_L)\big)}{q\big(x_1^f(\theta_L)\big)} + \mathbb{E}[\theta_L - \theta \mid \theta \leq \theta_L] = 0. \tag{17}$$

The optimal threshold is $\theta_L = \max\left\{\theta_L^{\text{uc}}, \theta_L^{\text{bc}}\right\}$, and the corresponding debt limit is $x_1^L = x_1^f(\theta_L)$.

- *If $\theta_L^{\text{uc}} > \theta_L^{\text{bc}}$, the financing constraint is slack, and $\delta = 1$.*
- *If $\theta_L^{\text{uc}} \leq \theta_L^{\text{bc}}$, the financing constraint binds, and $\delta = -\frac{x_0}{\mathbb{E}[\theta_L^{\text{bc}} - \theta \mid \theta \leq \theta_L^{\text{bc}}]} \cdot \frac{q'\left(x_1^f(\theta_L^{\text{bc}})\right)}{q\left(x_1^f(\theta_L^{\text{bc}})\right)}$.*

The threshold θ_L^{uc} captures the debt limit that would be optimal absent the initial financing requirement. If $x_0 > x_1^*(\mathbb{E}[\theta])$, initial debt is already high relative to the ex ante efficient benchmark, so the unconstrained solution sets the tightest limit, $\theta_L^{\text{uc}} = \bar{\theta}$. Otherwise, θ_L^{uc} is characterized by (17), which balances the welfare loss from constraining low-θ borrowers against the over-borrowing of high-θ borrowers. According to Proposition 1, the optimal policy takes $\theta_L = \max\{\theta_L^{\text{uc}}, \theta_L^{\text{bc}}\}$, acknowledging that the threshold must be at least θ_L^{bc} to satisfy the initial financing requirement. When $\theta_L^{\text{uc}} > \theta_L^{\text{bc}}$, the financing constraint does not bind. When $\theta_L^{\text{uc}} < \theta_L^{\text{bc}}$, the constraint binds and generates a positive Lagrange multiplier, reflecting the shadow cost of the financing requirement. If $\theta_L^{\text{uc}} = \theta_L^{\text{bc}}$ at an interior threshold, the constraint binds, but $\delta = 1$ and $\lambda = 0$.

Proposition 1 optimizes within the class of debt limits. The next proposition establishes when this candidate is globally optimal among all incentive-compatible debt schedules for a given x_0.[6] We construct a cumulative multiplier for the continuum of incentive constraints and show that the proposed debt limit and multiplier form a saddle point of the Lagrangian. Condition (18) guarantees the required monotonicity of that multiplier.

Proposition 2 (Sufficient Conditions for Optimality). *Consider the debt limit in Proposition 1. Let $\delta = 1/(1+\lambda)$ where λ is the Lagrange multiplier of the financing constraint in* (16)*. If the*

[6] The continuation problem is written as (31) in the appendix.

function

$$\delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta) \tag{18}$$

is non-decreasing for $\theta \in [\theta_L, \bar{\theta}]$. *Then,* $x_1^{n*}(\theta) = \min\{x_1^f(\theta), x_1^f(\theta_L)\}$ *solves the mechanism design problem* (12).

Condition (18) is a verification condition on the debt-limit schedule above the cutoff type. It ensures that, once the cutoff θ_L is chosen, leaving types above the cutoff unconstrained maximizes the Lagrangian. The linear example below translates this monotonicity condition into a primitive restriction on the density of types, allowing the sufficient condition to be determined without solving the full mechanism design problem.

Linear Example. *Under non-exclusive lending,*

$$x_1^f(\theta, x_0) = \frac{(1-\theta)Y + x_0}{2-\theta}, \qquad q(x_1^f(\theta, x_0)) = \frac{Y - x_0}{c}\frac{1}{2-\theta}.$$

A sufficient condition for (18) *is*

$$\sup_{\theta\in\Theta}(2-\theta)\frac{g'(\theta)}{g(\theta)} \leq 1.$$

In what follows, we restrict attention to debt limits. If condition (18) holds for every feasible x_0, this restriction is without loss; otherwise, our results characterize the optimal debt limit, which need not solve the unrestricted mechanism-design problem.

3.3 Initial Debt and Implementation

We now proceed to the second step and find the optimal value of x_0 by solving the following problem:

$$\begin{aligned} &\max_{x_0}\ W_0(x_0) \\ &\text{subject to } x_0 q(x_0) \geq I, \end{aligned} \tag{19}$$

where $W_0(x_0)$ has been defined in (12). The constraint $x_0 q(x_0) \geq I$ is both sufficient and necessary for the feasible set in problem (12) to be non-empty. This constraint holds if and only if $x_0 \in [x_0^{\min}, x_0^{\max}]$, where $x_0^{\min} \leq x_0^{\max}$ are the lower and upper solutions to $x_0 q(x_0) = I$. The solutions are distinct when $I < I^{\max}$ and coincide when $I = I^{\max}$.[7]

Figure 3 illustrates the problem in (19). The function $\theta_L^{\mathrm{bc}}(x_0)$ provides the maximum level of flexibility that allows us to fund the project by issuing a claim promising x_0 at $t = 2$. The borrower's

[7]Assumption 2 guarantees existence. Because $q' < 0$ and $q'' \leq 0$, the function $x_0 q(x_0)$ is strictly concave on the interior of its domain.

indifference curves represent the borrower's preferences over pairs (x_0, θ_L), with darker curves indicating higher payoff. For a given θ_L (and thus a fixed debt limit), increasing x_0 always lowers the borrower's payoff because it introduces more distortion from over-borrowing. By contrast, for a given x_0, the borrower's payoff is maximized when θ_L (and hence the debt limit) is neither too high nor too low. This is because a high debt limit offers excessive financial flexibility and leads to over-borrowing, whereas a low limit restricts flexibility and results in under-borrowing. If the financing constraint binds, the optimal solution is given by the tangency point of the indifference curve with the boundary $\theta_L^{\mathrm{bc}}(x_0)$.

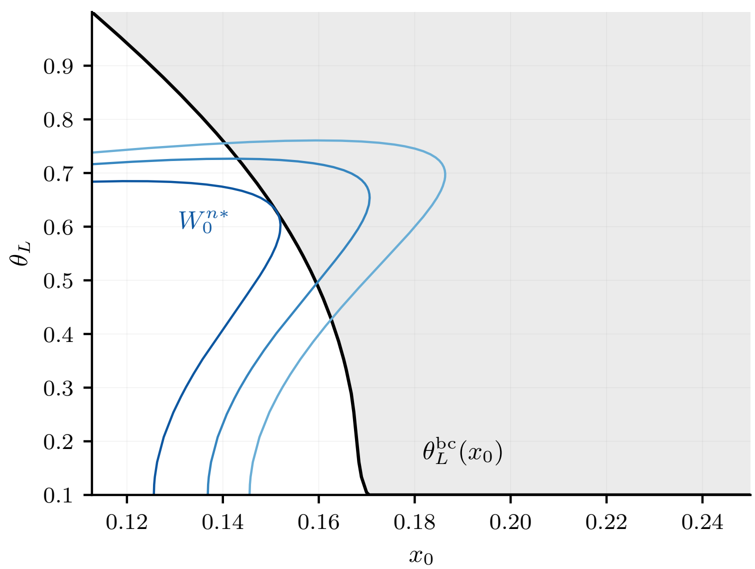

Figure 3: Indifference curves and optimal pair (x_0, θ_L) for $\theta \sim U[0.1, 1]$, $Y = 1$, $c(q) = \frac{1}{2}q^2$, and $I = 0.4 \times I^{\max}$.

The next proposition characterizes the optimal initial borrowing and its associated debt limit.

Proposition 3 (Necessary Conditions Initial Debt)**.** *Let x_0^{n*} be a solution to (19), and let θ_L^{n*} denote the cutoff associated with the debt limit $x_1^f(\theta_L^{n*})$ that solves (12) at x_0^{n*}. Define*

$$x_f^{n*}(\theta) := x_1^f(\theta, x_0^{n*}), \qquad x_L^{n*} := x_f^{n*}(\theta_L^{n*}).$$

1. *If $x_1^*(\mathbb{E}[\theta]) \leq x_0^{\min}$ there is no flexibility for additional financing at $t = 1$. In this case, $x_0^{n*} = x_0^{\min}$ and $\theta_L^{n*} = \bar{\theta} = 1$.*

2. *If $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$ there is flexibility for additional financing at $t = 1$. In this case, $\theta_L^{n*} < \bar{\theta} = 1$*

and $x_0^{n*} > x_0^{\min}$. *Specifically,* $\{x_0^{n*}, \theta_L^{n*}\}$ *jointly solve*

$$x_0^{n*}\mathbb{E}\left[q(x_L^{n*})\mathbf{1}_{\{\theta\le\theta_L^{n*}\}} + q(x_f^{n*}(\theta))\mathbf{1}_{\{\theta>\theta_L^{n*}\}}\right] = I$$

$$x_0^{n*} = \frac{(1-\delta)\mathbb{E}\left[q(x_L^{n*})\mathbf{1}_{\{\theta\le\theta_L^{n*}\}} + q(x_f^{n*}(\theta))\mathbf{1}_{\{\theta>\theta_L^{n*}\}}\right]}{-\mathbb{E}\left[q'(x_f^{n*}(\theta))\frac{\partial x_1^f(\theta,x_0^{n*})}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L^{n*}\}}\right]} \tag{20}$$

and

$$\delta = -\frac{x_0^{n*}}{\mathbb{E}[\theta_L^{n*}-\theta|\theta\le\theta_L^{n*}]}\frac{q'(x_L^{n*})}{q(x_L^{n*})}.$$

Proposition 3 includes two cases based on the comparison between $x_1^*(\mathbb{E}[\theta])$ and $x_0^{\min}$. In the first case, $x_1^*(\mathbb{E}[\theta]) \le x_0^{\min}$, so the optimal total debt without knowing θ falls below the minimum initial debt required to satisfy the financing constraint. Therefore, the financing constraint by itself already causes over-borrowing, and the borrower's incentive to dilute old debt at $t = 1$ further exacerbates the problem. The optimal policy therefore imposes a debt limit that eliminates any further incentives of over-borrowing, which leads to no financial flexibility. In the second case, $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$, so the minimum initial debt remains below the ex-ante optimal debt. The optimal policy preserves some financial flexibility to balance between over- and under-borrowing. At $t = 1$, the agent can still borrow flexibly when the liquidity shock is low (high θ).

A key feature of Proposition 3 is that the financing constraint always binds under non-exclusive lending. In other words, it is never optimal to borrow more than I at $t = 0$. Instead, the optimal strategy selects initial debt x_0 and debt limit θ_L that provide exactly enough capital to fund investment I and nothing more. The reason is straightforward: less borrowing at $t = 0$ reduces over-borrowing incentives at $t = 1$, creating greater financial flexibility.

Linear Example. *Let* $\hat{x}_0 := x_0/Y$, $\hat{x}_1^*(\theta) := x_1^*(\theta)/Y$, *and* $\kappa := cI/Y^2$,

$$\hat{x}_0^{\min} = \frac{1}{2}\left(1-\sqrt{1-4\kappa}\right), \qquad \hat{x}_1^*(\mathbb{E}[\theta]) = \frac{1-\mathbb{E}[\theta]}{2-\mathbb{E}[\theta]}.$$

Hence, interior financial flexibility requires

$$\mathbb{E}[\theta] < \frac{2\sqrt{1-4\kappa}}{1+\sqrt{1-4\kappa}}.$$

In the linear example, the debt capacity in (3) *is* $I^{\max} = Y^2/(4c)$, *so*

$$\kappa = \frac{1}{4}\frac{I}{I^{\max}}$$

measures the investment required relative to debt capacity. In Subsection 3.4, we refer to κ as the financing-needs parameter.

Proposition 3 provides necessary optimality conditions. To fully characterize the solution to the borrower's optimization problem (19), it suffices to show that the system in equation (20) has a unique solution. The next proposition establishes a condition on the inverse-demand elasticity that makes the objective function single-peaked in the initial amount of debt; this condition implies that relaxing the debt limit is valuable when flexibility is scarce, but eventually the added dilution cost outweighs the benefit.

Proposition 4 (Sufficient Conditions Initial Debt)**.** *Define the elasticity of the inverse demand for debt by*

$$\varepsilon(x) := -\frac{xq'(x)}{q(x)}$$

and define

$$\Gamma(x) := \frac{x\varepsilon'(x) - \varepsilon(x)}{\varepsilon(x)^2}.$$

If Γ is nonincreasing on the relevant debt interval, then the necessary conditions in Proposition 3 are also sufficient. In particular, if $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$, then the system in (20) has a unique solution, and this solution is the unique maximizer of (19).

The curvature condition is easy to verify in the linear-quadratic example. The same argument applies to the power-cost specification $c(q) = cq^\eta$ with $\eta \geq 2$: in that case, $\Gamma(x) = \eta - 1$.[8] Thus Γ is constant, so it is nonincreasing, which means that Proposition 4 applies to the quadratic case and to all power-cost specifications with $\eta \geq 2$.

We conclude this subsection by discussing implementation. The optimal allocation can be implemented using initial long-term debt and a covenant. At $t = 0$, the borrower issues debt with face value x_0^{n*}, and the initial debt contract includes an incurrence covenant that caps total debt due at $t = 2$ at x_L^{n*}. The borrower remains free to borrow from new lenders at $t = 1$, provided that total debt does not exceed this limit. The covenant, therefore, protects initial lenders without requiring exclusive lending.

In a practitioner handbook explaining the basic structure and negotiation of typical high-yield covenant packages, the law firm Simpson Thacher describes the same tradeoff:

> High yield covenants always seek to strike a delicate balance . . . On the one hand, the covenants provide protection for high yield investors against an issuer's overextending

[8] The induced debt-price schedule is $q(x) = ((Y - x)/(c\eta))^{1/(\eta-1)}$. Thus $q(x) = A(Y - x)^p$ with $p = 1/(\eta - 1)$, so $\varepsilon(x) = px/(Y - x)$ and $\Gamma(x) = 1/p = \eta - 1$.

> itself or unwisely using cash. . . . On the other hand, the covenants must provide flexibility for the issuer to operate its business and grow. . . . In other words, the covenants protect the investors' ability to be paid principal and interest . . . while preserving the issuer's ability to run its business and grow without undue restrictions. (Simpson Thacher & Bartlett LLP, 2022, p. 2)

In our model, a cap on total debt captures this tradeoff: it protects investors from dilution while preserving the borrower's flexibility to issue additional debt up to the cap.

3.4 Comparative Statics

In this section, we consider comparative statics in the context of the linear example. In this case, after scaling by Y, the solution depends only on financing needs κ and the distribution of discount factor shock θ. The central comparative static concerns financing needs: a higher κ requires more initial debt and a tighter limit on future borrowing. For the comparative statics exercise, we will restrict attention to parameter values for which the optimal contract provides some flexibility, that is, $\theta_L^{n*} < \bar{\theta} = 1$. For this reason, the comparative statics provided will be local, in the sense that they apply to small changes in parameters that do not imply a change in regime (from a solution with some flexibility to one without any flexibility). For the comparative statics results with respect to the distribution of θ, we consider a smooth one-parameter family of distributions $g(\cdot;\gamma)$ on $\Theta = [\underline{\theta}, 1]$, with cumulative density function $G(\cdot;\gamma)$. We assume that $g(\theta;\gamma)$ is strictly log-supermodular in (θ, γ), so that the family is ordered according to the monotone likelihood ratio (MLR) order.

Proposition 5 (Comparative statics in the linear example)**.** *Consider the linear example and let $\hat{x}_0 \coloneqq x_0/Y$. For each feasible (κ, γ), let $\theta_L^*(\kappa, \gamma)$ and $\hat{x}_0^*(\kappa, \gamma)$ denote the unique interior binding solution to the optimality conditions in Proposition 3.*

- *Financing needs: Fix γ, and suppose that $x_1^*(\mathbb{E}_\gamma[\theta]) > x_0^{\min}$ for every κ under comparison. Then $\theta_L^*(\kappa, \gamma)$ and $\hat{x}_0^*(\kappa, \gamma)$ are nondecreasing in κ.*

- *MLR Comparative Statics Near the No-Flexibility Boundary: Fix γ, and let*

$$\kappa^1 \coloneqq \frac{1 - \mathbb{E}_\gamma[\theta]}{(2 - \mathbb{E}_\gamma[\theta])^2}$$

denote the no-flexibility level for κ given γ defined by $\theta_L^*(\kappa^1, \gamma) = 1$. There exists $\kappa^\dagger < \kappa^1$

such that, for every $\kappa \in (\kappa^\dagger, \kappa^1)$,

$$\frac{\partial \theta_L^*}{\partial \gamma}(\kappa, \gamma) > 0, \qquad \frac{\partial \hat{x}_0^*}{\partial \gamma}(\kappa, \gamma) < 0.$$

For the financing-needs parameter, a higher κ increases initial debt and leaves less flexibility for new issuance. Figure 4 shows how the optimal debt policy under multiple lenders varies with financing need parameter κ. As κ increases, the initial debt level x_0 also rises, amplifying the borrower's incentive to dilute initial lenders at $t = 1$. This exacerbates over-borrowing, requiring a tighter debt limit. As a result, a larger range of types faces binding constraints ex post. However, for unconstrained types, the debt level increases with κ.

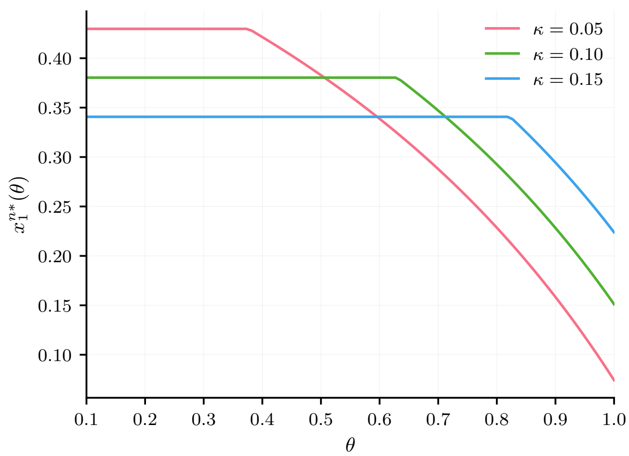

Figure 4: Comparative statics of face value at $t = 1$ for changes in financing need κ, with $\theta \sim U[0.1, 1]$, $Y = 1$, and $c(q) = \frac{1}{2}q^2$.

The effect of changes in the distribution on θ is ambiguous. As the example in Figure 5 illustrates, there is no comparative statics result that holds for every log-supermodular family of θ distributions and every financing need κ. The direct intuition is that an upward shift in θ reduces the borrower's need for additional financing at $t = 1$. The borrower then demands less flexibility and chooses a higher θ_L^*. Proposition 5 confirms this effect when flexibility is already low, so θ_L^* is close to its upper bound. This direct effect, however, ignores the price the borrower pays lenders for flexibility. An upward shift in θ also raises the price of debt at $t = 0$ because initial lenders are less exposed to future dilution. The lower flexibility premium can increase the borrower's demand for flexibility and, in some cases, can dominate the direct effect, reducing θ_L^* and increasing x_0.

Let $\chi(\theta_L)$ denote the low-debt binding schedule in Figure 3, corresponding to the inverse of $\theta_L^{\text{bc}}(\hat{x}_0)$ in the low-debt region. When the financing constraint binds, the function $\chi(\theta_L)$ represents the yield schedule that the borrower faces: for a fixed debt-limit threshold θ_L, it specifies the (scaled) initial debt the borrower must issue to raise the required funds, and therefore captures the yield of the debt issued at $t = 0$.[9] Because looser restrictions correspond to a lower θ_L, the schedule slopes downward. More flexibility requires a higher promised yield, so $\chi'(\theta_L)$ is the marginal price of flexibility. On the preference side, the borrower's objective defines an indifference curve in the $(\hat{x}_0, \theta_L)$ plane, with slope given by the marginal rate of substitution between x_0 (the cost of debt reflected in the yield) and θ_L (future issuance flexibility), denoted by $\text{MRS}_{\hat{x}_0,\theta_L}$. As in a standard demand problem, an interior optimum equates the marginal willingness to trade yield for flexibility with the marginal price of flexibility, that is $\text{MRS}_{\hat{x}_0,\theta_L} = \chi'(\theta_L)$. The Appendix verifies that an upward shift in the distribution of θ, in the sense of MLR, moves both sides of the optimality condition in the same direction. It makes initial debt safer and therefore makes flexibility cheaper, but it also reduces the borrower's marginal willingness to pay for flexibility. The comparative static is therefore ambiguous for the same reason that a demand response can be ambiguous with both substitution and income effects. The sign depends on whether the price effect or the preference effect dominates.

Figure 5 shows these two effects in the (x_0, θ_L) plane. In each panel, the black curves are the binding financing schedules, and the colored curves are the borrower's indifference curves. The optimal pair (x_0^*, θ_L^*) corresponds to the tangency point between these two curves. In the left panel, we consider the case where financing needs are high, and the direct effect on the value of flexibility dominates the pricing effect. The upward MLR shift, therefore, moves the tangency to a higher threshold θ_L^*. In the right panel, financing needs are low, and the price effect dominates, so the new tangency has a lower threshold θ_L^*.

4 Exclusive Lending

We now turn to the exclusive-lending environment, in which the borrower can commit to a long-term relationship with a single lender. Results in this section serve as a benchmark for evaluating the allocation's efficiency, as they correspond to the allocation a constraint planner would choose.

4.1 Complete Information

First, we will consider the case when there is no private information; so the only friction is the moral hazard problem at $t = 2$. Suppose that θ is observable and contractible. The borrower then chooses the long-term contract that maximizes her expected payoff subject to the lender's

[9] After multiplying by the return factor Y/I, it gives the yield x_0/I on initial debt.

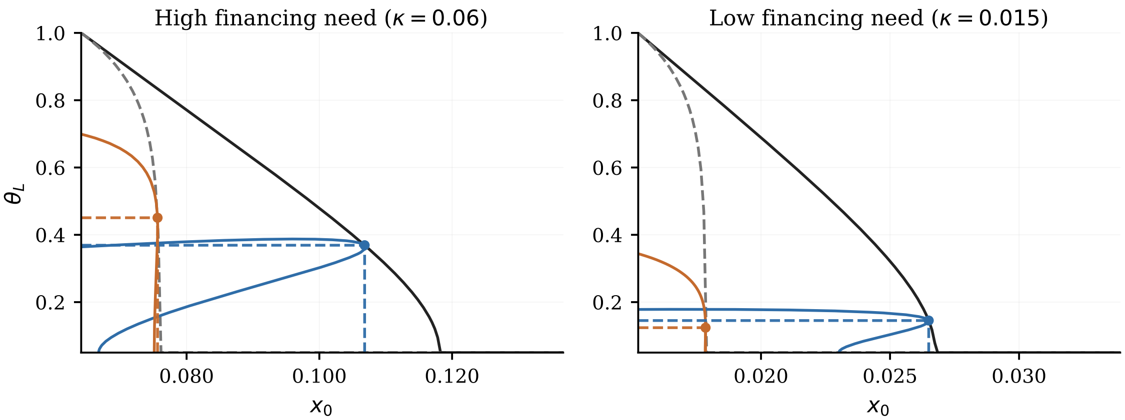

Figure 5: Indifference curves and binding financing schedules for changes in the distribution of θ, with $Y = 1$, $c(q) = \frac{1}{2}q^2$, and $g(\theta;\gamma) \propto e^{\gamma\theta}$ on $\Theta = [0.05, 1]$. The solid black financing schedule and blue indifference curve use $\gamma = -5$; the dashed gray financing schedule and orange indifference curve use $\gamma = 5$. The left panel sets $\kappa = 0.06$; the right panel sets $\kappa = 0.015$.

participation constraint and the basic financing and feasibility constraints.

$$\begin{aligned}
&\max_{\{\tau_0,\tau_1(\theta),x_1(\theta)\}} \quad \tau_0 - I + \mathbb{E}\Big[\tau_1(\theta) + \theta V_2\left(x_1(\theta)\right)\Big] \\
&\text{subject to} \\
&\mathbb{E}\Big[x_1(\theta)q\left(x_1(\theta)\right) - \tau_1(\theta)\Big] - \tau_0 \geq 0 \\
&\tau_0 \geq I, \qquad \tau_1(\theta) \geq 0, \qquad x_1(\theta) \in [0, Y].
\end{aligned} \tag{21}$$

The lender's participation constraint is always binding, and it is without loss of generality to set $\tau_1(\theta) = 0$. The remaining question is whether the borrower can finance the initial investment at the ex-post efficient debt level $x_1^*(\theta)$.

Lemma 3 (Complete Information)**.** *Let*

$$I^c := \mathbb{E}\Big[x_1^*(\theta)q\big(x_1^*(\theta)\big)\Big].$$

Suppose that θ is observable and contractible.

- *If $I^c \geq I$, the financing constraint is slack. The optimal allocation satisfies $x_1^{c*}(\theta) = x_1^*(\theta)$ and $\tau_0 = I^c$.*

- *If $I^c < I$, the financing constraint binds. Let $\delta^{c*} \in (0,1)$ be the solution to*

$$\mathbb{E}\Big[x_1^*(\delta^{c*}\theta)\, q\big(x_1^*(\delta^{c*}\theta)\big)\Big] = I.$$

  Then the optimal allocation satisfies $x_1^{c*}(\theta) = x_1^*(\delta^{c*}\theta)$ and $\tau_0 = I$.

The term $\mathbb{E}[x_1^*(\theta)q(x_1^*(\theta))]$ is the ex-ante pledgeable income at $t = 0$ under the ex-post efficient debt schedule. If it covers the initial investment I, the financing constraint is slack and the efficient schedule remains feasible and optimal. If it does not, the financing constraint binds, and the borrower must pledge more. The second case of Lemma 3 shows that this allocation is equivalent to the efficient schedule evaluated at the scaled shock $\delta^{c*}\theta$: when the financing constraint binds, it acts like a lower realization of the discount-factor shock and pushes total debt above the ex post efficient level. The endogenous discount factor δ^{c*} summarizes the tightness of the financing constraint: when the constraint binds, a lower δ^{c*} corresponds to a larger Lagrange multiplier and therefore to a tighter financing distortion.

4.2 Private Information under Exclusive Lending

We now turn to the case where the borrower privately observes the discount-factor shock θ before refinancing; we refer to this solution as the *information-constrained optimal* allocation. By the revelation principle, we can restrict attention to direct revelation mechanisms $\{\tau_0, \tau_1(\theta), x_1(\theta)\}$. The optimal mechanism can be found by solving the problem in (21) with the additional truth-telling IC constraint

$$\theta \in \arg\max_{\hat{\theta}} \; \tau_1(\hat{\theta}) + \theta V_2(x_1(\hat{\theta})). \tag{22}$$

Using the envelope theorem, we can rewrite (22) as

$$\begin{aligned} &\tau_1(\theta) + \theta V_2\left(x_1(\theta)\right) = \tau_1(\bar{\theta}) + \bar{\theta} V_2(x_1(\bar{\theta})) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta}, \;\forall \theta \in \Theta \\ &x_1(\theta) \text{ is non-increasing.} \end{aligned} \tag{23}$$

As before, the lender's $t = 0$ participation constraint must bind. The lender's payoff is $x_1(\theta)q(x_1(\theta)) - \tau_1(\theta) - \tau_0$, and a simple calculation shows that the initial financing constraint becomes[10]

$$\tau_0 = \mathbb{E}\left[W_1(x_1(\theta),\theta) + \frac{G(\theta)}{g(\theta)}V_2(x_1(\theta))\right] - \tau_1(\bar{\theta}) - \bar{\theta}V_2(x_1(\bar{\theta})) \geq I. \tag{24}$$

Substituting τ_0 and $\tau_1(\theta)$ into the objective function, the optimal contract solves[11]

$$\begin{aligned} &\max_{\{x_1(\theta),\tau_1(\theta)\geq 0\}} \mathbb{E}\left[W_1(x_1(\theta),\theta)\right] - I \\ &\text{subject to (23) and (24)} \end{aligned} \tag{25}$$

We make the following assumption to ensure that the previous optimization problem is convex:

Assumption 5. *For all $\theta \in \Theta$, the function*

$$W_1(x_1,\theta) + \frac{G(\theta)}{g(\theta)}V_2(x_1)$$

is concave in x_1. Under Assumption 1, this holds if

$$2 + \frac{x_1 q''(x_1)}{q'(x_1)} \geq \bar{\theta} + \frac{1}{g(\bar{\theta})}.$$

Linear Example. *Consider the linear example with quadratic cost of effort. Under the assumption that $G(\theta)$ is log-concave, Assumption 5 holds whenever*

$$(2-\bar{\theta})g(\bar{\theta}) \geq 1.$$

Note that problem (25) differs from the standard mechanism design problem in that it has an additional constraint $\tau_1(\theta) > 0$. Lemma 16 in the appendix proves that $\tau_1(\theta)$ must be non-increasing. Intuitively, borrowers with greater financing needs (i.e., lower θ) have incentives to borrow more. This result implies that we only need to evaluate the constraint at $\theta = \bar{\theta}$. In addition, it is optimal to set $\tau_1(\bar{\theta}) = 0$. Otherwise, the borrower can reduce $\tau_1(\theta)$ for all θ by the same amount and borrow more at $t = 0$, which relaxes the financing constraint.[12]

---

[10]This involves the following simplification by changing the order of integration:

$$\int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))g(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\tilde{\theta}} V_2(x_1(\tilde{\theta}))g(\theta)d\theta d\tilde{\theta} = \int_{\underline{\theta}}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))G(\tilde{\theta})d\tilde{\theta}.$$

The problem also substitutes the envelope condition (23) in the participation constraint.

[11]The constraint $x_1(\theta) \in [0, Y]$ will always be slack and hence omitted.

[12]If the financing constraint is slack, the borrower is indifferent between borrowing at $t = 0$ and $t = 1$ due to no

We solve the problem in two steps. First, let λ denote the Lagrange multiplier on the financing constraint $\tau_0 \geq I$, and define $\delta \coloneqq 1/(1+\lambda)$. For a given δ, define the virtual type

$$\phi_\delta(\theta) \coloneqq \theta + (1-\delta)\frac{G(\theta)}{g(\theta)}. \tag{26}$$

The distortion from incentive provision is captured by evaluating payoffs at the virtual type, since

$$W_1(x_1, \phi_\delta(\theta)) = W_1(x_1, \theta) + (1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1).$$

Given this, the optimal contract solves

$$\begin{aligned} &\max_{\{x_1(\theta)\}} \ \mathbb{E}\left[W_1(x_1(\theta), \phi_\delta(\theta))\right] - (1-\delta)\bar{\theta}V_2(x_1(\bar{\theta})) \\ &\text{subject to} \\ &x_1(\theta) \text{ is non-increasing.} \end{aligned} \tag{27}$$

The pointwise optimizer inside the expectation is $x_1^*(\phi_\delta(\theta))$. When $\delta = 1$, $\phi_1(\theta) = \theta$, so the solution coincides with the efficient rule $x_1^*(\theta)$. More generally, Assumption 1 ensures that $x_1^*(\phi_\delta(\theta))$ is non-increasing. However, pointwise maximization does not solve (27) because the objective also includes the final term involving $\bar{\theta}$.[13] When the constraint $\tau_1(\theta) \geq 0$ is slack, the term $(1-\delta)G(\theta)/g(\theta)$ is the cost of providing incentives for truthful reporting. The principal therefore chooses debt as if the borrower had a higher effective discount factor than her realized type. Since a borrower with a higher discount factor values equity more relative to debt, $x_1^*(\phi_\delta(\theta))$ assigns less debt than the efficient schedule for the same realized θ. The final term in (27) captures the rent left to the highest type by the limited liability constraint $\tau_1(\theta) \geq 0$, and that term is what eventually creates bunching at the top of the schedule.

By applying the method from the global theory of constrained optimization, we have the following result.

Proposition 6. *For a given δ, let $x_1^{s*}(\theta)$ denote the optimal debt schedule solving problem (27).*

discount between the two dates. As long as there is a slight preference for $t = 0$ borrowing, $\tau_1(\bar{\theta}) = 0$ would hold.

[13] To see this, note that $x_1(\bar{\theta})$ is chosen to maximize

$$W_1(x_1(\bar{\theta}), \bar{\theta}) + (1-\delta)\frac{G(\bar{\theta})}{g(\bar{\theta})}V_2(x_1(\bar{\theta})) - (1-\delta)\bar{\theta}V_2(x_1(\bar{\theta})),$$

so that the point-wise maximum solution will lead to a solution where $x_1(\theta)$ has an upward jump at $\bar{\theta}$ unless $\delta = 1$.

- *If $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$, the solution to problem (27) is*

$$x_1^{s*}(\theta) = \max\{x_1^*(\phi_\delta(\theta)), x_1^*(\phi_\delta(\theta_H))\},$$

  where $\theta_H \in \Theta$ solves:

$$\int_{\theta_H}^{\bar{\theta}} \frac{\partial W_1(x_1^*(\phi_\delta(\theta_H)), \theta)}{\partial x_1} dG(\theta) + (1-\delta) q\left(x_1^*(\phi_\delta(\theta_H))\right)\left(\theta_H G(\theta_H) + \int_{\theta_H}^{\bar{\theta}} \theta dG(\theta)\right) = 0. \quad (28)$$

- *If $\delta < \underline{\theta}/\mathbb{E}[\theta]$, $x_1^{s*}(\theta) = x_1^*(\delta\mathbb{E}[\theta])$ for all $\theta \in \Theta$.*

The optimal transfers $\{\tau_0, \tau_1(\theta)\}$ satisfy (23) and (24).

Proposition 6 describes the optimal level of debt under different degrees of financing constraint. When $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$, the financing constraint is not too tight and may even be slack. In this case, the optimal debt level initially follows $x_1^*(\phi_\delta(\theta))$ and therefore decreases with θ. Once θ reaches θ_H, the constraint $\tau_1(\theta) \geq 0$ binds, so the schedule can no longer follow the point-wise solution and instead remains flat at $x_1^*(\phi_\delta(\theta_H))$. In contrast, when $\delta < \underline{\theta}/\mathbb{E}[\theta]$, the financing constraint becomes very tight and the solution is fully pooled. In that case, the common debt level is $x_1^*(\delta\mathbb{E}[\theta])$, which is the efficient debt allocation evaluated at the lower aggregate type $\delta\mathbb{E}[\theta]$.

Borrowers with higher θ have lower financing needs at $t = 1$, so $x_1^*(\phi_\delta(\theta))$ gives them less debt and lower interim transfers. When the borrower must raise more funds at $t = 0$, the principal would like to lower transfers further and pledge more cash flow. For lower and intermediate types, that pressure operates through shifting the discount factor from θ to $\phi_\delta(\theta)$ in equation (26), which distorts the debt schedule relative to $x_1^*(\theta)$. For sufficiently high types, however, the limited liability constraint $\tau_1(\theta) \geq 0$ binds. At that point the mechanism cannot reduce transfers any further, so incentive compatibility requires the face value of debt to remain constant for all types above θ_H.

We now proceed to the second step.

Proposition 7. *Let*

$$I^s := \mathbb{E}\left[W_1\left(x_1^*(\theta), \theta\right) + V_2\left(x_1^*(\theta)\right)\frac{G(\theta)}{g(\theta)}\right] - \bar{\theta} V_2\left(x_1^*(\bar{\theta})\right)$$

and

$$I^p := x_1^*(\underline{\theta}) q\left(x_1^*(\underline{\theta})\right),$$

where $x_1^*(\theta)$ is defined in (5). Then:

- *If $I \leq I^s$, the optimal policy is $x_1^{s*}(\theta) = x_1^*(\theta)$, and $\delta^{s*} = 1$.*

- *If $I^s < I < I^p$, the optimal policy is the partially separating schedule*

$$x_1^{s*}(\theta) = \max\left\{x_1^*\left(\phi_{\delta^{s*}}(\theta)\right), x_1^*\left(\phi_{\delta^{s*}}(\theta_H)\right)\right\}.$$

The values $\delta^{s*} \in (\underline{\theta}/\mathbb{E}[\theta], 1)$ and $\theta_H \in (\underline{\theta}, \bar{\theta})$ jointly solve (28) and $\tau_0 = I$.

- *If $I^p \leq I \leq I^{\max}$, the optimal policy is fully pooled:*

$$x_1^{s*}(\theta) = x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right) \quad \text{for all } \theta \in \Theta,$$

where $\delta^{s*} \in [0, \underline{\theta}/\mathbb{E}[\theta]]$ solves

$$x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right) q\left(x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right)\right) = I.$$

At $I = I^p$, $\delta^{s*} = \underline{\theta}/\mathbb{E}[\theta]$; at $I = I^{\max}$, $\delta^{s*} = 0$.

Under Assumption 4, $I^s = 0$, in which case for every positive initial investment, the financing constraint binds.

The threshold I^s equals $I^c - \mathbb{E}[\tau_1(\theta)]$ under the efficient debt schedule. When lenders expect to provide the interim transfer $\tau_1(\theta)$, they reduce initial lending by $\mathbb{E}[\tau_1(\theta)]$ relative to the complete-information benchmark. If $I \leq I^s$, financing needs do not distort the debt schedule, which remains ex post efficient. Private information may still generate information rents, but those rents do not distort debt when financing is slack (Baron and Besanko, 1984).

If $I^s < I < I^p$, the financing constraint binds and the optimal schedule is partially separating. Debt follows the virtual-type allocation for lower and intermediate types, while higher types pool above θ_H because their limited-liability constraints bind. If $I \geq I^p$, financing needs are sufficiently high that all types pool at a common debt level. Incentive compatibility then implies zero interim transfers for every type, and the common debt level rises as financing needs increase.

4.3 Implementation of the Exclusive Benchmark

The exclusive-lending allocation can be implemented as a long-term contract signed at $t = 0$ together with a state-contingent credit line at $t = 1$. Let $x_0^s \coloneqq x_1^{s*}(\bar{\theta})$ denote the face value of outstanding debt for the type with the lowest refinancing need. If the solution is fully pooled, then $x_1^{s*}(\theta) = x_0^s$ for all θ, so there is no active refinancing margin at $t = 1$. Otherwise, each type on the separating region $[\underline{\theta}, \theta_H]$ chooses additional debt

$$\Delta x_1^s(\theta) \coloneqq x_1^{s*}(\theta) - x_0^s.$$

Proposition 8 (Implementation of the Exclusive Benchmark)**.** *If $\theta_H > \underline{\theta}$, let $\theta^s(\Delta x_1)$ denote the inverse of $\Delta x_1^s(\theta)$ on $[\underline{\theta}, \theta_H]$. Then the exclusive allocation can be implemented by the nonlinear price schedule*

$$p_1^s(\Delta x_1) \coloneqq \frac{1}{\Delta x_1} \int_0^{\Delta x_1} \theta^s(z) q(x_0^s + z)\, dz.$$

Under this schedule,

$$\tau_1^{s*}(\theta) = \Delta x_1^s(\theta)\, p_1^s(\Delta x_1^s(\theta)) \quad \forall \theta \in [\underline{\theta}, \theta_H],$$

and

$$p_1^s(\Delta x_1) < q(x_0^s + \Delta x_1) \quad \textit{for every } \Delta x_1 > 0.$$

Proposition 8 shows that the exclusive allocation takes the form of a credit line with nonlinear pricing of additional debt at $t = 1$. For a draw with face value Δx_1, the lender advances $\Delta x_1\, p_1^s(\Delta x_1)$ in cash, while the borrower promises success-contingent repayment with face value Δx_1. Thus, $p_1^s(\Delta x_1)$ is the average cash price per unit of face value along the exclusive allocation. Because $p_1^s(\Delta x_1) < q(x_0^s + \Delta x_1)$, the borrower receives less cash per unit of face value from the existing lender's line than from competitive spot borrowing. Equivalently, the line charges a higher implicit interest rate, and that wedge compensates the existing lender for the effect of refinancing on the legacy claim.

The exclusive benchmark is implementable under two conditions. First, it must restrict access to outside borrowing at $t = 1$; otherwise, the borrower can avoid the existing lender's pricing schedule and issue the same additional debt Δx_1 to new lenders at the market price $q(x_0^s + \Delta x_1)$. Second, the existing lender must be able to commit to supply state-contingent future lending through the credit line.

In practice, initial lenders can use covenants to restrict outside borrowing and protect against dilution. However, they may be unable to commit to providing the state-contingent financing required under a credit line, especially when they are small and dispersed or face liquidity shortages of their own. Section 3 therefore studies the best feasible contract when lending is non-exclusive and only simple debt contracts are available.

4.4 Comparison: Exclusive vs. Non-Exclusive Lending

Under exclusive lending, the incumbent lender sets a state-contingent price for additional credit to take into account how additional borrowing affects the existing claim, so the borrower bears the full cost of additional debt. Under non-exclusive lending with simple debt contracts, the initial lender cannot impose this state-contingent pricing schedule and can only restrict total debt. The debt limit binds for low values of θ and becomes slack for high values of θ, generating the single-crossing pattern illustrated in Figure 6 and formalized in the following proposition.

Proposition 9. *Suppose that Assumption 4 holds and that* $\theta_L^{n*}, \theta_H^{s*} \in (\underline{\theta}, \bar{\theta})$*. There exists a unique* $\theta^\dagger \in (\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\})$ *such that* $x_1^{n*}(\theta^\dagger) = x_1^{s*}(\theta^\dagger)$*, and*

$$x_1^{n*}(\theta) < x_1^{s*}(\theta) \quad \textit{for } \theta < \theta^\dagger, \qquad x_1^{n*}(\theta) > x_1^{s*}(\theta) \quad \textit{for } \theta > \theta^\dagger.$$

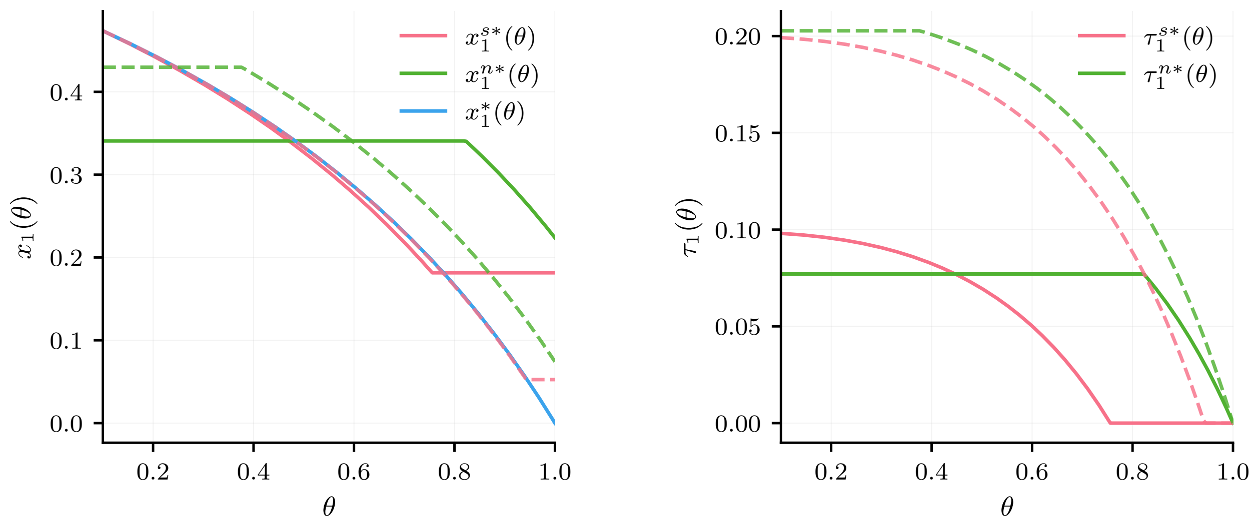

Figure 6: Optimal contract single lender and multiple lender. We assume that $\theta \sim U[0.1, 1]$, $Y = 1$, and $c(q) = \frac{1}{2}q^2$. Solid lines correspond to $I = 0.6 \times I^{\max}$, while dashed lines correspond to $I = 0.2 \times I^{\max}$, where $I^{\max} = \max_{x_1 \in [0,Y]} \; x_1 q(x_1)$.

Figure 6 compares the single-lender allocation with the best simple debt contract under non-exclusive lending. In the debt panel, the non-exclusive debt limit binds for low types, so non-exclusive debt starts below the single-lender allocation; once the limit becomes slack, the borrower borrows more because she does not internalize dilution. The transfer panel shows that the transfer comparison need not have the same crossing pattern. In the solid case, non-exclusive transfers are lower for low types and higher for high types. In the dashed case, the debt comparison still crosses, but non-exclusive transfers remain above exclusive-lending transfers.

The comparison makes clear what simple debt contracts lack: state-contingent compensation to existing lenders. Exclusive lending distorts borrowing only because of private information and the financing requirement. Non-exclusive lending adds an additional distortion because the borrower ignores how new debt reduces the value of old debt. With a simple debt contract, a debt limit is the best available response; however, in the next section, we show that the single-lender allocation can be achieved if we allow for richer instruments.

5 Richer Contracting Mechanisms and Implementation

So far, we have considered mechanisms that restrict future borrowing without affecting the terms of the initial debt contract. In this section, we consider clauses that modify the terms of the initial debt contract that can undo the distortion created by non-exclusive borrowing: performance-sensitive debt and contingent prepayment. We show that these richer mechanisms can implement the information-constrained single-lender allocation by compensating existing lenders for the dilution externality created by additional debt issuance. The mechanism follows the contract-contingent logic in Peters and Szentes (2012) and its extension to common agency in Szentes (2015). When contracts can condition on one another, a party can commit to how it will respond if another party deviates. Because the deviation then changes the responses it induces, these response rules can support allocations that are unavailable when contracts cannot condition on one another. In our setting, this logic operates *sequentially*: new borrowing triggers compensation for existing lenders, thereby making the borrower internalize the dilution it creates. Performance-sensitive debt adjusts the existing debt claim, whereas sweep-like clauses redirect part of the new proceeds to repay existing lenders. The two instruments relax different restrictions of simple debt contracts: performance-sensitive debt keeps repayment at $t = 2$ but allows the existing debt claim to vary with new issuance, whereas contingent prepayment keeps the existing claim fixed but requires new issuance at $t = 1$ to trigger repayment to existing lenders.

5.1 Performance-Sensitive Debt

First, we consider the case in which the existing debt claim x_0 can be state contingent, while all payments to existing lenders are still made at $t = 2$. The borrower can still issue new debt at $t = 1$, but the amount owed to existing lenders changes with the amount of new debt issued; thus, the interest rate on existing debt from $t = 1$ to $t = 2$ is state contingent. This adjustment compensates existing lenders when new borrowing dilutes their claim and makes the borrower internalize the dilution cost of new debt. We ask whether this performance-sensitive interest rate allows implementing the exclusive-lending allocation under non-exclusive lending.

Performance-sensitive debt allows payments on existing debt to be adjusted in response to verifiable changes in the borrower's condition. In loan agreements, this adjustment is typically implemented through performance-based pricing grids. These grids link the loan spread to measures such as credit ratings, Debt/EBITDA, or coverage ratios. Asquith et al. (2005) document that performance-pricing grids appear in 41% of their sample loans and 54% of loan volume in a sample of 8,761 bank loans issued over 1995–1998. They note that the widespread use of performance pricing coincided with the expansion of syndicated lending and show that syndicated loans are more likely to use interest-decreasing grids. They argue that syndication makes later renegotiation more costly, so the contract specifies rate reductions in advance. Our analysis points to a complementary role: under non-exclusive lending, performance-sensitive terms can preserve future financing while compensating existing lenders for dilution.

To implement the single-lender allocation, we adjust the initial lender's claim so that each type raises the same cash and ends with the same total debt as under exclusive lending. Let $\{\tau_0^{s*}, \tau_1^{s*}(\theta), x_1^{s*}(\theta)\}$ denote the exclusive-lending allocation characterized in Section 4, and let $x_0^s := x_1^{s*}(\bar{\theta})$ denote the baseline existing debt claim. For each report θ, define the existing debt claim

$$x_0^{\mathrm{ps}}(\theta) := x_1^{s*}(\theta) - \frac{\tau_1^{s*}(\theta)}{q(x_1^{s*}(\theta))}$$

and the new debt issued at $t = 1$ by

$$\Delta x_1^{\mathrm{ps}}(\theta) := x_1^{s*}(\theta) - x_0^{\mathrm{ps}}(\theta) = \frac{\tau_1^{s*}(\theta)}{q(x_1^{s*}(\theta))}.$$

The contract must therefore translate observed new issuance into the existing debt claim that implements the correct total debt for that type. The next proposition shows that performance-sensitive debt reproduces the exclusive-lending allocation for every issuance level on the equilibrium path.[14]

Proposition 10 (On-Path Replication with Nonlinear Performance-Sensitive Debt)**.** *Suppose the exclusive benchmark is not fully pooled, that is,* $\theta_H^{s*} > \underline{\theta}$*. On the separating region* $\theta \in [\underline{\theta}, \theta_H^{s*}]$*,* $\Delta x_1^{\mathrm{ps}}(\theta)$ *is invertible. Let* $\theta^{\mathrm{ps}}(\Delta x_1) := (\Delta x_1^{\mathrm{ps}})^{-1}(\Delta x_1)$ *and define the issuance-indexed existing debt claim by*

$$x_0^{\mathrm{ps}}(\Delta x_1) := x_1^{s*}(\theta^{\mathrm{ps}}(\Delta x_1)) - \Delta x_1.$$

After an issue of face value Δx_1*, set the existing debt claim equal to* $x_0^{\mathrm{ps}}(\Delta x_1)$*. For every issuance level in the image of* Δx_1^{ps}*, this schedule reproduces the corresponding total debt, interim transfer, and continuation payoff. No type gains by choosing another issuance level in this image. Hence, the schedule replicates the exclusive-lending allocation along the prescribed issuance path.*

Along the implemented path, total debt is split between the claim sold to new lenders and the adjusted existing debt claim:

$$x_1^{s*}(\theta) = \Delta x_1^{\mathrm{ps}}(\theta) + x_0^{\mathrm{ps}}(\theta) = x_0^s + \Delta x_1^{\mathrm{ps}}(\theta) \; + \; [x_0^{\mathrm{ps}}(\theta) - x_0^s]\,.$$

[14] One possible off-path completion is to cap observable aggregate issuance at the largest issue prescribed on path. Another is to continue the contractual response using the lowest type's incentive constraint, provided the continuation satisfies the relevant feasibility bounds.

The term $\Delta x_1^{\text{ps}}(\theta)$ is new debt issued at $t = 1$, while the bracketed term is the performance-sensitive adjustment to existing debt.

Performance-sensitive existing debt is the non-exclusive analogue of the nonlinear credit line in Proposition 8. In the credit-line implementation, the borrower chooses a draw and the single lender prices that draw below the competitive spot price, so the borrower pays for the effect of refinancing on the existing debt claim. Here new lenders still price the new issue competitively. The contract with existing lenders disciplines issuance directly: if the borrower issues the amount associated with report θ', existing debt is adjusted to $x_0^{\text{ps}}(\theta')$, so changing issuance changes both cash from new lenders and compensation to existing lenders. The borrower therefore faces the same deviations over the size of the draw as under the credit line, and incentive compatibility of the single-lender allocation rules out those deviations. The adjustment changes who receives repayment, not the amount of effort, because effort at $t = 2$ depends on total debt. Performance-sensitive debt therefore implements the same effort allocation as the credit line. Figure 7 illustrates the nonlinear performance-sensitive debt schedule in the linear example.

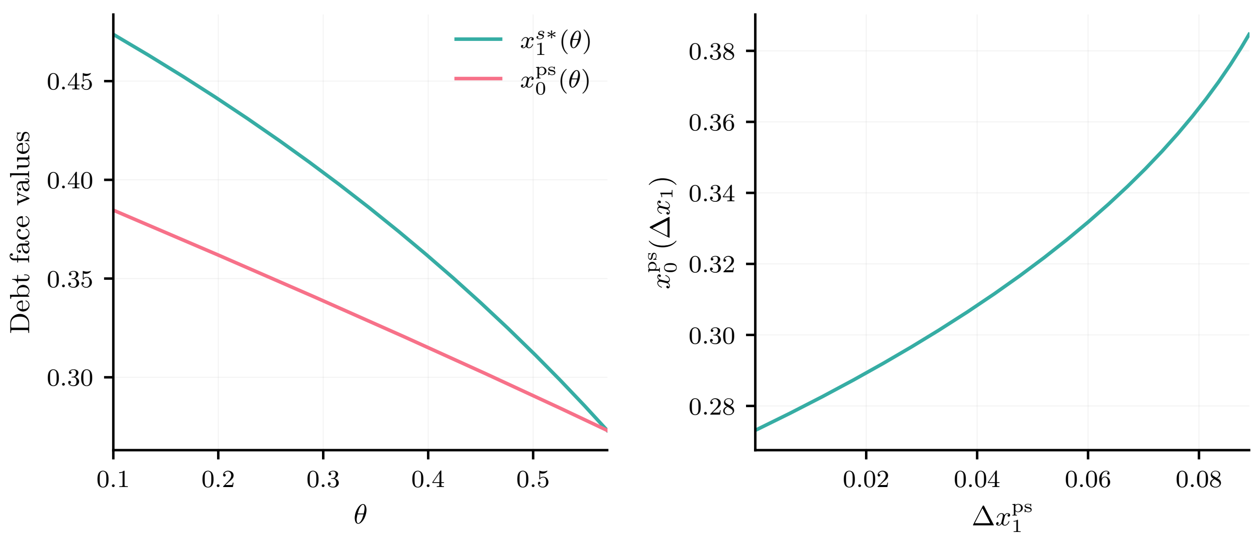

Figure 7: Implementation of the optimal contract using nonlinear performance-sensitive debt. We assume that $\theta \sim U[0.1, 1]$, $Y = 1$, $c(q) = \frac{1}{2}q^2$, and $I = 0.8 \times I^{\max}$. The left panel plots the separating region $[\underline{\theta}, \theta_H]$; on $[\theta_H, \bar{\theta}]$, there is no new issuance.

This mechanism differs from existing explanations for performance-sensitive debt. Manso et al. (2010) develop an asymmetric-information model in which performance-sensitive debt screens high-growth borrowers from low-growth borrowers. Bensoussan et al. (2021) study how performance-sensitive debt can mitigate debt overhang by changing shareholders' payoff from exercising growth options. Chaigneau et al. (2021) show that performance-sensitive debt can arise when additional performance signals are contractible. Relatedly, Diamond et al. (2022) study how liquidity and pledgeability shape performance pricing. In our paper, performance sensitivity prices the externality from non-exclusive borrowing: when the borrower issues new debt, the existing claim adjusts to compensate existing lenders for dilution.

In practice, designing and implementing the optimal performance-sensitive debt might be challenging. First, the contract must closely measure the amount of dilution to adjust the existing debt claim after new borrowing. That requires lenders to track total leverage, priority, liens, guarantees, leases, and other debt-like liabilities, not just debt reported in financial statements. This amount of monitoring is more natural in syndicated leveraged loans, where an administrative agent and concentrated lenders can collect information and administer pricing grids, than in high-yield bonds, where dispersed bondholders and trustees are less suited to continuous repricing of the claim. Second, the design of the repricing schedule requires more information than for a debt limit. A debt limit sets a cutoff, so the contract's shape remains simple even when lenders know little about the distribution of future financing shocks. Performance-sensitive debt must specify how the existing claim changes at each issuance level, so the contract must be calibrated to the distribution of shocks and to the price effect of additional debt. Given these challenges, lenders may prefer a simpler debt-incurrence covenant.

5.2 Contingent Prepayment

Second, we consider debt sweeps, which implement the same compensation through partial acceleration of the existing debt. They let the borrower issue new debt while requiring part of the proceeds to repay existing debt. Under a standard restrictive covenant, new debt issuance can trigger default and acceleration of the entire loan. A sweep instead makes prepayment conditional on the borrower's action. The borrower keeps access to outside debt, but the contract prices that access by redirecting some of the new proceeds to the existing debt claim.

Loan agreements use debt sweeps as part of a broader class of mandatory-prepayment provisions.[15] A *debt sweep* requires that a portion of the funds raised through new debt issuance be used to prepay outstanding loans. It need not apply to the full issue: the contract can require partial prepayment and still leave the borrower with new liquidity. Debt sweeps are a common feature in leveraged credit agreements, including private corporate loans, leveraged loans, and debtor-in-possession facilities. According to Bratton (2016), sweeps became prominent features of

[15] Other common sweep provisions include equity sweeps, which require a fixed share of proceeds from new equity issuance to repay existing debt, and asset sweeps, which direct part of asset-sale proceeds to debt repayment. Bradley and Roberts (2015) document that, from 1993 to 2001, the share of loans with an equity sweep rose from 24% to 75%, and the share with an asset sweep rose from 32% to 93%.

bank loan contracts during the 1990s; Bradley and Roberts (2015) document that, from 1993 to 2001, the share of loans with a debt sweep rose from 18% to 80%.[16] Eckbo et al. (2023) finds that debt-issuance sweeps are included in 26.2% of debtor-in-possession loan packages and 26.9% of matched leveraged-loan packages, compared with prepayment-sweep rates of 51.3% and 35.8%, respectively.[17]

We now show that a nonlinear debt sweep can reproduce the exclusive-lending allocation under non-exclusive lending. Let $x_0^s := x_1^{s*}(\bar{\theta})$ denote the existing debt level from the exclusive benchmark. If type θ is to end $t = 1$ with total debt $x_1^{s*}(\theta)$, then a prepayment of $b(\theta)$ requires gross new issuance of face value

$$\Delta x_1^{\mathrm{ds}}(\theta) := x_1^{s*}(\theta) - x_0^s + b(\theta).$$

The borrower receives the proceeds from this new issue and uses $b(\theta)$ to repay existing debt. To reproduce the exclusive-lending transfer, define

$$b(\theta) := \frac{(x_1^{s*}(\theta) - x_0^s)\, q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta)}{1 - q(x_1^{s*}(\theta))}. \tag{29}$$

The formula pins down the cash-flow transfer needed to reproduce the exclusive-lending payoff. It is not automatically a feasible sweep, however, because the contract can accelerate only debt that is already outstanding. The upper bound $b(\theta) \leq x_0^s$ is therefore an implementation condition rather than an incentive constraint. If this bound fails, the candidate schedule asks the borrower to prepay more legacy debt than exists, so it cannot be interpreted as a debt sweep.

The next proposition shows that a nonlinear debt sweep reproduces the exclusive-lending allocation for every issuance level on the equilibrium path.[18]

Proposition 11 (On-Path Replication with a Nonlinear Debt Sweep)**.** *Suppose the exclusive benchmark is not fully pooled. Assume the endpoint condition that ensures the sweep never requires prepayment above existing debt, $b(\theta) \leq x_0^s$:*

$$x_1^{s*}(\underline{\theta})q(x_1^{s*}(\underline{\theta})) - \tau_1^{s*}(\underline{\theta}) \leq x_0^s. \tag{30}$$

[16] Bratton (2016) note that the trend toward looser covenants in more recent years has reduced the use of sweep provisions.

[17] While the most common arrangement requires that 100% of proceeds be applied to prepayment, contracts often required that only a fraction of the proceeds go to prepayment. For example, in a loan received by Avon Products on June 29, 2012, the credit agreement required the company to prepay fifty percent (50%) of the net cash proceeds from any debt issuance exceeding $500 million. (Avon Products, Inc., 2014)

[18] The same off-path completions described in footnote 14 apply to the debt-sweep schedule, subject to the prepayment feasibility bounds.

On the separating region, $\Delta x_1^{\mathrm{ds}}(\theta)$ is invertible. Define

$$\theta^{\mathrm{ds}}(\Delta x_1) \coloneqq (\Delta x_1^{\mathrm{ds}})^{-1}(\Delta x_1).$$

After an issue of face value Δx_1, the debt-sweep prepayment is

$$b^{\mathrm{ds}}(\Delta x_1) \coloneqq b(\theta^{\mathrm{ds}}(\Delta x_1)).$$

For every issuance level in the image of Δx_1^{ds}, the debt sweep requires the borrower to use the fraction

$$\alpha^{\mathrm{ds}}(\Delta x_1) \;=\; \frac{b^{\mathrm{ds}}(\Delta x_1)}{q\Big(x_0^s + \Delta x_1 - b^{\mathrm{ds}}(\Delta x_1)\Big)\Delta x_1}$$

of the cash proceeds from new debt issuance to prepay existing debt. This rule reproduces the corresponding post-prepayment debt, interim transfer, and continuation payoff. No type gains by choosing another issuance level in this image. Hence, the rule replicates the exclusive-lending allocation along the prescribed issuance path.

Proposition 11 implements the same credit-line wedge through prepayment rather than an adjustment to the existing face value. With exclusive lending, the lender prices each draw from the credit line below the competitive spot price, so the borrower pays for the dilution imposed on the existing debt claim. With a debt sweep, new lenders still price the new claim competitively, but the contract diverts a nonlinear share $\alpha^{\mathrm{ds}}(\Delta x_1)$ of the proceeds toward repayment of existing debt.

The sweep works because the prepayment share varies with the size of the new issue. Larger issues correspond to lower θ states and create more dilution of the existing debt claim. The nonlinear schedule makes the borrower internalize that dilution while preserving access to outside debt. As a result, the borrower chooses the same post-prepayment debt level $x_1^{s*}(\theta)$ as in the exclusive benchmark.

Figure 8 illustrates the nonlinear debt sweep in the linear example. Contingent prepayment is most useful when new financing proceeds can be verified and redirected to existing lenders, so that the prepayment rule $b(\Delta x_1)$ is enforceable. This is more natural in private credit agreements, leveraged loans, and debtor-in-possession financing, where lenders can monitor issuance proceeds and payment waterfalls. When proceeds are hard to trace, creditors are dispersed, or prepayment is constrained by call protection or liquidity needs, implementing $b(\Delta x_1)$ becomes difficult. A total debt limit is easier to enforce because it only requires observing whether aggregate indebtedness x_1 exceeds a specified cap. The model therefore predicts that contingent prepayment and borrowing covenants are substitutes: sweeps should be more common when proceeds can be verified and

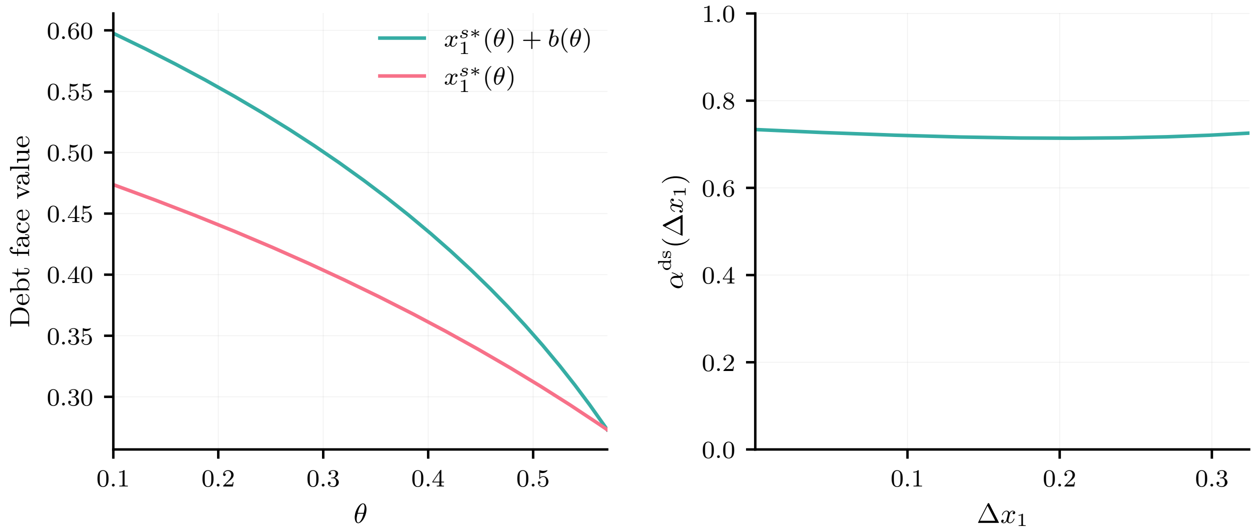

Figure 8: Implementation of the optimal contract using a nonlinear debt sweep. We assume that $\theta \sim U[0.1, 1]$, $Y = 1$, $c(q) = \frac{1}{2}q^2$, and $I = 0.8 \times I^{\max}$. The left panel plots the separating region $[\underline{\theta}, \theta_H]$; on $[\theta_H, \bar{\theta}]$, there is no new issuance.

redirected, while covenants should be more prominent when only total debt is easy to monitor.

5.3 Short-Term Debt

As a contrast to these transfer-like clauses, we now introduce short-term debt. Debt maturity is not the paper's central focus, but the exercise is useful because it shows that changing maturity alone does not provide state-contingent compensation to existing lenders.

Specifically, we allow the borrower to finance the upfront investment at $t = 0$ using a combination of short-term debt b that matures at $t = 1$, and long-term debt x_0 that matures at $t = 2$. Short-term debt is due after the discount-factor shock θ is realized. Because the borrower is financially constrained, she must issue new debt to repay the short-term debt b due at that time.

It becomes immediately clear that short-term debt must be riskless because there is no cash-flow risk at $t = 1$.[19] The next result shows that short-term debt crowds out long-term debt when rollover does not pool all types. Under full pooling, by contrast, maturity is indeterminate.

Proposition 12 (Short-Term Debt)**.** *Let* $I^p := x_1^*(\underline{\theta})q(x_1^*(\underline{\theta}))$*, and let* x_1^b *be the smaller solution to* $xq(x) = I$ *(and the unique solution when* $I = I^{\max}$*). Then:*

[19] A proof goes as follows. If the borrower were to default under some report $\hat{\theta}$ (resulting in zero payoff), she would instead choose a report $\hat{\theta}'$ that avoids default and yields a positive payoff. Thus, truthful reporting requires that default never occur, so that short-term debt must be riskless.

1. *If $I < I^p$, the unique optimal maturity structure is $x_0 = 0$ and $b = I$. At $t = 1$, total debt is*

$$x_1(\theta) = \max\{x_1^*(\theta), x_1^b\}.$$

2. *If $I^p \leq I \leq I^{\max}$, optimal total debt is fully pooled at x_1^b. Maturity is indeterminate: every*

$$x_0 \in [0, x_1^b], \qquad b = (x_1^b - x_0)q(x_1^b), \qquad x_1^L = x_1^b$$

   is optimal. In particular, the all-short-term contract $x_0 = 0$ and $b = I$ is optimal, but not unique.

When $I < I^p$, the borrower finances the entire investment with short-term debt. At $t = 1$, she chooses the efficient debt $x_1^*(\theta)$ whenever it raises enough to repay $b = I$ and otherwise borrows x_1^b. Issuing long-term debt creates dilution incentives on the states in which rollover does not bind. Short-term debt avoids this distortion, which makes $x_0 = 0$ uniquely optimal in this regime.

When $I \geq I^p$, the rollover requirement is high enough to pool all types at x_1^b. The same pooled allocation can be financed with any combination of short- and long-term debt in the proposition. Replacing one maturity with the other changes neither initial proceeds nor total debt at $t = 1$, so the maturity composition is not uniquely determined.

These results should be interpreted cautiously because the analysis abstracts away several costs and risks of short-term debt. Firms rarely rely only on short-term debt in practice. For instance, our model ignores transaction costs associated with rolling debt over and assumes that there are no uncertain intermediate cash flows. In a related setting with such uncertainty, Hu et al. (ming) show that borrowers issue long-term debt when short-term debt becomes risky. Outside the fully pooled regime, maturity alone also fails to reproduce the state-contingent compensation delivered by performance-sensitive debt or contingent prepayment.

6 Conclusion

Financial flexibility allows a firm to respond to future financing needs, but preserving that flexibility exposes existing creditors to dilution. We study how debt contracts balance these forces when a borrower cannot commit to refinance with its initial lender and privately observes its future liquidity need. Debt issued today finances investment, but the same debt changes the borrower's incentives when it returns to the market tomorrow. Because the borrower does not internalize the loss imposed on existing creditors, initial borrowing and future borrowing discretion must be designed jointly.

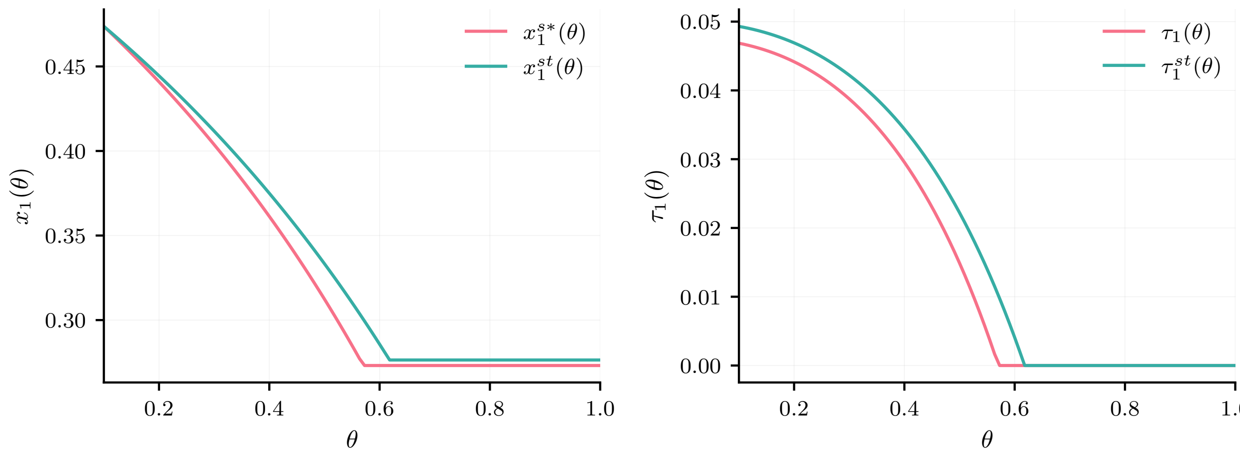

Figure 9: Optimal contract with short-term debt compared to the single lender contract. We assume that $\theta \sim U[0.1, 1]$, $Y = 1$, $c(q) = \frac{1}{2}q^2$, and $I = 0.8 \times I^{\max}$.

Under simple debt contracts, this joint design problem becomes a delegation problem: the initial contract cannot compensate existing lenders according to the borrower's later financing need, so it controls future borrowing by restricting the choices available to the borrower. The optimal restriction is a limit on total debt, which preserves access to refinancing but cannot simultaneously prevent over-borrowing by unconstrained types and under-borrowing by constrained types. Borrowers with severe liquidity needs borrow too little when the limit binds, while borrowers with mild needs borrow too much when it does not. Contracts that allow state-contingent compensation resolve the tradeoff differently. Exclusive lending prices additional financing to reflect the dilution of the existing claim, while performance-sensitive debt and, when feasible, contingent prepayment can provide the same compensation under non-exclusive lending. Short-term debt, by contrast, changes when claims are repaid but does not reproduce this state-contingent pricing.

More broadly, the form of financial flexibility depends on what lenders can commit to provide and what contracts can monitor and enforce. A state-contingent credit line requires the incumbent lender to commit to providing future financing, which may be difficult when lenders are dispersed or face liquidity constraints. Performance-sensitive debt requires detailed monitoring and calibration of how new borrowing affects the existing claim, while contingent prepayment requires issuance proceeds to be verifiable and redirectable to existing lenders. When these state-contingent adjustments are difficult to implement, a simple incurrence covenant limiting total debt may be more practical because aggregate indebtedness is easier to monitor and verify. Thus, richer contracts preserve more flexibility when contingent compensation can be implemented, while debt limits become valuable when only aggregate indebtedness can be reliably monitored.

References

Admati, A. R., P. M. Demarzo, M. F. Hellwig, and P. Pfleiderer (2018). The leverage ratchet effect. Journal of Finance 73(1), 145–198.

Aguiar, M., M. Amador, H. Hopenhayn, and I. Werning (2019). Take the short route: Equilibrium default and debt maturity. Econometrica 87(2), 423–462.

Amador, M. and K. Bagwell (2013). The theory of optimal delegation with an application to tariff caps. Econometrica 81(4), 1541–1599.

Amador, M., K. Bagwell, and A. Frankel (2018, October). A note on interval delegation. Economic Theory Bulletin 6(2), 239–249.

Amador, M., I. Werning, and G. M. Angeletos (2006). Commitment vs. flexibility. Econometrica 74(2), 365–396.

Asquith, P., A. Beatty, and J. Weber (2005, December). Performance pricing in bank debt contracts. Journal of Accounting and Economics 40(1-3), 101–128.

Asriyan, V. and V. Vanasco (2024, March). Security design in non-exclusive markets with asymmetric information. Review of Economic Studies 91(2), 690–719.

Attar, A., C. Casamatta, A. Chassagnon, and J.-P. Décamps (2019, May). Multiple Lenders, Strategic Default, and Covenants. American Economic Journal: Microeconomics 11(2), 98–130.

Attar, A., T. Mariotti, and F. Salanié (2011). Nonexclusive competition in the market for lemons. Econometrica 79(6), 1869–1918.

Attar, A., T. Mariotti, and F. Salanié (2014). Nonexclusive competition under adverse selection. Theoretical Economics 9(1), 1–40.

Attar, A., T. Mariotti, and F. Salanié (2020). The social costs of side trading. Economic Journal 130(630), 1608–1622.

Avon Products, Inc. (2014). Form 10-K, Annual Report for the fiscal year ended December 31, 2013. https://www.sec.gov/Archives/edgar/data/8868/000000886814000013/avp10k2013.htm. U.S. Securities and Exchange Commission; accessed August 25, 2025.

Bagnoli, M. and T. Bergstrom (2005). Log-Concave Probability and Its Applications. Economic Theory 26(2), 445–469.

Baron, D. P. and D. Besanko (1984). Regulation and information in a continuing relationship. Information Economics and policy 1(3), 267–302.

Bensoussan, A., B. Chevalier-Roignant, and A. Rivera (2021, October). Does performance-sensitive debt mitigate debt overhang? Journal of Economic Dynamics and Control 131, 104203.

Bisin, A. and D. Guaitoli (2004). Moral Hazard and Nonexclusive Contracts. The RAND Journal of Economics 35(2), 306–328.

Bisin, A. and A. A. Rampini (2006, February). Exclusive contracts and the institution of bankruptcy. Economic Theory 27(2), 277–304.

Bizer, D. S. and P. M. DeMarzo (1992). Sequential Banking. Journal of Political Economy 100(1), 41–61.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. Journal of Political Economy 81(3), 637–654.

Bradley, M. and M. R. Roberts (2015, June). The Structure and Pricing of Corporate Debt Covenants. Quarterly Journal of Finance 05(02), 1550001.

Bratton, W. W. (2016, October). Bond and loan covenants, theory and practice. Capital Markets Law Journal 11(4), 461–485.

Chaigneau, P., A. Edmans, and D. Gottlieb (2021). When is (performance-sensitive) debt optimal?

DeMarzo, P., Z. He, and F. Tourre (2023, October). Sovereign debt ratchets and welfare destruction. Journal of Political Economy 131(10), 2825–2892.

DeMarzo, P. M. (2019). Presidential Address: Collateral and Commitment. Journal of Finance 74(4), 1587–1619.

DeMarzo, P. M. and Z. He (2021). Leverage dynamics without commitment. Journal of Finance 76(3), 1195–1250.

Diamond, D. W., Y. Hu, and R. G. Rajan (2022). Liquidity, pledgeability, and the nature of lending. Journal of Financial Economics 143(3), 1275–1294.

Donaldson, J. R., D. Gromb, and G. Piacentino (2025). Conflicting priorities: A theory of covenants and collateral. Journal of Finance 80(3), 1739–1768.

Eckbo, B. E., K. Li, and W. Wang (2023). Loans to chapter 11 firms: contract design, repayment risk, and pricing. The Journal of Law and Economics 66(3), 465–509.

Fama, E. F. and M. Miller (1972). The Theory of Finance. Holt Rinehart & Winston.

Graham, J. R. (2022). Presidential address: Corporate finance and reality. The Journal of Finance 77(4), 1975–2049.

Halac, M. and P. Yared (2014). Fiscal rules and discretion under persistent shocks. Econometrica 82(5), 1557–1614.

Halac, M. and P. Yared (2018, August). Fiscal rules and discretion in a world economy. American Economic Review 108(8), 2305–2334.

Halac, M. and P. Yared (2022). Fiscal rules and discretion under limited enforcement. Econometrica 90(5), 2093–2127.

Halac, M. and P. Yared (2025). A theory of fiscal responsibility and irresponsibility. Journal of Political Economy 133(5), 1574–1620.

Hart, O. D. and J. Tirole (1988). Contract renegotiation and coasian dynamics. Review of Economic Studies 55(4), 509–540.

Holmström, B. (1984). On the theory of delegation. In M. Boyer and R. E. Kihlstrom (Eds.), Bayesian Models in Economic Theory, pp. 115–141. Amsterdam: North-Holland.

Hu, Y., F. Varas, and C. Ying (forthcoming). Debt maturity management. The Review of Financial Studies.

Jensen, M. C. (1986). Agency costs of free cash flow, corporate finance, and takeovers. The American economic review 76(2), 323–329.

Kahn, C. M. and D. Mookherjee (1998). Competition and Incentives with Nonexclusive Contracts. The RAND Journal of Economics 29(3), 443–465.

Luenberger, D. G. (1969). Optimization by Vector Space Methods. John Wiley & Sons.

Manso, G., B. Strulovici, and A. Tchistyi (2010). Performance-sensitive debt. The Review of Financial Studies 23(5), 1819–1854.

Milgrom, B. Y. P. and C. Shannon (1994). Monotone comparative statics. Econometrica 62(1), 157–180.

Peters, M. and B. Szentes (2012). Definable and contractible contracts. Econometrica 80(1), 363–411.

Segal, I. (1999). Contracting with Externalities. Quarterly Journal of Economics 114(2), 337–388.

Simpson Thacher & Bartlett LLP (2022). Leveraged finance 101: A covenant handbook. Technical report, Simpson Thacher & Bartlett LLP. Available at https://www.stblaw.com/about-us/publications/view/2022/09/12/simpson-thacher-publishes-em-leveraged-finance-101-a-covenant-handbook-em.

Smith, C. W. and J. B. Warner (1979, June). On financial contracting. Journal of Financial Economics 7(2), 117–161.

Szentes, B. (2015). Contractible contracts in common agency problems. The Review of Economic Studies 82(1 (290)), 391–422.

Appendix

A Notation and Standing Conventions

This appendix summarizes the notation used throughout the paper. We omit symbols that are used only within a single result or proof; those symbols are defined where they appear.

Object	Meaning
Primitives and technology	
I, Y	Initial investment at $t = 0$ and project cash flow at $t = 2$ following success.
θ, $\Theta = [\underline{\theta}, \bar{\theta}]$	Borrower's privately observed discount-factor shock at $t = 1$ and its support.
$G(\theta)$, $g(\theta)$	Cumulative distribution function and density of θ.
$c(q)$, $q(x_1)$	Effort cost and optimal effort, or project success probability, given total debt x_1.
$I^{\max}$	Maximum pledgeable income, $\max_{x_1 \in [0,Y]} x_1 q(x_1)$.
Debt, transfers, and values	
τ_0, $\tau_1(\theta)$	Cash transferred to the borrower by lenders at $t = 0$ and $t = 1$.
x_0, $x_1(\theta)$	Initial debt issued at $t = 0$ and total debt outstanding after refinancing at $t = 1$, both due at $t = 2$.
Δx_1	Face value of new debt issued at $t = 1$; under simple debt, $\Delta x_1 = x_1 - x_0$.
$V_2(x_1)$	Borrower's residual claim at $t = 2$, net of effort cost.
$W_1(x_1, \theta)$	Ex post firm value at $t = 1$, equal to debt value plus discounted equity value.
$V_1(x_1, x_0, \theta)$	Borrower's continuation payoff at $t = 1$ under non-exclusive lending, net of the legacy claim x_0.
$W_0(x_0)$	Optimal expected firm value for a fixed initial debt level x_0.
$x_1^*(\theta)$	Ex post-efficient total debt, which maximizes $W_1(x_1, \theta)$.
Non-exclusive lending and debt limits	
$x_1^f(\theta, x_0)$	Flexible debt choice that maximizes $V_1(x_1, x_0, \theta)$; the argument x_0 is suppressed when clear.
θ_L, x_1^L	Debt-limit cutoff and associated cap $x_1^L = x_1^f(\theta_L, x_0)$; types $\theta \le \theta_L$ face the cap.
θ_L^{uc}, θ_L^{bc}	Optimal debt-limit cutoff without the financing constraint and lowest cutoff that satisfies the financing constraint.
$x_0^{\min}$, $x_0^{\max}$	Lower and upper roots of $x_0 q(x_0) = I$, possibly coincident, which bound feasible initial debt.
x_0^{n*}, $x_1^{n*}(\theta)$	Optimal initial debt and total-debt schedule under non-exclusive lending with simple debt contracts.

Object	Meaning
$x_f^{n*}(\theta)$, x_L^{n*}	Flexible debt evaluated at x_0^{n*} and the corresponding optimal debt limit.
λ, δ	Multiplier on the financing constraint and endogenous discount factor $\delta = 1/(1+\lambda)$.
$\varepsilon(x)$, $\Gamma(x)$	Inverse-demand elasticity $-xq'(x)/q(x)$ and the curvature measure $[x\varepsilon'(x) - \varepsilon(x)]/\varepsilon(x)^2$.
$\hat{x}_0$, κ	Scaled initial debt x_0/Y and financing-needs parameter cI/Y^2 in the linear example.
$g(\theta;\gamma)$, $\mathbb{E}_\gamma$	Parametric type distribution used for MLR comparative statics and expectation under that distribution.
Exclusive-lending benchmark	
I^c, I^s	Financing thresholds below which the efficient debt schedule is feasible under complete information and private information.
I^p	Financing threshold at which the private-information debt schedule becomes fully pooled.
δ^{c*}, δ^{s*}	Endogenous discount factors in the complete-information and private-information exclusive-lending benchmarks.
$\phi_\delta(\theta)$	Virtual type $\theta + (1-\delta)G(\theta)/g(\theta)$ under exclusive lending with private information.
$x_1^{c*}(\theta)$	Optimal debt schedule under complete information.
$x_1^{s*}(\theta)$, $\tau_1^{s*}(\theta)$	Optimal debt schedule and interim transfer under exclusive lending with private information.
θ_H	Exclusive-lending cutoff above which the debt schedule is pooled and the interim transfer is zero.
x_0^s, $\Delta x_1^s(\theta)$, $p_1^s(\Delta x_1)$	Baseline existing debt, credit-line draw, and nonlinear average price per unit of face value that implement the exclusive benchmark.
Richer contracts and short-term debt	
$x_0^{\mathrm{ps}}(\theta)$, $\Delta x_1^{\mathrm{ps}}(\theta)$	Adjusted existing claim and new issuance under performance-sensitive debt.
$b(\theta)$, $\Delta x_1^{\mathrm{ds}}(\theta)$	Debt-sweep prepayment and gross new issuance that reproduce the exclusive allocation.
$b^{\mathrm{ds}}(\Delta x_1)$, $\alpha^{\mathrm{ds}}(\Delta x_1)$	Issuance-indexed sweep prepayment and fraction of new-debt proceeds used for prepayment.
b, x_1^b	Face value of short-term debt due at $t=1$ and the total debt required to roll it over; these symbols are used only in the short-term-debt section.

B Non-Exclusive Lending with Simple Debt Contracts

Let us first rewrite the problem in (11). Substituting $W_1(x_1, \theta) = x_0 q(x_1) + V_1(x_1, x_0, \theta)$ together with equation (10) into the objective function in (11) and applying integration by parts, we obtain the following problem:

$$
\begin{aligned}
W_0(x_0) = \max_{x_1(\theta) \geq x_0} \; & V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E}\left[x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta))\right] \\
& \text{subject to} \\
& V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall \theta \in \Theta \\
& x_0 \mathbb{E}\left[q(x_1(\theta))\right] \geq I \\
& x_1(\theta) \text{ is non-increasing.}
\end{aligned}
\tag{31}
$$

Proof of Lemma 2

Proof. Recall that

$$V_1(x_1, x_0, \theta) = W_1(x_1, \theta) - x_0 q(x_1).$$

For part (1), note that, by Lemma 1,

$$\frac{\partial^2 V_1(x_1, x_0, \theta)}{\partial x_1 \partial \theta} = V_2'(x_1) = -q(x_1) \leq 0,$$

with strict inequality for $x_1 < Y$. Thus V_1 has decreasing differences in (x_1, θ). Since V_1 is concave in x_1, the monotone-comparative-statics argument of Milgrom and Shannon (1994) implies that $x_1^f(\theta, x_0)$ is nonincreasing in θ. Similarly, on $x_1 < Y$,

$$\frac{\partial^2 V_1(x_1, x_0, \theta)}{\partial x_1 \partial x_0} = -q'(x_1) > 0,$$

so V_1 has increasing differences in (x_1, x_0), which implies that $x_1^f(\theta, x_0)$ is nondecreasing in x_0.

For part (2), $x_1^*(\theta)$ maximizes $W_1(x_1, \theta)$, so

$$\frac{\partial W_1(x_1^*(\theta), \theta)}{\partial x_1} = 0.$$

Evaluating the derivative of V_1 at $x_1 = x_1^*(\theta)$ yields

$$\frac{\partial V_1(x_1^*(\theta), x_0, \theta)}{\partial x_1} = -x_0 q'(x_1^*(\theta)) \geq 0.$$

Since V_1 is concave in x_1, its maximizer lies at or to the right of $x_1^*(\theta)$, so $x_1^f(\theta, x_0) \geq x_1^*(\theta)$.

For part (3), evaluating the derivative of V_1 at $x_1 = x_0$ gives

$$\frac{\partial V_1(x_0, x_0, \theta)}{\partial x_1} = (1-\theta)q(x_0) \geq 0,$$

where the expression follows from direct differentiation. Again by concavity, the maximizer lies at or to the right of x_0, so $x_1^f(\theta, x_0) \geq x_0$. If $\bar{\theta} = 1$, then

$$\frac{\partial V_1(x_0, x_0, \bar{\theta})}{\partial x_1} = 0.$$

Thus x_0 satisfies the first-order condition, and concavity implies $x_1^f(\bar{\theta}, x_0) = x_0$. □

Proof of Proposition 1

Proof. Letting λ be the multiplier of the investment constraint, we write

$$\max_{\theta_L} \int_{\underline{\theta}}^{\theta_L} \Big(W_1(x_1^f(\theta_L), \theta) + \lambda x_0 q(x_1^f(\theta_L)) \Big)\, dG(\theta) + \int_{\theta_L}^{\bar{\theta}} \Big(W_1(x_1^f(\theta), \theta) + \lambda x_0 q(x_1^f(\theta)) \Big)\, dG(\theta).$$

Dividing by $1+\lambda$ and defining $\delta \coloneqq 1/(1+\lambda)$, we can rewrite the previous optimization problem in the following equivalent form

$$\max_{\theta_L} \int_{\underline{\theta}}^{\theta_L} \Big((1-\delta)x_0 q(x_1^f(\theta_L)) + \delta W_1(x_1^f(\theta_L), \theta) \Big)\, dG(\theta) + \int_{\theta_L}^{\bar{\theta}} \Big((1-\delta)x_0 q(x_1^f(\theta)) + \delta W_1(x_1^f(\theta), \theta) \Big)\, dG(\theta).$$

Next, we consider the first-order condition for θ_L, which is given by

$$\begin{aligned} \frac{\partial \text{ObjFun}}{\partial \theta_L} &= x_1^{f\prime}(\theta_L) \int_{\underline{\theta}}^{\theta_L} \Big(x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L)) \Big)\, dG(\theta) \\ &= x_1^{f\prime}(\theta_L) G(\theta_L) q(x_1^f(\theta_L)) \left\{ x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \mathbb{E}[\theta_L - \theta | \theta \leq \theta_L] \right\} = 0 \end{aligned}$$

Assumptions 3 and 1 imply that $G(\theta)$ and $q(x)$ are log-concave; accordingly, the functions $E[\theta_L - \theta|\theta \leq \theta_L]$ and $\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$ are increasing in θ_L. It follows that the objective function is quasi-concave, so any local maximum is also a global one. The second-order condition, evaluated at a local maximum θ_L, is

$$\frac{\partial^2 \text{ObjFun}}{\partial \theta_L^2} = x_1^{f\prime}(\theta_L)\left[\delta G(\theta_L) + x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} g(\theta_L)\right] q(x_1^f(\theta_L)) + \\ + \left(x_1^{f\prime}(\theta_L)\right)^2 G(\theta_L)\left[x_0 q''(x_1^f(\theta_L)) + \delta\mathbb{E}[\theta_L - \theta|\theta \leq \theta_L] q'(x_1^f(\theta_L))\right].$$

First, we verify that $\theta_L > \underline{\theta}$. Evaluating the second order condition at $\theta_L = \underline{\theta}$, we get that $\frac{\partial \text{ObjFun}}{\partial \theta_L} = 0$ and $\frac{\partial^2 \text{ObjFun}}{\partial \theta_L^2} > 0$, which means that $\theta_L = \underline{\theta}$ is a local minimum. Second, the solution is interior (that is, $\theta_L < \bar{\theta}$) only if $\left.\frac{\partial \text{ObjFun}}{\partial \theta_L}\right|_{\theta_L = \bar{\theta}} \leq 0$. As $x_1^{f'}(\theta_L) < 0$, this is the case only if

$$x_0 q'(x_1^f(\bar{\theta})) + \delta\mathbb{E}\left[\bar{\theta} - \theta\right] q(x_1^f(\bar{\theta})) \geq 0.$$

In this case, θ_L is given by the solution to

$$x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta\mathbb{E}[\theta_L - \theta|\theta \leq \theta_L] = 0. \tag{32}$$

On the other hand, if

$$x_0 q'(x_1^f(\bar{\theta})) + \delta\mathbb{E}\left[\bar{\theta} - \theta\right] q(x_1^f(\bar{\theta})) < 0,$$

then we get that $\left.\frac{\partial \text{ObjFun}}{\partial \theta_L}\right|_{\theta_L = \bar{\theta}} > 0$, so $\theta_L = \bar{\theta} = 1$.

The next step is to consider the financing constraint. If the constraint is slack, then $\delta = 1$. Let θ_L^{uc} the solution when $\delta = 1$. The solution is interior only if

$$x_0 q'(x_1^f(\bar{\theta})) + \left(\bar{\theta} - \mathbb{E}[\theta]\right) q(x_1^f(\bar{\theta})) \geq 0.$$

Noting that Lemma 2 establishes that $x_1^f(\bar{\theta}) = x_0$ when $\bar{\theta} = 1$, this is equivalent to saying that

$$\left.\frac{\partial W_1(x_1, \mathbb{E}[\theta])}{\partial x_1}\right|_{x_1 = x_0} \geq 0,$$

which means that $x_1^*(\mathbb{E}[\theta]) \geq x_0$. If this is the case, the solution from equation (32) becomes

$$x_0 \frac{q'(x_1^f(\theta_L^{\mathrm{uc}}))}{q(x_1^f(\theta_L^{\mathrm{uc}}))} + \mathbb{E}[\theta_L^{\mathrm{uc}} - \theta | \theta \leq \theta_L^{\mathrm{uc}}] = 0$$

On the other hand, if the constraint is binding. Then, $\theta_L = \theta_L^{\mathrm{bc}}$, where θ_L^{bc} satisfies

$$x_0 \mathbb{E}\left[q\big(x_1^f(\theta_L^{\mathrm{bc}})\big)\,\mathbf{1}_{\{\theta \leq \theta_L^{\mathrm{bc}}\}} + q\big(x_1^f(\theta)\big)\,\mathbf{1}_{\{\theta > \theta_L^{\mathrm{bc}}\}}\right] = I,$$

and from equation (32), we get that

$$\delta = -\frac{x_0}{\mathbb{E}[\theta_L^{\mathrm{bc}} - \theta | \theta \leq \theta_L^{\mathrm{bc}}]} \frac{q'(x_1^f(\theta_L^{\mathrm{bc}}))}{q(x_1^f(\theta_L^{\mathrm{bc}}))}.$$

Finally, note that for any $\theta_L > \theta_L^{\mathrm{bc}}$, Lemma 2 establishes that $x_1^f(\theta_L, x_0) \leq x_1^f(\theta_L^{\mathrm{bc}}, x_0)$, so, because q is decreasing,

$$x_0 \mathbb{E}\left[q\big(x_1^f(\theta_L)\big)\,\mathbf{1}_{\{\theta \leq \theta_L\}} + q\big(x_1^f(\theta)\big)\,\mathbf{1}_{\{\theta > \theta_L\}}\right] > I,$$

we conclude that the solution to the optimization problem is $\max\{\theta_L^{\mathrm{bc}}, \theta_L^{\mathrm{uc}}\}$.

□

Proof of Proposition 2

Proof. Our verification analysis follows the approach in Amador et al. (2006) and Amador and Bagwell (2013). In particular, we find a Lagrange multiplier such that the debt limit and the multiplier are a saddle-point of the Lagrangian. The optimality of the policy then follows from the sufficient conditions in Theorem 2 in (Luenberger, 1969, p. 221). First, we will consider a Lagrange relaxation of the optimization problem in (31). Letting λ be the Lagrange multiplier of the financial constraint, we can consider the problem

$$\begin{aligned}
&\max_{x_1(\theta)} \; V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E}\left[(1+\lambda)x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\tilde{\theta}))\right] - \lambda I \\
&\text{subject to} \\
&V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall \theta \in \Theta \\
&x_1(\theta) \geq x_0 \\
&x_1(\theta) \text{ is non-increasing.}
\end{aligned} \tag{33}$$

Dividing by $(1+\lambda)$ and defining $\delta \coloneqq 1/(1+\lambda)$, we consider the problem

$$\begin{aligned}
&\max_{x_1(\theta)} \; \delta V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E}\left[x_0 q(x_1(\theta)) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\tilde{\theta}))\right] \\
&\text{subject to} \\
&V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall \theta \in \Theta \\
&x_1(\theta) \geq x_0 \\
&x_1(\theta) \text{ is non-increasing.}
\end{aligned} \tag{34}$$

If we let $\Lambda(\theta)$ be the cumulative Lagrange multiplier for the constraint

$$V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall \theta \in \Theta,$$

then, the Lagrangian for our problem is

$$\begin{aligned}
\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q\left(x_1(\theta)\right) g(\theta) - \delta G(\theta) V_2(x_1(\theta))\right] d\theta \\
+ \int_{\underline{\theta}}^{\bar{\theta}} \left(V_1(x_1(\theta), x_0, \theta) + \int_{\theta}^{\bar{\theta}} V_2\left(x_1(\tilde{\theta})\right) d\tilde{\theta}\right) d\Lambda(\theta)
\end{aligned}$$

If we change the order of integration, we obtain that

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} V_2\left(x_1(\tilde{\theta})\right) d\tilde{\theta} d\Lambda(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \Lambda(\theta) V_2(x_1(\theta)) d\theta.$$

Substituting in $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ we get the following expression for the Lagrangean.

$$\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q\left(x_1(\theta)\right) g(\theta) + \left(\Lambda(\theta) - \delta G(\theta)\right) V_2(x_1(\theta))\right] d\theta + \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta) \quad (35)$$

Next, we consider the following optimization problem

$$\begin{aligned} &\max_{x_1(\theta)} \ \mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) \\ &x_1(\theta) \geq x_0 \\ &x_1(\theta) \text{ is non-increasing.} \end{aligned}$$

Denoting the partial derivative with respect to x_1 by $V_1'(x_1, x_0, \theta) := \frac{\partial V_1(x_1, x_0, \theta)}{\partial x_1}$, we get that the directional derivative of $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ in the direction $\mathbf{h}$ is:

$$\nabla \mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}; \mathbf{h}) = V_1'(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) h(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q'\left(x_1(\theta)\right) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1(\theta))\right] h(\theta) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} V_1'(x_1(\theta), x_0, \theta) h(\theta) d\Lambda(\theta) \quad (36)$$

Evaluating the gradient in equation (36) at

$$x_1^l(\theta) := \begin{cases} x_1^f(\theta) & \text{if } \theta \geq \theta_L \\ x_1^f(\theta_L) & \text{if } \theta < \theta_L, \end{cases}$$

we get

$$\nabla \mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \int_{\theta_L}^{\bar{\theta}} \left[x_0 q'(x_1^f(\theta)) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1^f(\theta))\right] h(\theta) d\theta + \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1^f(\theta_L))\right] h(\theta) d\theta + \int_{\underline{\theta}}^{\theta_L} V_1'(x_1^f(\theta_L), x_0, \theta) h(\theta) d\Lambda(\theta)$$

Let

$$\Lambda(\theta) = \begin{cases} \delta & \text{if } \theta = \bar{\theta} \\ \delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta) & \text{if } \theta \in [\theta_L, \bar{\theta}) \\ 0 & \text{if } \theta \in [\underline{\theta}, \theta_L). \end{cases} \quad (37)$$

Note that from the FOC of x_1^f, we can get $V_1'(x_1^f(\theta_L), x_0, \theta) = (\theta_L - \theta)q(x_1^f(\theta_L))$ and

$$\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L))g(\theta) + \delta G(\theta)q(x_1^f(\theta_L))\right] h(\theta)d\theta$$

We want to rewrite the directional derivative $\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h})$ in terms of the increments of $\mathbf{h}(\theta)$, so we can incorporate the monotonicity constraint directly the first order conditions. With this objective in mind, we let

$$\begin{aligned}\Phi(\theta) &:= \int_{\underline{\theta}}^{\theta} \left[x_0 q'(x_1^f(\theta_L))g(\tilde{\theta}) + \delta G(\tilde{\theta})q(x_1^f(\theta_L))\right] d\tilde{\theta} \\ &= x_0 q'(x_1^f(\theta_L))G(\theta) + \delta q(x_1^f(\theta_L)) \int_{\underline{\theta}}^{\theta} G(\tilde{\theta})d\tilde{\theta}\end{aligned}$$

and use integration by parts to rewrite the directional derivative as

$$\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \Phi(\theta_L)h(\theta_L) - \int_{\underline{\theta}}^{\theta_L} \Phi(\theta)dh(\theta).$$

Evaluating $\Phi(\theta)$ at θ_L we get

$$\begin{aligned}\Phi(\theta_L) &= x_0 q'(x_1^f(\theta_L))G(\theta_L) + \delta q(x_1^f(\theta_L)) \int_{\underline{\theta}}^{\theta_L} G(\tilde{\theta})d\tilde{\theta} \\ &= G(\theta_L)q(x_1^f(\theta_L)) \left[x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta\mathbb{E}[\theta_L - \tilde{\theta} \mid \tilde{\theta} \leq \theta_L]\right].\end{aligned}$$

Suppose first that $\theta_L < \bar{\theta}$. Proposition 1 then implies that this expression is equal to zero. Hence, we can write the directional derivative as

$$\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = -\int_{\underline{\theta}}^{\theta_L} \Phi(\theta)dh(\theta).$$

If $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ is concave (we return to this below), and we let P be the set of all non-increasing functions on Θ that satisfy $x_1(\theta) \geq x_0$, then Lemma 1 in (Luenberger, 1969, p. 227) provides the following necessary and sufficient condition for $\mathbf{x}_1^l$ to maximize $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$:

$$\begin{aligned}&\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \tilde{\mathbf{x}}_1) \leq 0 \quad \forall\ \tilde{\mathbf{x}}_1 \in P \\ &\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{x}_1^l) = 0,\end{aligned}$$

As $d\tilde{x}_1(\theta) \le 0$, we need that $\Phi(\theta) \le 0$, which is equivalent to requiring that

$$x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \int_{\underline{\theta}}^{\theta} \frac{G(\tilde{\theta})}{G(\theta)} d\tilde{\theta} \le 0, \ \forall \theta \in [\underline{\theta}, \theta_L). \tag{38}$$

Using integration by parts, we get

$$\int_{\underline{\theta}}^{\theta} \frac{G(\tilde{\theta})}{G(\theta)} d\tilde{\theta} = \theta - E[\tilde{\theta} | \tilde{\theta} \le \theta]$$

so can write equation (38) as

$$0 \ge \frac{x_0 q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta E[\theta - \tilde{\theta} | \tilde{\theta} \le \theta], \quad \forall \theta \in [\underline{\theta}, \theta_L).$$

Substituting $\frac{x_0 q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$ from Proposition 1, we get

$$1 \ge \frac{\mathbb{E}[\tilde{\theta} | \tilde{\theta} \le \theta_L] - \mathbb{E}[\tilde{\theta} | \tilde{\theta} \le \theta]}{\theta_L - \theta}, \quad \forall \theta \in [\underline{\theta}, \theta_L).$$

As $G(\theta)$ is log-concave (Assumption 1), the function $\theta - \mathbb{E}[\tilde{\theta} | \tilde{\theta} \le \theta]$ is increasing in θ (Bagnoli and Bergstrom, 2005), which means that the previous inequality is satisfied for all $\theta < \theta_L$.

We next consider $\theta_L = \bar{\theta} = 1$. In this case, Lemma 2 implies that $x_1^l(\theta) = x_0$ for all θ. The one-sided optimality condition for the debt-limit problem is

$$x_0 \frac{q'(x_0)}{q(x_0)} + \delta \mathbb{E}[1 - \theta] \le 0.$$

Since $\theta - \mathbb{E}[\tilde{\theta} \mid \tilde{\theta} \le \theta]$ is increasing in θ, this condition implies

$$\Phi(\theta) = G(\theta) q(x_0) \left[x_0 \frac{q'(x_0)}{q(x_0)} + \delta \mathbb{E}[\theta - \tilde{\theta} \mid \tilde{\theta} \le \theta] \right] \le 0 \quad \text{for all } \theta \in (\underline{\theta}, 1],$$

while $\Phi(\underline{\theta}) = 0$ by definition. For any feasible schedule $\tilde{\mathbf{x}}_1$, let $\mathbf{h} = \tilde{\mathbf{x}}_1 - \mathbf{x}_1^l$. The constraint $\tilde{x}_1(\theta) \ge x_0$ implies that $h(1) \ge 0$, while monotonicity implies that $dh(\theta) \le 0$. It follows that

$$\nabla \mathcal{L}(\mathbf{x}_1^l, \boldsymbol{\Lambda}; \mathbf{h}) = \Phi(1) h(1) - \int_{\underline{\theta}}^{1} \Phi(\theta) dh(\theta) \le 0.$$

It remains to verify the concavity of $\mathcal{L}(\mathbf{x}_1, \boldsymbol{\Lambda})$. Substituting $\boldsymbol{\Lambda}$ from equation (37) in the Lagrangian in (35), we get

$$\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = \int_{\underline{\theta}}^{\theta_L} \left[x_0 q\left(x_1(\theta)\right) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right] g(\theta) d\theta$$
$$+ \int_{\theta_L}^{\bar{\theta}} \left[x_0 q\left(x_1(\theta)\right) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} V_2(x_1(\theta)) \right] g(\theta) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta) \quad (39)$$

Assumption 3 implies that $q(x_1)$ is concave, so the two functions inside the Lebesgue integrals and $V_1(x_1, x_0, \theta)$ are concave in x_1 for every $\theta \in \Theta$. Hence, it suffices that $\Lambda(\theta)$ is non-decreasing. If $\theta_L = \bar{\theta} = 1$, then $\Lambda(\theta) = 0$ for $\theta < 1$ and $\Lambda(1) = \delta$, so this condition holds. Suppose instead that $\theta_L < \bar{\theta}$. On $[\theta_L, \bar{\theta})$, $\Lambda(\theta)$ is non-decreasing by condition (18). At θ_L, the first-order condition in Proposition 1 implies that $\Lambda(\theta_L) \geq \Lambda(\theta_L -) = 0$ if

$$1 - \mathbb{E}[\theta_L - \theta \mid \theta \leq \theta_L] \frac{g(\theta_L)}{G(\theta_L)} = \frac{\partial}{\partial \theta_L} \left[\frac{\int_{\underline{\theta}}^{\theta_L} G(\theta) d\theta}{G(\theta_L)} \right] \geq 0.$$

This inequality follows from the log-concavity of $G(\theta)$. Finally, $q' < 0$ implies that $\Lambda(\bar{\theta}) = \delta \geq \Lambda(\bar{\theta}-)$, completing the verification. □

Proof of Proposition 3

We consider the optimization problem (19). Assumptions 2 and 3 imply that $x_0 q(x_0) = I$ has lower and upper solutions $x_0^{\min} \leq x_0^{\max}$ and that $x_0 q(x_0) \geq I$ if and only if $x_0 \in [x_0^{\min}, x_0^{\max}]$. The solutions are distinct when $I < I^{\max}$ and coincide when $I = I^{\max}$. Hence, we can write problem (19) as

$$\max_{x_0 \in [x_0^{\min}, x_0^{\max}]} W_0(x_0). \quad (40)$$

If $I = I^{\max}$, then $x_0^{\min} = x_0^{\max} = x_1^*(0)$. For any $\theta_L < 1$, the flexible-debt choice satisfies $x_1^f(\max\{\theta, \theta_L\}, x_0) > x_0$ with positive probability. Since q is decreasing,

$$x_0 \mathbb{E}\left[q\left(x_1^f(\max\{\theta, \theta_L\}, x_0) \right) \right] < x_0 q(x_0) = I^{\max},$$

so such a cutoff is infeasible. Thus, $\theta_L = 1$ and the first case of Proposition 3 applies. The remaining argument concerns $I < I^{\max}$, for which the feasible interval is nondegenerate. Let $\theta \vee \theta_L := \max\{\theta, \theta_L\}$. Since Lemma 2 establishes that $x_1^f(\theta, x_0)$ is nonincreasing in θ, we can write

$\min\{x_1^f(\theta), x_1^f(\theta_L)\} = x_1^f(\theta \vee \theta_L)$. Given Proposition 2, we can write $W_0(x_0)$ as

$$\begin{aligned} W_0(x_0) = \max_{\theta_L \in \Theta} \ & \mathbb{E}\left[W_1\left(x_1^f(\theta \vee \theta_L), \theta\right)\right] \\ & \text{subject to} \\ & x_0 \mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L)\right)\right] \geq I \end{aligned} \tag{41}$$

Let $W_1'(x_1, \theta) \coloneqq \frac{\partial W_1(x_1,\theta)}{\partial x_1}$ and let $x_1^L = x_1^f(\theta_L, x_0)$ be the debt limit in Proposition 1. By the envelope theorem, we have that for any $x_0 \in [x_0^{\min}, x_0^{\max}]$ the derivative $W_0'(x_0)$ is

$$\begin{aligned} W_0'(x_0) = & \, \mathbb{E}\left[W_1'(x_1^f(\theta \vee \theta_L, x_0), \theta) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right] \\ & + \lambda \mathbb{E}\left[q(x_1^f(\theta \vee \theta_L, x_0)) + x_0 q'(x_1^f(\theta \vee \theta_L, x_0)) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right], \end{aligned} \tag{42}$$

where $\lambda = \delta^{-1} - 1$ is the multiplier of the financing constraint $x_0 \mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L)\right)\right] \geq I$, and

$$\frac{\partial x_1^f(\theta, x_0)}{\partial x_0} = \begin{cases} \frac{q'(x_1^f(\theta, x_0))}{V_1''(x_1^f(\theta, x_0), x_0, \theta)} & \text{if } \theta < \bar{\theta} = 1 \\ 1 & \text{if } \theta = \bar{\theta} = 1. \end{cases}$$

For $\theta > \theta_L$, we have that

$$W_1'(x_1^f(\theta, x_0), \theta) = x_0 q'(x_1^f(\theta, x_0)) + V_1'(x_1^f(\theta, x_0), x_0, \theta) = x_0 q'(x_1^f(\theta, x_0));$$

while for $\theta < \theta_L$ we have

$$W_1'(x_1^f(\theta_L, x_0), \theta) = x_0 q'(x_1^f(\theta_L, x_0)) + V_1'(x_1^f(\theta_L, x_0), x_0, \theta) = x_0 q'(x_1^f(\theta_L, x_0)) + (\theta_L - \theta) q(x_1^f(\theta_L, x_0))$$

Hence, we can write the equation for $W_0'(x_0)$ as

$$\begin{aligned} W_0'(x_0) = & \, \mathbb{E}\left[(\theta_L - \theta) \mathbf{1}_{\theta \leq \theta_L}\right] q(x_1^f(\theta_L, x_0)) \frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0} \\ & + \mathbb{E}\left[\lambda q(x_1^f(\theta \vee \theta_L, x_0)) + (1 + \lambda) x_0 q'(x_1^f(\theta \vee \theta_L, x_0)) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right]. \end{aligned}$$

Noting that $\mathbb{E}[(\theta_L - \theta)\mathbf{1}_{\{\theta<\theta_L\}}] = G(\theta_L)\mathbb{E}[\theta_L - \theta|\theta \le \theta_L]$, and letting $\lambda = \delta^{-1} - 1$, we can write

$$\begin{aligned}
W_0'(x_0) = \delta^{-1}G(\theta_L)\left\{\delta\mathbb{E}[\theta_L - \theta|\theta \le \theta_L]q(x_1^f(\theta_L, x_0)) + x_0 q'(x_1^f(\theta_L, x_0))\right\}\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0} \\
+ G(\theta_L)(\delta^{-1} - 1)q(x_1^f(\theta_L, x_0)) \\
+ \mathbb{E}\left[(\delta^{-1} - 1)q(x_1^f(\theta, x_0))\mathbf{1}_{\theta>\theta_L} + \delta^{-1}x_0 q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\theta>\theta_L}\right].
\end{aligned}$$

When $\theta_L < \bar{\theta} = 1$, Proposition 1 gives the cutoff first-order condition

$$\delta\mathbb{E}[\theta_L - \theta|\theta \le \theta_L]q(x_1^f(\theta_L, x_0)) + x_0 q'(x_1^f(\theta_L, x_0)) = 0,$$

so we get that

$$W_0'(x_0) = \delta^{-1}\left\{(1-\delta)q(x_1^f(\theta_L, x_0))G(\theta_L) + \mathbb{E}\left[(1-\delta)q(x_1^f(\theta, x_0))\mathbf{1}_{\theta>\theta_L} + x_0 q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\theta>\theta_L}\right]\right\}.$$

Hence, when $\theta_L < \bar{\theta} = 1$, we can write $W_0'(x_0)$ as

$$W_0'(x_0) = \delta^{-1}\mathbb{E}\left[(1-\delta)q(x_1^f(\theta \vee \theta_L, x_0)) + x_0 q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\theta>\theta_L}\right]. \tag{43}$$

If instead $\theta_L = 1$, then implemented debt equals x_0 for every θ, and therefore

$$W_0(x_0) = W_1(x_0, \mathbb{E}[\theta]), \qquad W_0'(x_0) = q(x_0)\left[1 - \mathbb{E}[\theta] - \varepsilon(x_0)\right].$$

The cutoff equality used above is unavailable because θ_L is at the boundary. If $\theta_L < \bar{\theta} = 1$ and the solution is interior, that is $x_0^* > x_0^{\min}$, then we have that $W_0'(x_0) = 0$, in which case we get the following necessary first order

$$x_0 = \frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta\le\theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]}.$$

We can conclude then that if this is the case, then (x_0, θ_L, δ) satisfy

$$x_0 = \frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta \leq \theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta > \theta_L\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta > \theta_L\}}\right]}$$

$$\delta = -\frac{x_0}{\mathbb{E}[\theta_L - \theta | \theta \leq \theta_L]}\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$$

$$x_0\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta \leq \theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta > \theta_L\}}\right] = I.$$

It remains to determine under what conditions the solution is interior. We proceed through a series of lemmas characterizing the solution to the optimization problem (40)

Lemma 4. *Let x_0^* be a solution to problem (40) and θ_L^* the associated solution to problem (41). If the financing constraint $x_0^*\mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L, x_0^*)\right)\right] \geq I$ is slack, then $x_0 \geq x_1^*(\mathbb{E}[\theta])$.*

Proof. We proceed by contradiction. As $x_1^L = x_1^f(\theta_L, x_0) > x_0$, and Lemma 2 establishes that $x_1^f(\bar{\theta}, x_0) = x_0$, it must be the case that $x_0 \in (x_0^{\min}, x_0^{\max})$ because $x_0 \in \{x_0^{\min}, x_0^{\max}\}$ does not satisfy the constraint $x_0\mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] \geq I$. This means, that x_0^* is in the interior $[x_0^{\min}, x_0^{\max}]$, so it must satisfy the necessary condition $W_0'(x_0) = 0$. This means that $\lambda = 0$ and $\theta_L < \bar{\theta}$, so the expression in equation (42) reduces to

$$W_0'(x_0) = \mathbb{E}\left[W_1'(x_1^f(\theta \vee \theta_L, x_0), \theta)\frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right].$$

From here we get that

$$\begin{aligned}
W_0'(x_0) &= \mathbb{E}\left[W_1'(x_1^f(\theta \vee \theta_L, x_0), \theta)\frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right] \\
&= \mathbb{P}(\theta > \theta_L)\mathbb{E}\left[W_1'(x_1^f(\theta, x_0), \theta)\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\bigg|\theta > \theta_L\right] \\
&\quad + \mathbb{P}(\theta \leq \theta_L)\mathbb{E}\left[W_1'(x_1^f(\theta_L, x_0), \theta)\Big|\theta \leq \theta_L\right]\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0}.
\end{aligned}$$

Suppose that $x_0 < x_1^*(\mathbb{E}[\theta])$. If the constraint is slack then equation (17) implies that

$$\mathbb{E}\left[W_1'(x_1^f(\theta_L, x_0), \theta)\Big|\theta \leq \theta_L\right] = 0.$$

Moreover, Lemma 2 establishes that $x_1^f(\theta, x_0) \geq x_1^*(\theta)$, so $W_1'(x_1^f(\theta, x_0), \theta) \leq 0$; on the relevant interior branch the inequality is strict. It follows then that

$$W_0'(x_0) = \mathbb{P}(\theta > \theta_L)\mathbb{E}\left[W_1'(x_1^f(\theta, x_0), \theta)\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\Big|\theta > \theta_L\right] < 0,$$

which yields a contradiction.

□

From here, we can conclude that if the financing constraint is slack. then it must be the case that $x_0 \geq x_1^*(\mathbb{E}[\theta])$. The next lemma shows that in any solution, the constraint must be binding.

Lemma 5. *If x_0^* is a solution to problem (40) and θ_L^* the associated solution to problem (41). Then, the financing constraint is binding, i.e., $x_0^*\mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L, x_0^*)\right)\right] = I$.*

Proof. We proceed by contradiction. Suppose that the constraint is slack. Then, it must be the case that $x_0^* \in (x_0^{\min}, x_0^{\max})$, and by Lemma 4 it must be the case that $x_0^* \geq x_1^*(\mathbb{E}[\theta])$. From Proposition 1, we get that $\theta_L^* = \bar{\theta}$ and $\lambda = 0$. Next, we show that x_0 cannot achieve a local maximum at x_0^*. As $W_1(x_1, \mathbb{E}[\theta])$ is concave in x_1, and achieves a maximum at $x_1^*(\mathbb{E}[\theta])$, we have that we have $W_0'(x_0) < 0$ for any $x_0 > x_1^*(\mathbb{E}[\theta])$, so $x_0^* > x_1^*(\mathbb{E}[\theta])$ cannot be optimal. Hence, if $x_0^* \geq x_1^*(\mathbb{E}[\theta])$ it must be equal to $x_1^*(\mathbb{E}[\theta])$. If $x_0^* = x_1^*(\mathbb{E}[\theta]) \in (x_0^{\min}, x_0^{\max})$, then the constraint must be slack for $x_0 = x_1^*(\mathbb{E}[\theta])$. By continuity, it is also slack for $x_0' = x_1^*(\mathbb{E}[\theta]) - \epsilon$ (for some $\epsilon > 0$), which means that

$$W_0'(x_0') = \mathbb{E}\left[W_1'(x_1^f(\theta \vee \theta_L, x_0'), \theta)\frac{\partial x_1^f(\theta \vee \theta_L, x_0')}{\partial x_0}\right] < 0.$$

Taking the limit when $\epsilon \to 0$, we get that $W_0'(x_1^*(\mathbb{E}[\theta])-) < 0$, which means that $x_1^*(\mathbb{E}[\theta])$ cannot be optimal. We conclude that the constraint cannot be slack at the optimal solution.

□

The final step is to characterize the conditions under which $x_0^* > x_0^{\min}$.

Lemma 6. *Let x_0^* be a solution to the problem (40). If $x_0^{\min} \geq x_1^*(\mathbb{E}[\theta])$; then $x_0^* = x_0^{\min}$. If $x_0^{\min} < x_1^*(\mathbb{E}[\theta])$ then $x_0^* > x_0^{\min}$.*

Proof. We consider two cases: $x_0^{\min} \geq x_1^*(\mathbb{E}[\theta])$ and $x_0^{\min} < x_1^*(\mathbb{E}[\theta])$. If $x_0^{\min} \geq x_1^*(\mathbb{E}[\theta])$, then Proposition 1 implies that for any $x_0 \in [x_0^{\min}, x_0^{\max}]$, we have $\theta_L = \bar{\theta} = 1$, which means that $x_0^* = \arg\max_{x_0 \in [x_0^{\min}, x_0^{\max}]} W_1(x_0, \mathbb{E}[\theta])$. As $W_1(x_0, \mathbb{E}[\theta])$ is a concave function (by Assumption 3), achieving its maximum at $x_1^*(\mathbb{E}[\theta]) \leq x_0^{\min}$. It follows that $x_0^* = x_0^{\min}$. On the other hand, if $x_0^{\min} < x_1^*(\mathbb{E}[\theta])$, then $W_1(x_0, \mathbb{E}[\theta])$ is increasing at $x_0 = x_0^{\min}$, which means that $x_0^* > x_0^{\min}$. □

Proof of Proposition 4

The proof focuses on the case with interior financial flexibility, so the financing constraint binds and $\theta_L < 1$. For fixed initial debt x_0 and threshold θ_L, the implemented total debt is $x_1^f(\theta \vee \theta_L, x_0)$: low-$\theta$ borrowers are capped at $x_1^f(\theta_L, x_0)$, while high-θ borrowers choose their flexible debt level. Write

$$\rho(\theta_L) \coloneqq \mathbb{E}[\theta_L - \theta \mid \theta \leq \theta_L].$$

Also, define the expected debt price as,

$$Q(x_0, \theta_L) \coloneqq \mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L, x_0)\right)\right].$$

The first step is to identify the debt region that can contain an optimum. Let

$$x^m \coloneqq \arg\max_{x \in [0,Y]} xq(x)$$

be the unique maximizer of the value of total debt, characterized by $\varepsilon(x^m) = 1$. Assumption 3 implies

$$x\varepsilon'(x) - \varepsilon(x) = \frac{x^2\left((q'(x))^2 - q''(x)q(x)\right)}{q(x)^2} > 0 \qquad \text{for } x > 0,$$

so ε is strictly increasing and $\Gamma \geq 0$ on the relevant debt interval. Define the financing surplus by

$$\mathcal{F}(x_0, \theta_L) \coloneqq x_0 Q(x_0, \theta_L) - I.$$

Financing capacity need not be monotone in x_0, so the binding equation $\mathcal{F}(x_0, \theta_L) = 0$ may have multiple roots for a given θ_L. We first show that every solution to the initial-debt problem lies in the low-debt region $x_L < x^m$, where the binding root is unique.

Consider a candidate solution with $\theta_L \in (\underline{\theta}, 1)$, and let $x_L \coloneqq x_1^f(\theta_L, x_0)$. At any binding point,

$$\mathcal{F}_{\theta_L}(x_0, \theta_L) = x_0 G(\theta_L) q'(x_L) \frac{\partial x_1^f(\theta_L, x_0)}{\partial \theta_L} > 0.$$

Thus, the constraint qualification holds even at a point where $\mathcal{F}_{x_0} = 0$. The joint first-order conditions therefore apply to every candidate with an interior cutoff. Since the endogenous discount factor δ from the Lagrange multiplier satisfies $\delta \leq 1$, the first-order condition for the debt limit gives

$$\frac{x_0}{x_L}\varepsilon(x_L) = \delta\rho(\theta_L) \leq \rho(\theta_L).$$

Combining this inequality with the first-order condition for flexible debt at type θ_L,

$$\left(1 - \frac{x_0}{x_L}\right) \varepsilon(x_L) = 1 - \theta_L,$$

gives

$$\varepsilon(x_L) = 1 - \theta_L + \frac{x_0}{x_L}\varepsilon(x_L) \leq 1 - \theta_L + \rho(\theta_L) < 1.$$

The strict inequality follows because $G(\theta_L) > 0$ and $\Theta \subseteq (0, 1]$, so $\mathbb{E}[\theta \mid \theta \leq \theta_L] > 0$ and therefore $\rho(\theta_L) < \theta_L$. Hence, every candidate with an interior cutoff satisfies $x_L < x^m$. This conclusion also holds at the boundary between binding and slack financing, where $\delta = 1$.

The cutoff endpoints do not yield a solution outside the low-debt region. At the lower endpoint, take $\tilde{\theta}_L > \underline{\theta}$ sufficiently close to $\underline{\theta}$. Both financing capacity and the borrower's payoff increase with $\tilde{\theta}_L$, since

$$\mathcal{F}_{\tilde{\theta}_L}(x_0, \tilde{\theta}_L) = x_0 G(\tilde{\theta}_L) q'(x_1^f(\tilde{\theta}_L, x_0)) \frac{\partial x_1^f(\tilde{\theta}_L, x_0)}{\partial \tilde{\theta}_L} > 0$$

and

$$\frac{\partial}{\partial \tilde{\theta}_L} \mathbb{E}\left[W_1\left(x_1^f(\theta \vee \tilde{\theta}_L, x_0), \theta\right)\right] = \frac{\partial x_1^f(\tilde{\theta}_L, x_0)}{\partial \tilde{\theta}_L} G(\tilde{\theta}_L) q(x_1^f(\tilde{\theta}_L, x_0)) \left\{x_0 \frac{q'(x_1^f(\tilde{\theta}_L, x_0))}{q(x_1^f(\tilde{\theta}_L, x_0))} + \rho(\tilde{\theta}_L)\right\} > 0.$$

The last inequality follows because $\rho(\tilde{\theta}_L) \to 0$, while $q' < 0$ and $\partial x_1^f(\tilde{\theta}_L, x_0)/\partial \tilde{\theta}_L < 0$. Thus, $\theta_L = \underline{\theta}$ is not optimal. If $\theta_L < 1$ and $x_0 \in \{x_0^{\min}, x_0^{\max}\}$, then $x_1^f(\theta \vee \theta_L, x_0) > x_0$ with positive probability. Since q is decreasing,

$$Q(x_0, \theta_L) < q(x_0),$$

and therefore $x_0 Q(x_0, \theta_L) < x_0 q(x_0) = I$, so neither initial-debt endpoint is feasible. Thus, if a cutoff $\theta_L < 1$ is strictly feasible, $\mathcal{F}(\cdot, \theta_L)$ is negative at both endpoints and positive at some interior x_0. By continuity, the binding equation then has at least two roots. Finally, suppose that $\theta_L = 1$ and $I < I^{\max}$. For every feasible $x_0 \geq x^m$, a small decrease in x_0 preserves feasibility. It also raises the borrower's payoff because

$$\frac{\partial W_1(x_0, \mathbb{E}[\theta])}{\partial x_0} = q(x_0)\left[1 - \mathbb{E}[\theta] - \varepsilon(x_0)\right] < 0$$

whenever $x_0 \geq x^m$. Hence, no point with $\theta_L = 1$ and $x_0 \geq x^m$ is optimal. If $I = I^{\max}$, the feasible initial-debt interval collapses to x^m, so there is no feasible point with $\theta_L = 1$ and $x_0 > x^m$. Moreover, the interior-flexibility case does not arise because $x_1^*(\mathbb{E}[\theta]) < x^m$. We can therefore restrict attention to the low-debt region $x_L < x^m$.

The flexible-debt first-order condition implies that $x_L = x^m$ when $x_0 = \theta_L x^m$. Since $x_1^f(\theta_L, x_0)$ is strictly increasing in x_0, $x_L < x^m$ is equivalent to $x_0 < \theta_L x^m$. In this low-debt region, every implemented debt level satisfies

$$x_0 \le x_1^f(\theta \vee \theta_L, x_0) \le x_L < x^m.$$

Consequently,

$$\mathcal{F}_{x_0}(x_0, \theta_L) = \mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L, x_0)\right) \times \left(1 - \frac{x_0}{x_1^f(\theta \vee \theta_L, x_0)} \varepsilon\left(x_1^f(\theta \vee \theta_L, x_0)\right) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right)\right] > 0,$$

where the inequality uses $\varepsilon(x_1^f(\theta \vee \theta_L, x_0)) < 1$ and

$$\frac{\partial x_1^f(\theta, x_0)}{\partial x_0} = \frac{1}{1 + (1-\theta)\Gamma(x_1^f(\theta, x_0))} \le 1.$$

Thus, for each θ_L, the binding financing equation has at most one solution in the low-debt region. At $\theta_L = 1$, this unique low-debt root is $\chi(1) = x_0^{\min}$. By the implicit-function theorem, the unique low-debt root continues through the cutoff domain unless it reaches $x_L = x^m$. At that boundary, $x_0 = \theta_L x^m$, and the cutoff first-order condition would require

$$\delta = \frac{\theta_L}{\rho(\theta_L)} > 1.$$

Hence, the low-debt root exists throughout the relevant interval where $\delta \le 1$. We denote this continuation by $x_0 = \chi(\theta_L)$, so that

$$\chi(\theta_L) Q(\chi(\theta_L), \theta_L) = I.$$

The implicit-function theorem gives

$$\chi'(\theta_L) = -\frac{\mathcal{F}_{\theta_L}(\chi(\theta_L), \theta_L)}{\mathcal{F}_{x_0}(\chi(\theta_L), \theta_L)} < 0.$$

Below, the relevant interval for θ_L corresponds to the values for which the first-order condition for the debt limit gives an admissible multiplier, $\delta \le 1$. Hence, the left endpoint is characterized by $\delta = 1$.

Lemma 7 (Upper-tail marginal cost)**.** *Fix a type θ, if Γ is nonincreasing, then*

$$-q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}$$

is increasing in x_0.

Proof. Let $x = x_1^f(\theta, x_0)$. The first-order condition flexible-debt allocation (15) can be written as

$$\left(1 - \frac{x_0}{x}\right)\varepsilon(x) = 1 - \theta.$$

Differentiating with respect to x_0 gives

$$\frac{\partial x_1^f(\theta, x_0)}{\partial x_0} = \frac{\varepsilon(x)}{\frac{x_0}{x}\varepsilon(x) + \left(1 - \frac{x_0}{x}\right)x\varepsilon'(x)} = \frac{1}{1 + (1-\theta)\Gamma(x)}.$$

Hence the marginal-cost term is

$$-q'(x)\frac{\partial x}{\partial x_0} = \frac{-q'(x)}{1 + (1-\theta)\Gamma(x)}.$$

Assumption 3 implies $q'' \leq 0$, so $-q'(x)$ is increasing in x. Since Γ is nonincreasing, the denominator is nonincreasing in x. The ratio is therefore increasing in x. Lemma 2 implies that $x_1^f(\theta, x_0)$ is nondecreasing in x_0, so the upper-tail marginal-cost term is increasing in x_0. □

Along the low-debt binding schedule, where $x_0 = \chi(\theta_L)$, define

$$N(x_0, \theta_L) \coloneqq -\mathbb{E}\left[q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta > \theta_L\}}\right].$$

Using the binding initial financing condition to substitute

$$Q(\chi(\theta_L), \theta_L) = \mathbb{E}\left[q\left(x_1^f(\theta \vee \theta_L, \chi(\theta_L))\right)\right] = \frac{I}{\chi(\theta_L)},$$

we can write the first-order condition for the initial debt as the zero of the residual

$$\mathcal{S}(\theta_L) \coloneqq 1 - \frac{\chi(\theta_L)^2 N(\chi(\theta_L), \theta_L)}{I} - \frac{\chi(\theta_L)}{x_1^f(\theta_L, \chi(\theta_L))}\frac{\varepsilon(x_1^f(\theta_L, \chi(\theta_L)))}{\rho(\theta_L)},$$

The term $1 - \chi(\theta_L)^2 N(\chi(\theta_L), \theta_L)/I$ is the endogenous discount factor δ from the Lagrange multiplier implied by the envelope condition

$$x_0 = \frac{(1-\delta)\mathbb{E}\left[q(x_L)\mathbf{1}_{\{\theta\le\theta_L\}} + q(x_f(\theta,x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]}{-\mathbb{E}\left[q'(x_f(\theta,x_0))\frac{\partial x_1^f(\theta,x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]}.$$

The last term is the endogenous discount factor δ implied by the Lagrange multiplier from the first-order condition for the debt limit

$$\delta = -\frac{x_0}{\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]}\frac{q'(x_L)}{q(x_L)}.$$

Thus $\mathcal{S}(\theta_L)=0$ is exactly the condition that these two first-order conditions agree.

Lemma 8 (Single crossing of the residual)**.** *If Γ is nonincreasing, then $\mathcal{S}(\theta_L)$ is strictly increasing on the relevant interval for the threshold θ_L.*

Proof. Along the low-debt binding schedule, we have $\chi'(\theta_L)<0$. Hence, the debt limit $x_1^f(\theta_L,\chi(\theta_L))$ is decreasing as $x_1^f(\theta,x_0)$ is decreasing in θ and increasing in x_0. The last term in the definition of $\mathcal{S}(\theta_L)$ can be written as

$$\chi(\theta_L)\frac{\varepsilon(x_1^f(\theta_L,\chi(\theta_L)))}{x_1^f(\theta_L,\chi(\theta_L))}\frac{1}{\rho(\theta_L)}. \tag{44}$$

As $q''(x)\le 0$ by Assumption, 3, the middle factor

$$\frac{\varepsilon(x)}{x} = -\frac{q'(x)}{q(x)}$$

is increasing in x; hence, since $x_1^f(\theta_L,\chi(\theta_L))$ is decreasing in θ_L, this middle factor is decreasing in θ_L. Log-concavity of G implies that $\rho(\theta_L)$ is increasing, so the term (44) decreases with θ_L.

It remains to consider the envelope term

$$1-\frac{\chi(\theta_L)^2 N(\chi(\theta_L),\theta_L)}{I}. \tag{45}$$

Lemma 7 implies that the integrand in $N(\chi(\theta_L),\theta_L)$ is decreasing in θ_L, since $\chi(\theta_L)$ is decreasing. As the integration region $\{\theta>\theta_L\}$ shrinks when we increase θ_L, we conclude that $N(\chi(\theta_L),\theta_L)$ is decreasing in θ_L. Since $\chi(\theta_L)$ also decreases, it follows that the product $\chi(\theta_L)^2N(\chi(\theta_L),\theta_L)$ is strictly decreasing in θ_L for $\theta_L<1$. It follows that the term in (45) is strictly increasing in θ_L. It follows then that $\mathcal{S}(\theta_L)$ is strictly increasing. □

Conclusion. When the financing constraint is binding, the initial-debt problem can be reduced to maximizing the value along the low-debt binding schedule, $x_0 = \chi(\theta_L)$. The envelope calculation in the proof of Proposition 3 implies that the sign of the derivative of the value function is given by the sign of $\mathcal{S}(\theta_L)$, multiplied by

$$\chi'(\theta_L)\frac{Q(\chi(\theta_L),\theta_L)}{\delta}.$$

The expected debt price and δ are positive, while $\chi'(\theta_L) < 0$. Therefore the objective function increases when $\mathcal{S}(\theta_L) < 0$ and decreases when $\mathcal{S}(\theta_L) > 0$.

Lemma 8 shows that $\mathcal{S}(\theta_L)$ is strictly increasing. At the left end of the relevant threshold interval, the financing constraint just starts to bind, so the Lagrange multiplier is zero and the endogenous discount factor δ from the Lagrange multiplier is one; hence, the term in (44) is one. Since $N(\chi(\theta_L),\theta_L) > 0$ whenever $\theta_L < 1$, we can conclude that $\mathcal{S}(\theta_L) < 0$ at this point. At the no-flexibility endpoint $(x_0,\theta_L) = (x_0^{\min},1)$, the term in equation (45) is one, so

$$\mathcal{S}(1) = 1 - \frac{\varepsilon(x_0^{\min})}{1-\mathbb{E}[\theta]}.$$

Because $\varepsilon(x_1^*(\mathbb{E}[\theta])) = 1 - \mathbb{E}[\theta]$ and ε is nondecreasing, the condition $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$ implies $\mathcal{S}(1) \geq 0$. If equality held, the unique zero of $\mathcal{S}(\theta_L)$ occurs at the no-flexibility endpoint. Hence, on the interior-flexibility region, $\mathcal{S}(1) > 0$. It follows that $\mathcal{S}$ crosses zero exactly once. The value function then rises up to that crossing and then starts to fall. Since any interior optimum must satisfy the necessary condition, and any such solution must be on the decreasing low-debt binding schedule parameterized by χ, the unique crossing is the unique maximizer of (19). □

Linear-Example Details for Section 3

This subsection records the algebra behind the linear example used in Section 3. The example keeps the debt-limit logic from the general problem but makes the two margins explicit. Initial debt x_0 affects the borrower's refinancing choice because it is the legacy claim that can be diluted at $t = 1$, while θ affects the value of preserving cash flows at $t = 2$. We therefore keep θ_L as the debt-limit type: borrowers with $\theta \leq \theta_L$ are capped at $x_1^f(\theta_L, x_0)$, and borrowers with $\theta > \theta_L$ refinance at their flexible choice.

Under non-exclusive lending in the linear example,

$$V_1(x_1,x_0,\theta) = (x_1 - x_0)\frac{Y - x_1}{c} + \frac{\theta}{2}\frac{(Y-x_1)^2}{c}, \qquad x_1^f(\theta,x_0) = \frac{(1-\theta)Y + x_0}{2-\theta},$$

The expression for x_1^f shows the leverage-ratchet force in closed form. Holding θ fixed, larger initial debt raises flexible refinancing because the borrower captures the proceeds from new debt but does not internalize the full loss imposed on the legacy claim. Holding x_0 fixed, larger θ lowers refinancing because the borrower puts more weight on continuation value. Substituting the flexible debt choice into the debt price gives

$$q(x_1^f(\theta, x_0)) = \frac{Y - x_0}{c} \frac{1}{2-\theta}$$

and the financing capacity under a threshold debt limit is

$$x_0 \mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] = \frac{x_0(Y - x_0)}{c} \mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta \le \theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta > \theta_L\}}\right].$$

This expectation is the initial lenders' $t = 0$ debt value under the implemented debt schedule. In low-θ states the limit binds, so the price is evaluated at θ_L; in high-θ states the borrower is unconstrained, so the price is evaluated at the realized θ. This is the linear version of the financing constraint in (8).

The verification condition in Proposition 2 also becomes a primitive restriction on the density. Condition (18) is satisfied whenever

$$\sup_{\theta \in \Theta} (2-\theta) \frac{g'(\theta)}{g(\theta)} \le 1.$$

The scaled variables

$$\hat{x}_0 := \frac{x_0}{Y}, \qquad \hat{x}_1 := \frac{x_1}{Y}, \qquad \kappa := \frac{cI}{Y^2},$$

remove Y and c from the optimality system. The parameter κ measures the required investment in units of Y^2/c. In these scaled variables, the minimum feasible initial debt and the ex ante efficient benchmark satisfy

$$\hat{x}_0^{\min} = \frac{1}{2}\left(1 - \sqrt{1-4\kappa}\right), \qquad \hat{x}_1^*(\mathbb{E}[\theta]) = \frac{1-\mathbb{E}[\theta]}{2-\mathbb{E}[\theta]}.$$

Hence, $\hat{x}_1^*(\mathbb{E}[\theta]) > \hat{x}_0^{\min}$ if and only if

$$\mathbb{E}[\theta] < \frac{2\sqrt{1-4\kappa}}{1+\sqrt{1-4\kappa}}.$$

This inequality is the interior-flexibility condition from Proposition 3. If it fails, the minimum initial debt needed to finance the project already exceeds the ex ante efficient debt level, so the optimal simple-debt contract leaves no room for additional borrowing at $t = 1$. If it holds, the borrower can retain some refinancing flexibility, and the interior first-order system applies.

In that case, $(\hat{x}_0, \theta_L, \delta)$ solve

$$\kappa = \hat{x}_0(1-\hat{x}_0)\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\le\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right],$$

$$\frac{\hat{x}_0}{1-\hat{x}_0} = \frac{(1-\delta)\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\le\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right]}{\mathbb{E}\left[\frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right]},$$

$$\delta = \frac{2-\theta_L}{\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]}\frac{\hat{x}_0}{1-\hat{x}_0}.$$

The three equations have the same interpretation as in the general problem. The first equation is the binding initial financing constraint. The second equation is the initial-debt envelope condition, which equates the marginal value of relaxing the financing constraint with the marginal dilution cost created by more legacy debt. The third equation is the debt-limit condition for capped low-θ borrowers and pins down the endogenous discount factor δ from the Lagrange multiplier. Thus the linear example reduces the non-exclusive-lending problem to three scalar unknowns: initial debt, the debt-limit type, and the financing multiplier.

Proof of Proposition 5

The proof works directly on the binding frontier. First, we show that the optimal solution satisfies $\hat{x}_0 \in [0, 1/2]$, so the binding constraint gives a unique frontier choice $\hat{x}_0 = \chi(\theta_L; \gamma, \kappa)$ for each feasible θ_L. Second, Proposition 3 reduces the interior binding optimum to a one-dimensional equation in θ_L. Comparative statics then follow from the monotonicity of the one-dimensional equation.

For any function $h(\theta)$ and $\gamma \in [\underline{\gamma}, \overline{\gamma}]$ let

$$\mathbb{E}_\gamma[h(\theta)] := \int_{\underline{\theta}}^1 h(\theta)g(\theta;\gamma)\,d\theta,$$

and let

$$\hat{U}(\hat{x}_0, \theta_L; \gamma) = \int_{\underline{\theta}}^1 \hat{u}(\hat{x}_0, \theta_L, \theta)g(\theta;\gamma)\,d\theta,$$

$$\hat{C}(\hat{x}_0, \theta_L; \gamma) = \hat{x}_0(1-\hat{x}_0)K(\theta_L;\gamma)$$

with

$$K(\theta_L;\gamma) := \mathbb{E}_\gamma\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\le\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right] = \int_{\underline{\theta}}^1 \frac{g(\theta;\gamma)}{2-\max\{\theta,\theta_L\}}\,d\theta,$$

and

$$\hat{u}(\hat{x}_0, \theta_L, \theta) = \begin{cases} \dfrac{1-\hat{x}_0}{2-\theta_L} - \dfrac{(1-\theta/2)(1-\hat{x}_0)^2}{(2-\theta_L)^2}, & \theta \le \theta_L, \\ \dfrac{1-\hat{x}_0^2}{2(2-\theta)}, & \theta > \theta_L. \end{cases}$$

The normalization has the following interpretation. For a pair $(\hat{x}_0, \theta_L)$, the implemented total debt is $x_1^f(\theta_L, x_0)$ when the debt limit binds, $\theta \le \theta_L$, and $x_1^f(\theta, x_0)$ when the borrower is unconstrained, $\theta > \theta_L$. Thus, $\hat{u}$ is $\frac{c}{Y^2}$ times the interim firm value W_1 evaluated at the implemented debt level, and $\hat{U}$ is its expectation under $g(\cdot;\gamma)$. Similarly, K is the expected normalized debt-price term on the same debt schedule, so $\hat{C}$ equals $\frac{c}{Y^2}$ times the initial financing capacity $x_0\mathbb{E}_\gamma[q(x_1^f(\max\{\theta, \theta_L\}, x_0))]$. The borrower problem at time zero in equation (19) can be written as

$$\max_{(\hat{x}_0, \theta_L)\in[0,1]\times\Theta} \hat{U}(\hat{x}_0, \theta_L; \gamma) \quad \text{s.t.} \quad \hat{C}(\hat{x}_0, \theta_L; \gamma) \ge \kappa.$$

Lemma 9. *For any feasible pair $(\hat{x}_0, \theta_L)$ with $\hat{x}_0 > 1/2$, the mirrored pair $(1-\hat{x}_0, \theta_L)$ is feasible and yields weakly higher objective. Hence an optimizer exists in*

$$\mathcal{X}^- := [0, 1/2] \times \Theta.$$

Proof. Let $\hat{x}_0' > 1/2$ and $\hat{x}_0^- := 1 - \hat{x}_0' \in [0, 1/2)$. Since

$$\hat{x}_0^-(1-\hat{x}_0^-) = \hat{x}_0'(1-\hat{x}_0'),$$

we have $\hat{C}(\hat{x}_0^-, \theta_L; \gamma) = \hat{C}(\hat{x}_0', \theta_L; \gamma)$, so feasibility is unchanged. Now compare payoffs pointwise. For $\theta > \theta_L$:

$$\hat{u}(\hat{x}_0^-, \theta_L, \theta) - \hat{u}(\hat{x}_0', \theta_L, \theta) = \frac{2\hat{x}_0' - 1}{2(2-\theta)} > 0.$$

For $\theta \le \theta_L$:

$$\hat{u}(\hat{x}_0^-, \theta_L, \theta) - \hat{u}(\hat{x}_0', \theta_L, \theta) = (2\hat{x}_0' - 1)\frac{1-\theta_L+\theta/2}{(2-\theta_L)^2} \ge 0.$$

Taking expectation with respect to θ, we get

$$\hat{U}(\hat{x}_0^-, \theta_L; \gamma) \ge \hat{U}(\hat{x}_0', \theta_L; \gamma).$$

$\square$

Lemma 10. *Fix (κ, γ) and restrict attention to $\mathcal{X}^- = [0, 1/2]$. Define*

$$\Theta(\gamma, \kappa) \coloneqq \{\theta_L \in \Theta : K(\theta_L; \gamma) \geq 4\kappa\}$$
$$\chi(\theta_L; \gamma, \kappa) \coloneqq \frac{1 - \sqrt{1 - \frac{4\kappa}{K(\theta_L;\gamma)}}}{2}$$
$$V(\theta_L; \gamma, \kappa) \coloneqq \hat{U}(\chi(\theta_L; \gamma, \kappa), \theta_L; \gamma).$$

Then:

1. *$K_\theta(\theta_L; \gamma) = G(\theta_L; \gamma)/(2 - \theta_L)^2 > 0$, so $\Theta(\gamma, \kappa)$ is an interval.*

2. *If $(\hat{x}_0, \theta_L) \in \mathcal{X}^-$ is binding, then $\theta_L \in \Theta(\gamma, \kappa)$ and*

$$\hat{x}_0 = \chi(\theta_L; \gamma, \kappa).$$

3. *If the optimum on $\mathcal{X}^-$ is binding and unique, then its second coordinate is the unique maximizer of $V(\cdot; \gamma, \kappa)$ on $\Theta(\gamma, \kappa)$.*

Proof. Using

$$K(\theta_L; \gamma) = \int_{\underline{\theta}}^{\theta_L} \frac{g(\theta; \gamma)}{2 - \theta_L}\, d\theta + \int_{\theta_L}^{1} \frac{g(\theta; \gamma)}{2 - \theta}\, d\theta,$$

Leibniz' rule gives

$$K_\theta(\theta_L; \gamma) = \frac{G(\theta_L; \gamma)}{(2 - \theta_L)^2} > 0.$$

Hence $K(\cdot; \gamma)$ is strictly increasing, so $\Theta(\gamma, \kappa)$ is an interval. On $[0, 1/2]$, the map $x \mapsto x(1 - x)$ is strictly increasing. Therefore, the binding equation $\hat{x}_0(1 - \hat{x}_0)K(\theta_L; \gamma) = \kappa$ has the unique solution $\hat{x}_0 = \chi(\theta_L; \gamma, \kappa)$ on $\mathcal{X}^-$. The last claim is immediate from this one-to-one parameterization of the binding frontier. $\square$

We can reduce the system in Proposition 3 to a one-dimensional equation in θ_L. Define

$$A(\theta_L; \gamma) \coloneqq \mathbb{E}_\gamma \left[\frac{1}{2 - \theta} \mathbf{1}_{\{\theta > \theta_L\}} \right] = \int_{\theta_L}^{1} \frac{g(\theta; \gamma)}{2 - \theta}\, d\theta$$
$$\rho(\theta_L; \gamma) \coloneqq \mathbb{E}_\gamma[\theta_L - \theta \mid \theta \leq \theta_L]$$
$$\alpha(\theta_L; \gamma) \coloneqq \frac{2 - \theta_L}{\rho(\theta_L; \gamma)}$$
$$\Psi(\theta_L; \gamma) \coloneqq \frac{K(\theta_L; \gamma)}{A(\theta_L; \gamma) + (1 + \alpha(\theta_L; \gamma))K(\theta_L; \gamma)}$$

Also define

$$
\begin{aligned}
R(\theta_L;\gamma,\kappa) &\coloneqq \chi(\theta_L;\gamma,\kappa) - \Psi(\theta_L;\gamma) \\
B(\theta_L;\gamma) &\coloneqq \frac{A(\theta_L;\gamma)}{K(\theta_L;\gamma)} + \alpha(\theta_L;\gamma) \\
\beta(\theta_L;\gamma,\kappa) &\coloneqq \frac{1+\sqrt{1-\frac{4\kappa}{K(\theta_L;\gamma)}}}{1-\sqrt{1-\frac{4\kappa}{K(\theta_L;\gamma)}}} \\
F(\theta_L;\gamma,\kappa) &\coloneqq B(\theta_L;\gamma) - \beta(\theta_L;\gamma,\kappa).
\end{aligned}
$$

With this notation, we can write the system of equations in Proposition 3 as

$$
\begin{aligned}
\kappa &= \hat{x}_0(1-\hat{x}_0)K(\theta_L;\gamma) \\
\frac{\hat{x}_0}{1-\hat{x}_0} &= \frac{(1-\delta)K(\theta_L;\gamma)}{A(\theta_L;\gamma)} \\
\delta &= \alpha(\theta_L;\gamma)\frac{\hat{x}_0}{1-\hat{x}_0}
\end{aligned}
\tag{46}
$$

Lemma 11 (Scalar equation for the interior binding optimum)**.** *Suppose the solution is characterized by the binding system (46). Then, on $\Theta(\gamma,\kappa)$:*

1. *$(\chi(\theta_L;\gamma,\kappa),\theta_L)$ solves the interior binding system if and only if*

$$R(\theta_L;\gamma,\kappa) = 0.$$

2. *Equivalently,*

$$R(\theta_L;\gamma,\kappa) = 0 \quad \Longleftrightarrow \quad B(\theta_L;\gamma) = \beta(\theta_L;\gamma,\kappa).$$

Proof. From the second and third equations in (46), with $r \coloneqq \hat{x}_0/(1-\hat{x}_0)$, we have

$$r = \frac{(1-\alpha r)K}{A} \quad \Longrightarrow \quad r = \frac{K}{A+\alpha K}.$$

Hence

$$\hat{x}_0 = \frac{r}{1+r} = \frac{K}{A+(1+\alpha)K} = \Psi(\theta_L;\gamma).$$

The first equation in (46) gives $\hat{x}_0 = \chi(\theta_L;\gamma,\kappa)$ on $\mathcal{X}^-$. Therefore the interior binding system is equivalent to $R(\theta_L;\gamma,\kappa)=0$. Next,

$$\Psi(\theta_L;\gamma)=\frac{1}{1+B(\theta_L;\gamma)}$$

because

$$\frac{1}{1+B}=\frac{1}{1+A/K+\alpha}=\frac{K}{A+(1+\alpha)K}.$$

Likewise,

$$\chi(\theta_L;\gamma,\kappa)=\frac{1}{1+\beta(\theta_L;\gamma,\kappa)}.$$

Therefore

$$R(\theta_L;\gamma,\kappa)=\frac{B(\theta_L;\gamma)-\beta(\theta_L;\gamma,\kappa)}{(1+B(\theta_L;\gamma))(1+\beta(\theta_L;\gamma,\kappa))},$$

so

$$R(\theta_L;\gamma,\kappa)=\frac{F(\theta_L;\gamma,\kappa)}{(1+B(\theta_L;\gamma))(1+\beta(\theta_L;\gamma,\kappa))}.$$

Thus $R(\theta_L;\gamma,\kappa)=0$ if and only if $B(\theta_L;\gamma)=\beta(\theta_L;\gamma,\kappa)$. □

The next lemma establishes that the one-dimensional equation determining θ_L is monotonic.

Lemma 12. *Fix (κ,γ) and suppose $G(\cdot;\gamma)$ is log-concave on Θ. Then, on the interior of $\Theta(\gamma,\kappa)$,*

$$\begin{aligned}\chi_\theta(\theta_L;\gamma,\kappa)&<0\\ \Psi_\theta(\theta_L;\gamma)&>0\\ R_\theta(\theta_L;\gamma,\kappa)&<0.\end{aligned}$$

In particular, $R(\cdot;\gamma,\kappa)$ has at most one zero.

Proof. Since $K_\theta>0$,

$$\chi_\theta(\theta_L;\gamma,\kappa)=-\frac{\kappa K_\theta(\theta_L;\gamma)}{K(\theta_L;\gamma)^2\sqrt{1-\frac{4\kappa}{K(\theta_L;\gamma)}}}<0.$$

Now let $D:=A+(1+\alpha)K$. Then

$$\Psi_\theta(\theta_L;\gamma)=\frac{AK_\theta-KA_\theta-\alpha_\theta K^2}{D^2},$$

with

$$K_\theta(\theta_L;\gamma)=\frac{G(\theta_L;\gamma)}{(2-\theta_L)^2}>0,\qquad A_\theta(\theta_L;\gamma)=-\frac{g(\theta_L;\gamma)}{2-\theta_L}<0,$$

and, under log-concavity of $G(\cdot;\gamma)$, we also have $\rho(\theta_L;\gamma) > 0$ and $\rho_\theta(\theta_L;\gamma) \geq 0$ for interior θ_L, because $\rho(\cdot;\gamma)$ is the mean-advantage-over-inferior function associated with $G(\cdot;\gamma)$, which is increasing under log-concavity of $G(\cdot;\gamma)$ (Bagnoli and Bergstrom, 2005, Theorem 5). It follows that

$$\alpha_\theta(\theta_L;\gamma) = -\frac{1}{\rho(\theta_L;\gamma)} - \frac{(2-\theta_L)\rho_\theta(\theta_L;\gamma)}{\rho(\theta_L;\gamma)^2} < 0.$$

Therefore $\Psi_\theta(\theta_L;\gamma) > 0$, and hence

$$R_\theta(\theta_L;\gamma,\kappa) = \chi_\theta(\theta_L;\gamma,\kappa) - \Psi_\theta(\theta_L;\gamma) < 0.$$

□

Lemma 13. *Fix (θ_L,γ) and suppose $\theta_L \in \Theta(\gamma,\kappa') \subseteq \Theta(\gamma,\kappa)$ with $\kappa' \geq \kappa$. Then*

$$\chi(\theta_L;\gamma,\kappa') \geq \chi(\theta_L;\gamma,\kappa).$$

Proof. For fixed (θ_L,γ), $\chi(\theta_L;\gamma,\kappa)$ is increasing in κ by inspection of the closed form. □

Lemma 14. *Fix an interior zero θ^* of $F(\cdot;\gamma,\kappa)$. Then*

$$F_\theta(\theta^*;\gamma,\kappa) < 0.$$

If the distribution is varied along a differentiable local path and

$$F_\gamma(\theta^*;\gamma,\kappa) > 0,$$

then the corresponding local solution satisfies

$$\frac{\partial\theta_L^*}{\partial\gamma}(\kappa,\gamma) > 0.$$

Moreover, at an interior zero,

$$F_\gamma(\theta^*;\gamma,\kappa) = B_\gamma(\theta^*;\gamma) - \frac{B(\theta^*;\gamma)\big(B(\theta^*;\gamma)+1\big)}{B(\theta^*;\gamma)-1}\,\partial_\gamma \log K(\theta^*;\gamma).$$

Proof. At a zero of F, Lemma 11 gives

$$R_\theta(\theta^*;\gamma,\kappa) = \frac{F_\theta(\theta^*;\gamma,\kappa)}{(1+B(\theta^*;\gamma))(1+\beta(\theta^*;\gamma,\kappa))}.$$

The denominator is positive, and Lemma 12 gives $R_\theta < 0$, so $F_\theta < 0$. The implicit-function theorem applied to

$$F(\theta_L^*(\kappa,\gamma);\gamma,\kappa) = 0$$

then gives

$$\frac{\partial \theta_L^*}{\partial \gamma}(\kappa,\gamma) = -\frac{F_\gamma(\theta_L^*(\kappa,\gamma);\gamma,\kappa)}{F_\theta(\theta_L^*(\kappa,\gamma);\gamma,\kappa)}.$$

Thus $F_\gamma > 0$ implies $\partial_\gamma \theta_L^* > 0$.

It remains only to compute the derivative of β. Write

$$z = \sqrt{1 - \frac{4\kappa}{K(\theta^*;\gamma)}}.$$

Then $\beta = (1+z)/(1-z)$ and

$$\partial_\gamma \beta = \frac{4\kappa}{zK(1-z)^2}\,\partial_\gamma \log K.$$

At a zero, $B = \beta = (1+z)/(1-z)$, so

$$z = \frac{B-1}{B+1}, \qquad \frac{4\kappa}{K} = 1 - z^2.$$

Substitution yields

$$\partial_\gamma \beta = \frac{B(B+1)}{B-1}\,\partial_\gamma \log K,$$

which proves the displayed formula for $F_\gamma = B_\gamma - \beta_\gamma$. □

Lemma 15. *Suppose that increasing γ shifts the density upward in the MLR order, in the sense that*

$$\frac{\partial^2}{\partial\theta\,\partial\gamma}\log g(\theta;\gamma) > 0.$$

Let $\mu_\gamma := \mathbb{E}_\gamma[\theta]$. At $\theta_L = 1$,

$$K(1;\gamma) = 1, \qquad K_\gamma(1;\gamma) = 0,$$

and

$$B(1;\gamma) = \frac{1}{1-\mu_\gamma}, \qquad B_\gamma(1;\gamma) = \frac{\partial_\gamma \mu_\gamma}{(1-\mu_\gamma)^2} > 0.$$

Consequently, $F_\gamma(1;\gamma,\kappa) > 0$ for every $\kappa < 1/4$ for which $\beta(1;\gamma,\kappa)$ is defined.

Proof. When $\theta_L = 1$, the debt-price term is identically equal to one:

$$K(1;\gamma) = \int_{\underline{\theta}}^{1} g(\theta;\gamma)\, d\theta = 1.$$

Differentiating the identity gives $K_\gamma(1;\gamma) = 0$. Also $A(1;\gamma) = 0$ and

$$\rho(1;\gamma) = \mathbb{E}_\gamma[1-\theta] = 1 - \mu_\gamma,$$

so $B(1;\gamma) = 1/(1-\mu_\gamma)$ and

$$B_\gamma(1;\gamma) = \frac{\partial_\gamma \mu_\gamma}{(1-\mu_\gamma)^2}.$$

The strict cross-partial condition makes $\partial_\gamma \log g(\theta;\gamma)$ strictly increasing in θ. Hence

$$\partial_\gamma \mu_\gamma = \int_{\underline{\theta}}^{1} \theta \partial_\gamma \log g(\theta;\gamma) g(\theta;\gamma)\, d\theta = \mathrm{Cov}_\gamma(\theta, \partial_\gamma \log g(\theta;\gamma)) > 0,$$

where the last inequality follows from positive association because both arguments are increasing and the score is strictly increasing. Since $K_\gamma(1;\gamma) = 0$, the derivative of $\beta(1;\gamma,\kappa)$ with respect to γ is zero. Hence $F_\gamma(1;\gamma,\kappa) = B_\gamma(1;\gamma) > 0$. $\square$

Financing-needs part: By Lemmas 9 and 10, the unique binding optimum on $\mathcal{X}^-$ is the unique zero of $R(\cdot;\gamma,\kappa)$ on $\Theta(\gamma,\kappa)$.

Fix γ and let $\kappa' \geq \kappa$. If $\theta_L^*(\kappa,\gamma) \notin \Theta(\gamma,\kappa')$, then, because $\Theta(\gamma,\kappa')$ is an upper interval, every feasible cutoff for κ' lies weakly to the right of $\theta_L^*(\kappa,\gamma)$. Thus $\theta_L^*(\kappa',\gamma) \geq \theta_L^*(\kappa,\gamma)$. If instead $\theta_L^*(\kappa,\gamma) \in \Theta(\gamma,\kappa')$, Lemma 13 gives

$$R(\theta_L^*(\kappa,\gamma);\gamma,\kappa') \geq R(\theta_L^*(\kappa,\gamma);\gamma,\kappa) = 0,$$

because Ψ does not depend on κ. Since $R(\cdot;\gamma,\kappa')$ is strictly decreasing by Lemma 12, its unique zero lies weakly to the right:

$$\theta_L^*(\kappa',\gamma) \geq \theta_L^*(\kappa,\gamma).$$

At the interior binding optimum,

$$\hat{x}_0^*(\kappa,\gamma) = \Psi(\theta_L^*(\kappa,\gamma);\gamma).$$

Lemma 12 gives $\Psi_\theta > 0$, so $\hat{x}_0^*(\kappa,\gamma)$ is also nondecreasing in κ.

Local MLR part: Fix γ and define

$$\mu_\gamma = \mathbb{E}_\gamma[\theta], \qquad \kappa^1(\gamma) = \frac{1-\mu_\gamma}{(2-\mu_\gamma)^2},$$

where $\kappa^1(\gamma) < 1/4$ because $\mu_\gamma > 0$. At the no-flexibility boundary,

$$F(1;\gamma,\kappa^1(\gamma)) = 0.$$

Indeed, at $\theta_L = 1$, $B(1;\gamma) = 1/(1-\mu_\gamma)$ and

$$\beta(1;\gamma,\kappa^1(\gamma)) = \frac{1+\frac{\mu_\gamma}{2-\mu_\gamma}}{1-\frac{\mu_\gamma}{2-\mu_\gamma}} = \frac{1}{1-\mu_\gamma}.$$

Lemma 15 gives $F_\gamma(1;\gamma,\kappa^1(\gamma)) > 0$ under the strict cross-partial condition. By continuity, the same strict inequality holds for all interior roots sufficiently close to the endpoint. Since $F_\theta < 0$ at interior roots by Lemma 14, the implicit-function theorem implies

$$\partial_\gamma \theta_L^*(\kappa,\gamma) > 0$$

whenever the interior root is in this endpoint neighborhood. The roots converge to the endpoint as $\kappa \uparrow \kappa^1(\gamma)$, again by continuity and uniqueness of the scalar equation. Hence we can choose $\kappa^\dagger < \kappa^1(\gamma)$ so that, for every $\kappa \in (\kappa^\dagger, \kappa^1(\gamma))$, the interior root lies in this endpoint neighborhood. Therefore,

$$\frac{\partial \theta_L^*}{\partial \gamma}(\kappa,\gamma) > 0 \qquad \text{for every } \kappa \in (\kappa^\dagger, \kappa^1(\gamma)).$$

Finally, because the financing constraint binds at the interior optimum,

$$\hat{x}_0^*(\kappa,\gamma) = \chi(\theta_L^*(\kappa,\gamma);\gamma,\kappa).$$

Lemma 12 gives $\chi_\theta < 0$. The local MLR order also gives $K_\gamma(\theta_L;\gamma) \geq 0$ for each fixed θ_L, because the kernel $1/(2-\max\{\theta,\theta_L\})$ is increasing in θ. Since χ is decreasing in K, $\chi_\gamma \leq 0$. Therefore, along the local solution,

$$\frac{\partial}{\partial \gamma}\hat{x}_0^*(\kappa,\gamma) = \chi_\theta\, \partial_\gamma \theta_L^*(\kappa,\gamma) + \chi_\gamma < 0,$$

which proves

$$\frac{\partial \hat{x}_0^*}{\partial \gamma}(\kappa,\gamma) < 0 \qquad \text{for every } \kappa \in (\kappa^\dagger, \kappa^1(\gamma)).$$

Price and MRS effects of distributional shifts

This section separates the two effects behind the distributional comparative static in the linear example. The first effect is a price effect: an upward shift in θ changes the financing frontier because it changes the expected price of initial debt. The second effect is a preference effect: the same shift changes the borrower's marginal rate of substitution between initial debt and the debt-limit threshold.

Fix an interior threshold $\theta_L \in (\underline{\theta}, 1)$ and a smooth one-parameter family $g(\cdot;\gamma)$ on $\Theta = [\underline{\theta}, 1]$. Let

$$s(\theta;\gamma) := \frac{\partial}{\partial\gamma} \log g(\theta;\gamma)$$

denote the score. Assume that the family is ordered by a smooth upward MLR shift, so $s(\theta;\gamma)$ is increasing in θ. Since $g(\cdot;\gamma)$ is a density, $\mathbb{E}_\gamma[s(\theta;\gamma)] = 0$.

The normalized financing frontier is

$$\kappa = \hat{x}_0(1-\hat{x}_0)K(\theta_L;\gamma), \qquad K(\theta_L;\gamma) = \mathbb{E}_\gamma\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\le\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right].$$

On the branch $\hat{x}_0 \in (0, 1/2)$, write the solution of the binding frontier as $\hat{x}_0 = \chi(\theta_L;\gamma,\kappa)$. Also define

$$A(\theta_L;\gamma) := \mathbb{E}_\gamma\left[\frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right], \qquad \rho(\theta_L;\gamma) := \mathbb{E}_\gamma[\theta_L - \theta \mid \theta \le \theta_L].$$

Remark 1 (Price and MRS effects of an upward MLR shift)**.** *Suppose the smooth upward MLR condition above holds. On the binding frontier,*

$$\chi' < 0, \qquad \frac{\partial}{\partial\gamma}\chi' \ge 0.$$

At any admissible interior point satisfying the debt-limit first-order condition, the slope of the borrower's indifference curve,

$$\mathrm{MRS}_{\hat{x}_0,\theta_L} := \left.\frac{d\hat{x}_0}{d\theta_L}\right|_{\hat{U}},$$

satisfies

$$\frac{\partial}{\partial\gamma}\mathrm{MRS}_{\hat{x}_0,\theta_L} \ge 0,$$

whenever the MRS is well defined.

Proof. First consider the binding frontier. Leibniz' rule gives

$$K'(\theta_L;\gamma) = \frac{G(\theta_L;\gamma)}{(2-\theta_L)^2}.$$

Thus

$$\frac{\partial}{\partial \gamma} \log\left(\frac{K'}{K}\right) = \frac{\frac{\partial}{\partial \gamma} G(\theta_L; \gamma)}{G(\theta_L; \gamma)} - \frac{\frac{\partial}{\partial \gamma} K(\theta_L; \gamma)}{K(\theta_L; \gamma)}.$$

The first term is

$$\frac{\frac{\partial}{\partial \gamma} G(\theta_L; \gamma)}{G(\theta_L; \gamma)} = \mathbb{E}_\gamma[s(\theta; \gamma) \mid \theta \le \theta_L] \le 0,$$

because the score is increasing and has unconditional mean zero. The second term is nonnegative because

$$\frac{\partial}{\partial \gamma} K(\theta_L; \gamma) = \mathbb{E}_\gamma\left[\frac{s(\theta; \gamma)}{2 - \max\{\theta, \theta_L\}}\right] \ge 0,$$

where the inequality follows from positive association. Both the score and the debt-price kernel are increasing in θ. Hence

$$\frac{\partial}{\partial \gamma} \log\left(\frac{K'}{K}\right) \le 0.$$

Implicit differentiation of the binding frontier gives

$$\chi' = -\frac{\chi(1-\chi)}{1-2\chi} \frac{K'}{K} < 0.$$

Let $H(\chi) := \chi(1-\chi)/(1-2\chi)$. On $\chi \in (0, 1/2)$, $H'(\chi) > 0$. Since

$$\frac{\partial}{\partial \gamma} \chi = -H(\chi) \frac{\frac{\partial}{\partial \gamma} K}{K} \le 0,$$

we have

$$\frac{\partial}{\partial \gamma} \chi' = -H'(\chi) \frac{\partial \chi}{\partial \gamma} \frac{K'}{K} - H(\chi) \frac{\partial}{\partial \gamma}\left(\frac{K'}{K}\right) \ge 0.$$

An upward MLR shift therefore makes the binding-frontier slope less negative.

Now consider the MRS. For the normalized objective $\hat{U}$, direct differentiation gives

$$\hat{U}_{\hat{x}_0} = \frac{G(\theta_L; \gamma) D_U}{(2-\theta_L)^2} - \hat{x}_0 A(\theta_L; \gamma), \qquad \hat{U}_{\theta_L} = -\frac{(1-\hat{x}_0) G(\theta_L; \gamma) D_U}{(2-\theta_L)^3},$$

where

$$D_U := (1-\hat{x}_0)\rho(\theta_L; \gamma) - \hat{x}_0(2-\theta_L).$$

Define

$$\mathcal{L}_U := \frac{G(\theta_L; \gamma) D_U}{(2-\theta_L)^2}.$$

Then

$$\mathrm{MRS}_{\hat{x}_0,\theta_L} = -\frac{\hat{U}_{\theta_L}}{\hat{U}_{\hat{x}_0}} = \frac{1-\hat{x}_0}{2-\theta_L}\frac{\mathcal{L}_U}{\mathcal{L}_U - \hat{x}_0 A}.$$

Holding $(\hat{x}_0, \theta_L)$ fixed and differentiating with respect to γ gives

$$\frac{\partial}{\partial\gamma}\mathrm{MRS}_{\hat{x}_0,\theta_L} = \frac{\hat{x}_0(1-\hat{x}_0)}{2-\theta_L}\frac{\mathcal{L}_U\frac{\partial A}{\partial\gamma} - A\frac{\partial \mathcal{L}_U}{\partial\gamma}}{(\mathcal{L}_U - \hat{x}_0 A)^2}.$$

The MLR order gives $\partial A/\partial\gamma \geq 0$, $\partial G/\partial\gamma \leq 0$, and

$$\frac{\partial}{\partial\gamma}\rho(\theta_L;\gamma) = -\operatorname{Cov}_\gamma(\theta, s(\theta;\gamma) \mid \theta \leq \theta_L) \leq 0.$$

At an admissible interior point satisfying the debt-limit first-order condition,

$$\delta = \frac{2-\theta_L}{\rho(\theta_L;\gamma)}\frac{\hat{x}_0}{1-\hat{x}_0} \leq 1,$$

because $\delta = 1/(1+\lambda)$ with $\lambda \geq 0$. This is equivalent to $D_U \geq 0$. Therefore $\mathcal{L}_U \geq 0$ and $\partial\mathcal{L}_U/\partial\gamma \leq 0$, which implies

$$\frac{\partial}{\partial\gamma}\mathrm{MRS}_{\hat{x}_0,\theta_L} \geq 0.$$

At a tangency, the MRS equals $\chi' < 0$. Thus the price effect and the MRS effect both make their respective slopes less negative after an upward MLR shift. The total movement of the tangency depends on which slope shifts more. □

C Exclusive Lending

For notational convenience, let us define

$$\bar{V}_1 = \tau_1(\bar{\theta}) + \bar{\theta}V_2(x_1(\bar{\theta}))$$

as type $\bar{\theta}$'s continuation payoff at $t = 1$.

Lemma 16. *Any incentive-compatible pair $\{\tau_1(\theta), x_1(\theta)\}$ satisfying $\bar{V}_1 \geq \bar{\theta}V_2(x_1(\bar{\theta}))$ satisfies the constraint*

$$\tau_1(\theta) = \bar{V}_1 - \theta V_2\left(x_1(\theta)\right) - \int_\theta^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \geq 0 \quad \forall\theta \in \Theta.$$

Proof. We verify that it is sufficient to consider the non-negativity constraint of $\tau_1(\theta)$ for the highest type. Integration by parts implies that

$$\theta V_2\left(x_1(\theta)\right)+\int_\theta^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta}=\bar{\theta}V_2\left(x_1(\bar{\theta})\right)-\int_\theta^{\bar{\theta}}\tilde{\theta}V_2'(x_1(\tilde{\theta}))dx_1(\tilde{\theta})$$

Hence, we can write the envelope condition as

$$\tau_1(\theta)=\tau_1(\bar{\theta})+\int_\theta^{\bar{\theta}}\tilde{\theta}V_2'(x_1(\tilde{\theta}))dx_1(\tilde{\theta})$$

It follows that, for any $\theta''>\theta'$, we have

$$\tau_1(\theta'')=\tau_1(\theta')-\int_{\theta'}^{\theta''}\tilde{\theta}\underbrace{V_2'(x_1(\tilde{\theta}))}_{<0}\underbrace{dx_1(\tilde{\theta})}_{\leq 0}\leq\tau_1(\theta'),$$

which means that $\tau_1(\bar{\theta})\geq 0$ implies $\tau_1(\theta)\geq 0$ for all $\theta<\bar{\theta}$. Moreover, the constraint $\tau(\bar{\theta})\geq 0$ is equivalent to require that $\bar{V}_1\geq\bar{\theta}V_2\left(x_1(\bar{\theta})\right)$ □

Proof of Proposition 6

Proof. Let $\mathbf{x}_1:=\{x_1(\theta)\}_{\theta\in\Theta}$. For a fixed λ (and therefore a fixed δ) we have

$$\max_{\mathbf{x}_1}\mathcal{U}(\mathbf{x}_1):=\int_{\underline{\theta}}^{\bar{\theta}}\left[W_1(x_1(\theta),\theta)+(1-\delta)\frac{G(\theta)}{g(\theta)}V_2\left(x_1(\theta)\right)\right]dG(\theta)-(1-\delta)\bar{\theta}V_2\left(x_1(\bar{\theta})\right)$$

s.t. $\mathbf{x}_1$ is non-increasing.

Starting from $\mathbf{x}_1$, the directional derivative in the direction $h(\theta)$ is generally defined as

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h})=\frac{d}{dk}\mathcal{U}(\mathbf{x}+k\mathbf{h})\Big|_{k=0},$$

which, in our case, is given by

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h})=(1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)h(\bar{\theta})+\int_{\underline{\theta}}^{\bar{\theta}}\Big[\frac{\partial W_1(x_1(\theta),\theta)}{\partial x_1}g(\theta)-(1-\delta)G(\theta)q\left(x_1(\theta)\right)\Big]h(\theta)d\theta.$$

Letting

$$\Phi(\theta;\mathbf{x}_1):=\int_{\underline{\theta}}^{\theta}\Big[\frac{\partial W_1(x_1(\tilde{\theta}),\tilde{\theta})}{\partial x_1}g(\tilde{\theta})-(1-\delta)G(\tilde{\theta})q\left(x_1(\tilde{\theta})\right)\Big]d\tilde{\theta},$$

the directional derivative can be written as

$$\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{h}) = (1-\delta)\bar{\theta} q\left(x_1(\bar{\theta})\right) h(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \Phi'(\theta; \mathbf{x}_1) h(\theta) d\theta.$$

Using integration by parts, we obtain that $\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{h})$ can be written as

$$\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{h}) = \left[\Phi(\bar{\theta}; \mathbf{x}_1) + (1-\delta)\bar{\theta} q\left(x_1(\bar{\theta})\right)\right] h(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1) dh(\theta).$$

This alternative formulation for the directional derivative is convenient because it is written explicitly in terms of $dh(\theta)$ rather than $h(\theta)$. This allows us to incorporate the monotonicity constraint directly into the analysis of the first-order conditions.

Let P be set of all non-increasing functions on Θ. Lemma 1 in (Luenberger, 1969, p. 227) provides the following necessary and sufficient condition for $\mathbf{x}_1$ to maximize $\mathcal{U}$:

$$\begin{aligned} &\nabla \mathcal{U}(\mathbf{x}_1; \tilde{\mathbf{x}}_1) \leq 0 \quad \forall\ \tilde{\mathbf{x}}_1 \in P \\ &\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{x}_1) = 0. \end{aligned}$$

In our case, these two conditions become

$$\left[\Phi(\bar{\theta}; \mathbf{x}_1) + (1-\delta)\bar{\theta} q\left(x_1(\bar{\theta})\right)\right] \tilde{x}_1(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1) d\tilde{x}_1(\theta) \leq 0 \quad \forall\ \tilde{\mathbf{x}}_1 \in P \tag{47a}$$

$$\left[\Phi(\bar{\theta}; \mathbf{x}_1) + (1-\delta)\bar{\theta} q\left(x_1(\bar{\theta})\right)\right] x_1(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1) dx_1(\theta) = 0. \tag{47b}$$

For notational convenience in this proof, let

$$x_1^s(\theta) := x_1^*(\phi_\delta(\theta)).$$

We are going to construct a solution, $\mathbf{x}_1$, given by

$$x_1(\theta) = \begin{cases} x_1^s(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_H) \\ x_1^s(\theta_H) & \text{if } \theta \in [\theta_H, \bar{\theta}] \end{cases}$$

and show that it satisfies (47). By definition, $x_1^s(\theta)$ solves

$$\frac{\partial W_1(x_1, \theta)}{\partial x_1} - (1-\delta)\frac{G(\theta)}{g(\theta)} q\left(x_1\right) = 0.$$

Therefore, $\Phi(\theta;\mathbf{x}_1)=0$ for all $\theta\in[\underline{\theta},\theta_H]$. Moreover, we define $\theta_H\in[\underline{\theta},\bar{\theta}]$ as the solution to

$$\int_{\theta_H}^{\bar{\theta}}\Big[\frac{\partial W_1(x_1^s(\theta_H),\tilde{\theta})}{\partial x_1}g(\tilde{\theta})-(1-\delta)G(\tilde{\theta})q\left(x_1^s(\theta_H)\right)\Big]d\tilde{\theta}+(1-\delta)\bar{\theta}q\left(x_1^s(\theta_H)\right)=0. \tag{48}$$

We will show at the end of this proof that this solution exists. By construction, we have

$$\Phi(\bar{\theta};\mathbf{x}_1)+(1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)=0,$$

and (47b) holds. Moreover, (47a) becomes

$$-\int_{\underline{\theta}}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)d\tilde{x}_1(\theta)\leq 0\quad\forall\,\tilde{\mathbf{x}}_1\in P.$$

Assumption 5 and the monotonicity of $x_1^s(\theta)$ imply that

$$\frac{\partial W_1(x_1,\theta)}{\partial x_1}\Big|_{x_1=x_1^s(\theta_H)}-(1-\delta)\frac{G(\theta)}{g(\theta)}q\left(x_1^s(\theta_H)\right)\leq 0,\quad\forall\theta\in[\theta_H,\bar{\theta}]$$

so $\Phi(\theta;\mathbf{x}_1)\leq 0$ for all $\theta\in(\theta_H,\bar{\theta}]$. Therefore, the first order condition (47) reduces to

$$\int_{\theta_H}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)d\tilde{x}_1(\theta)\geq 0\quad\forall\,\tilde{\mathbf{x}}_1\in P,$$

which is satisfied for all non-increasing $\tilde{\mathbf{x}}_1$.

The final step is to consider the existence of a solution to equation (48). Note that

$$\frac{\partial^2 W_1(x_1^s(\theta_H),\theta)}{\partial\theta\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{\partial W_1(x_1^s(\theta_H),\theta)}{\partial\theta}\right)=\frac{\partial V_2(x_1^s(\theta_H))}{\partial x_1}=-q(x_1^s(\theta_H)).$$

Therefore, for $\theta\in[\theta_H,\bar{\theta}]$, we have

$$\frac{\partial W_1(x_1^s(\theta_H),\theta)}{\partial x_1}=\frac{\partial W_1(x_1^s(\theta_H),\theta_H)}{\partial x_1}+(\theta_H-\theta)q(x_1(\theta_H))=(\theta_H-\theta)q(x_1(\theta_H))+(1-\delta)\frac{G(\theta_H)}{g(\theta_H)}q\left(x_1(\theta_H)\right),$$

which allow us to simplify equation (48) in the following way

$$q(x_1^s(\theta_H))\int_{\theta_H}^{\bar{\theta}}\left[(\theta_H-\tilde{\theta})+(1-\delta)\frac{G(\theta_H)}{g(\theta_H)}-(1-\delta)\frac{G(\tilde{\theta})}{g(\tilde{\theta})}\right]g(\tilde{\theta})d\tilde{\theta}+(1-\delta)\bar{\theta}q\left(x_1^s(\theta_H)\right)=0.$$

Hence, we can cancel $q(x_1^s(\theta_H))$ and obtain an expression exclusively in terms of δ and $g(\theta)$

$$(1-\delta)\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)} + \int_{\theta_H}^{\bar{\theta}} \left[(\theta_H - \tilde{\theta})g(\tilde{\theta}) - (1-\delta)G(\tilde{\theta})\right] d\tilde{\theta} + (1-\delta)\bar{\theta} = 0.$$

Integrating by parts, we get

$$\int_{\theta_H}^{\bar{\theta}} \left[(\theta_H - \tilde{\theta})g(\tilde{\theta}) - G(\tilde{\theta})\right] d\tilde{\theta} = \theta_H - \bar{\theta}$$

so, we end with

$$H(\theta_H) \coloneqq (1-\delta)\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)} + \theta_H - \delta\left[\bar{\theta} - \int_{\theta_H}^{\bar{\theta}} G(\tilde{\theta})d\tilde{\theta}\right] = 0.$$

The function $H(\theta_H)$ satisfies the following properties

$$\begin{aligned} H(\underline{\theta}) &= \underline{\theta} - \delta\mathbb{E}[\theta] \\ H(\bar{\theta}) &= (1-\delta)\bar{\theta} \\ H'(\theta_H) &= \left[\frac{g(\theta_H)}{G(\theta_H)} + (1-\delta)\frac{\partial}{\partial\theta_H}\log\left(\frac{G(\theta_H)}{g(\theta_H)}\right)\right]\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)}. \end{aligned}$$

Log-concavity of G implies that $G(\theta)/g(\theta)$ is weakly increasing. The first term in brackets is strictly positive, so $H'(\theta_H) > 0$ on $(\underline{\theta}, \bar{\theta})$. Therefore, the partial-separation branch exists if and only if $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$. At equality, $H(\underline{\theta}) = 0$ and hence $\theta_H = \underline{\theta}$, so this branch collapses to full pooling; for $\delta < \underline{\theta}/\mathbb{E}[\theta]$, the solution is fully pooled.

If $\delta < \underline{\theta}/\mathbb{E}[\theta]$, let $\bar{x}_1 \coloneqq x_1^*(\delta\mathbb{E}[\theta])$. The first order condition is satisfied by setting $\theta_H = \underline{\theta}$ and $x_1(\theta) = \bar{x}_1$ such that

$$\Phi(\bar{\theta}; \mathbf{x}_1 = \bar{x}_1) + (1-\delta)\bar{\theta}q\left(\bar{x}_1\right) = 0,$$

which requires that

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[\frac{\partial W_1(\bar{x}_1, \tilde{\theta})}{\partial x_1} g(\tilde{\theta}) - (1-\delta)q(\bar{x}_1)G(\tilde{\theta})\right] d\tilde{\theta} + (1-\delta)\bar{\theta}q(\bar{x}_1) = 0.$$

Using integration by parts, we recover the first-order condition for $\bar{x}_1$:

$$\bar{x}_1 q'(\bar{x}_1) + \left(1 - \delta\mathbb{E}[\theta]\right) q(\bar{x}_1) = 0.$$

Noticing that $x_1^s(\underline{\theta})$ solves

$$x_1^s(\underline{\theta})q'(x_1^s(\underline{\theta})) + (1-\underline{\theta})\,q(x_1^s(\underline{\theta})) = 0$$

we realize that whenever $\underline{\theta} > \delta\mathbb{E}[\theta]$, we have that $\bar{x}_1 > x_1^s(\underline{\theta})$. This, in turn implies that

$$\frac{\partial W_1(\bar{x}_1, \theta)}{\partial x_1} g(\theta) - (1-\delta)q(\bar{x}_1)G(\theta) < 0$$

so $\Phi(\theta; \mathbf{x}_1 = \bar{x}_1) < 0$ which means that

$$-\int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1 = \bar{x}_1) d\tilde{x}_1(\theta) \leq 0 \quad \forall\, \tilde{\mathbf{x}}_1 \in P,$$

which means that $x_1(\theta) = \bar{x}_1$ is optimal. $\square$

Proof of Proposition 7

Proof. For each $\delta \in [0,1]$, let $x_1^{s*}(\theta;\delta)$ denote the actual optimal schedule characterized in Proposition 6. Set $\tau_1^{s*}(\bar{\theta};\delta) = 0$ and recover the remaining interim transfers from (23). The initial pledgeable income generated by this mechanism is

$$\begin{aligned}\mathcal{T}_0(\delta) &\coloneqq \mathbb{E}\left[x_1^{s*}(\theta;\delta)q\left(x_1^{s*}(\theta;\delta)\right) - \tau_1^{s*}(\theta;\delta)\right] \\ &= \mathbb{E}\left[W_1\left(x_1^{s*}(\theta;\delta),\theta\right) + \frac{G(\theta)}{g(\theta)}V_2\left(x_1^{s*}(\theta;\delta)\right)\right] - \bar{\theta}V_2\left(x_1^{s*}(\bar{\theta};\delta)\right),\end{aligned} \tag{49}$$

where the second equality follows from (24).

We first establish how pledgeable income varies with δ. For any feasible debt schedule x_1, let $\mathcal{S}(x_1) \coloneqq \mathbb{E}[W_1(x_1(\theta),\theta)]$, and let $\mathcal{T}_0(x_1)$ denote the corresponding expression in (49). The reduced objective in (27) can be written as

$$\delta\mathcal{S}(x_1) + (1-\delta)\mathcal{T}_0(x_1).$$

For $\delta > 0$, maximizing this expression is equivalent to maximizing $\mathcal{S}(x_1) + \lambda\mathcal{T}_0(x_1)$, where $\lambda = (1-\delta)/\delta$. Consider $\lambda' > \lambda$, and denote the corresponding optimal schedules by x_1' and x_1. Revealed preference gives

$$\begin{aligned}\mathcal{S}(x_1) + \lambda\mathcal{T}_0(x_1) &\geq \mathcal{S}(x_1') + \lambda\mathcal{T}_0(x_1'), \\ \mathcal{S}(x_1') + \lambda'\mathcal{T}_0(x_1') &\geq \mathcal{S}(x_1) + \lambda'\mathcal{T}_0(x_1).\end{aligned}$$

Adding these inequalities yields

$$(\lambda' - \lambda)\left(\mathcal{T}_0(x_1') - \mathcal{T}_0(x_1)\right) \geq 0.$$

Thus, $\mathcal{T}_0$ is weakly increasing in λ and, equivalently, weakly decreasing in δ.

The explicit schedules in Proposition 6 also imply continuity. For $\delta \geq \delta^p := \underline{\theta}/\mathbb{E}[\theta]$, the cutoff $\theta_H(\delta)$ is the unique root of $H(\theta_H) = 0$. The strict monotonicity of H in θ_H and the continuity of H in (θ_H, δ) imply that $\theta_H(\delta)$ is continuous. For $\delta \leq \delta^p$, the schedule is the continuous pooled allocation $x_1^*(\delta\mathbb{E}[\theta])$. The two branches agree at δ^p, where $\theta_H = \underline{\theta}$. Hence, $x_1^{s*}(\theta; \delta)$ is continuous in δ. Continuity of x_1^* and boundedness of the schedules allow dominated convergence in (49), so $\mathcal{T}_0(\delta)$ is also continuous.

At $\delta = 1$, Proposition 6 gives the efficient schedule, so

$$\mathcal{T}_0(1) = I^s.$$

At $\delta = \delta^p$, the schedule is fully pooled at $x_1^*(\underline{\theta})$. The envelope condition and $\tau_1^{s*}(\bar{\theta}; \delta^p) = 0$ then imply $\tau_1^{s*}(\theta; \delta^p) = 0$ for every type. Therefore,

$$\mathcal{T}_0(\delta^p) = x_1^*(\underline{\theta})q\left(x_1^*(\underline{\theta})\right) = I^p.$$

At $\delta = 0$, understood as the limiting case $\lambda \to \infty$, the reduced program directly gives the revenue-maximizing pooled allocation $x_1^*(0)$. Since $x_1^*(0)$ maximizes $W_1(x_1, 0) = x_1 q(x_1)$,

$$\mathcal{T}_0(0) = x_1^*(0)q\left(x_1^*(0)\right) = I^{\max}.$$

If $I \leq I^s$, the efficient schedule is feasible and optimal. The financing constraint is slack when $I < I^s$ and binds with a zero multiplier when $I = I^s$, so $\delta^{s*} = 1$. If $I^s < I < I^p$, continuity and monotonicity of $\mathcal{T}_0$ give a $\delta^{s*} \in (\delta^p, 1)$ such that $\mathcal{T}_0(\delta^{s*}) = I$. Proposition 6 then gives the partially separating schedule, with $\theta_H \in (\underline{\theta}, \bar{\theta})$ determined jointly by (28) and $\tau_0 = I$. This argument establishes existence but does not require uniqueness of the pair (θ_H, δ^{s*}).

Finally, if $I^p \leq I \leq I^{\max}$, continuity gives a $\delta^{s*} \in [0, \delta^p]$. In this range, Proposition 6 gives

$$x_1^{s*}(\theta) = x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right) \quad \text{for all } \theta \in \Theta,$$

and all interim transfers are zero. The binding financing constraint is therefore

$$x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right) q\left(x_1^*\left(\delta^{s*}\mathbb{E}[\theta]\right)\right) = I.$$

This equation has a unique solution in the pooled region. To see this, let $a = \delta\mathbb{E}[\theta]$. The first-order condition for $x_1^*(a)$ is

$$x_1^*(a)q'\left(x_1^*(a)\right) + (1-a)q\left(x_1^*(a)\right) = 0.$$

Differentiating this condition and using $q' < 0$ and $q'' \leq 0$ gives

$$x_1^{*\prime}(a) = \frac{q\left(x_1^*(a)\right)}{(2-a)q'\left(x_1^*(a)\right) + x_1^*(a)q''\left(x_1^*(a)\right)} < 0.$$

The same first-order condition gives

$$\frac{d}{dx_1}\left[x_1 q(x_1)\right]\bigg|_{x_1 = x_1^*(a)} = aq\left(x_1^*(a)\right).$$

Consequently, for $\delta > 0$,

$$\frac{d}{d\delta}\left[x_1^*\left(\delta\mathbb{E}[\theta]\right) q\left(x_1^*\left(\delta\mathbb{E}[\theta]\right)\right)\right] = \delta\mathbb{E}[\theta]^2 q\left(x_1^*\left(\delta\mathbb{E}[\theta]\right)\right) x_1^{*\prime}\left(\delta\mathbb{E}[\theta]\right) < 0.$$

The endpoint identities then give $\delta^{s*} = \delta^p$ at $I = I^p$ and $\delta^{s*} = 0$ at $I = I^{\max}$. $\square$

Proof of Proposition 8

Proof. For the lower bound, use the envelope condition (23) together with $\tau_1^{s*}(\bar{\theta}) = 0$. Integration by parts gives

$$\tau_1^{s*}(\theta) = \tau_1^{s*}(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} \tilde{\theta} V_2'(x_1^{s*}(\tilde{\theta}))\, dx_1^{s*}(\tilde{\theta}).$$

Because $x_1^{s*}(\theta)$ is non-increasing and $V_2'(x_1) = -q(x_1) \leq 0$, the integral is weakly positive, so $\tau_1^{s*}(\theta) \geq 0$.

If the solution is fully pooled, then $x_1^{s*}(\theta) = x_0^s$ for all θ, and the envelope condition implies $\tau_1^{s*}(\theta) = 0$ for all θ. Hence the upper bound holds trivially, and there is no active refinancing margin to represent with $\theta^s(\cdot)$.

Now consider the partially separating case. On the pooled region $[\theta_H, \bar{\theta}]$, debt is constant at x_0^s, so the envelope condition again implies $\tau_1^{s*}(\theta) = 0$. Hence

$$\tau_1^{s*}(\theta) = \left(x_1^{s*}(\theta) - x_0^s\right) q(x_1^{s*}(\theta)) = 0 \quad \forall \theta \in [\theta_H, \bar{\theta}].$$

On the separating region $[\underline{\theta}, \theta_H]$, define

$$B(\theta) \coloneqq \left(x_1^{s*}(\theta) - x_0^s\right) q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta).$$

Differentiating the envelope condition on that region yields

$$\tau_1^{s*\prime}(\theta) = \theta q(x_1^{s*}(\theta))x_1^{s*\prime}(\theta).$$

Therefore,

$$\begin{aligned} B'(\theta) &= \left[q(x_1^{s*}(\theta)) + (x_1^{s*}(\theta) - x_0^s)\, q'(x_1^{s*}(\theta))\right] x_1^{s*\prime}(\theta) - \tau_1^{s*\prime}(\theta) \\ &= \left[\frac{\partial W_1(x_1^{s*}(\theta),\theta)}{\partial x_1} - x_0^s q'(x_1^{s*}(\theta))\right] x_1^{s*\prime}(\theta). \end{aligned}$$

On the separating region, Proposition 6 gives

$$\frac{\partial W_1(x_1^{s*}(\theta),\theta)}{\partial x_1} = (1-\delta^{s*})\frac{G(\theta)}{g(\theta)}q(x_1^{s*}(\theta)) \geq 0.$$

Since $q'(x_1) < 0$, $x_0^s \geq 0$, and $x_1^{s*}(\theta)$ is strictly decreasing on $[\underline{\theta}, \theta_H]$, we have $B'(\theta) < 0$ on $[\underline{\theta}, \theta_H)$. Because $B(\theta_H) = 0$, it follows that $B(\theta) > 0$ for all $\theta < \theta_H$, which proves

$$\tau_1^{s*}(\theta) \leq (x_1^{s*}(\theta) - x_0^s)\, q(x_1^{s*}(\theta)) \quad \forall \theta \in \Theta,$$

with strict inequality on the separating region below θ_H.

To obtain the implementation formula, let $\theta^s(\Delta x_1)$ denote the inverse of $\Delta x_1^s(\theta)$ on the separating region and define

$$\tilde{\tau}_1^s(\Delta x_1) := \tau_1^{s*}(\theta^s(\Delta x_1)).$$

By the chain rule,

$$\frac{d\tilde{\tau}_1^s(\Delta x_1)}{d\Delta x_1} = \tau_1^{s*\prime}(\theta^s(\Delta x_1))\frac{d\theta^s(\Delta x_1)}{d\Delta x_1} = \theta^s(\Delta x_1)q(x_0^s + \Delta x_1).$$

Since $\tilde{\tau}_1^s(0) = \tau_1^{s*}(\theta_H) = 0$, integrating from 0 to Δx_1 yields

$$\tilde{\tau}_1^s(\Delta x_1) = \int_0^{\Delta x_1} \theta^s(z) q(x_0^s + z)\, dz.$$

Therefore, defining

$$p_1^s(\Delta x_1) := \frac{\tilde{\tau}_1^s(\Delta x_1)}{\Delta x_1},$$

we obtain

$$\tau_1^{s*}(\theta) = \Delta x_1^s(\theta)\, p_1^s(\Delta x_1^s(\theta)) \quad \forall \theta \in [\underline{\theta}, \theta_H].$$

Finally, the strict upper bound above implies

$$\tilde{\tau}_1^s(\Delta x_1) < \Delta x_1 q(x_0^s + \Delta x_1) \quad \forall \Delta x_1 > 0,$$

so dividing by $\Delta x_1 > 0$ gives $p_1^s(\Delta x_1) < q(x_0^s + \Delta x_1)$. □

D Comparison of Exclusive and Non-Exclusive Lending

Proof of Proposition 9

Proof. Write $x_s(\theta) \coloneqq x_1^{s*}(\theta)$ and $x_n(\theta) \coloneqq x_1^{n*}(\theta)$ inside this proof. Also define

$$\varepsilon(x) \coloneqq -\frac{xq'(x)}{q(x)}$$

as the positive elasticity of inverse demand for debt. By Lemma 1 and Assumption 3, $q'(x) < 0$ and $q''(x) \le 0$. Hence

$$\varepsilon'(x) = -\frac{q'(x)}{q(x)} + x\frac{q'(x)^2 - q''(x)q(x)}{q(x)^2} > 0,$$

so ε is strictly increasing.

We first show the useful order

$$x_0^s \le x_0^{\min} \le x_0^{n*}.$$

Let

$$L^s(\theta) \coloneqq x_s(\theta)q(x_s(\theta)) - \tau_1^{s*}(\theta)$$

be the exclusive lender's net state-contingent repayment. On the separating region, the envelope condition gives

$$\tau_1^{s*\prime}(\theta) = \theta q(x_s(\theta))x_s'(\theta).$$

Therefore,

$$L^{s\prime}(\theta) = \frac{\partial W_1(x_s(\theta),\theta)}{\partial x_1}x_s'(\theta).$$

Using the exclusive-lending first-order condition on the separating region,

$$\frac{\partial W_1(x_s(\theta),\theta)}{\partial x_1} = (1-\delta^{s*})\frac{G(\theta)}{g(\theta)}q(x_s(\theta)),$$

and $x_s'(\theta) \leq 0$, we obtain $L^{s\prime}(\theta) \leq 0$. On the pooled region, $L^s(\theta) = x_0^s q(x_0^s)$. Thus

$$L^s(\theta) \geq x_0^s q(x_0^s) \quad \forall\theta.$$

Since the exclusive lender's participation constraint and the financing constraint bind, $I = \mathbb{E}[L^s(\theta)]$. Hence

$$I \geq x_0^s q(x_0^s).$$

Moreover,

$$x_0^s = x_1^*(\phi_{\delta^{s*}}(\theta_H^{s*})) \leq x_1^*(0),$$

because $\phi_{\delta^{s*}}(\theta_H^{s*}) \geq 0$ and $x_1^*(\cdot)$ is decreasing. The function $xq(x)$ is increasing on $[0, x_1^*(0)]$, so $I \geq x_0^s q(x_0^s)$ implies $x_0^s \leq x_0^{\min}$. Feasibility of the non-exclusive initial debt gives $x_0^{n*} \in [x_0^{\min}, x_0^{\max}]$, so $x_0^{n*} \geq x_0^{\min}$. Since $\theta_L^{n*} < \bar{\theta}$, the no-flexibility case in Proposition 3 is ruled out. Hence $x_0^{n*} > x_0^{\min}$.

We now prove the debt crossing. Let δ^{n*} be the multiplier associated with the optimal non-exclusive debt-limit problem. The debt-limit first-order condition gives

$$\frac{x_0^{n*}}{x_L^{n*}}\varepsilon(x_L^{n*}) = \delta^{n*}\mathbb{E}[\theta_L^{n*} - \theta \mid \theta \leq \theta_L^{n*}] < \theta_L^{n*} - \underline{\theta},$$

where the strict inequality uses $\delta^{n*} \leq 1$ and the fact that the conditional mean gap is smaller than the largest possible gap. The flexible-debt first-order condition at the debt limit gives

$$\varepsilon(x_L^{n*}) = 1 - \theta_L^{n*} + \frac{x_0^{n*}}{x_L^{n*}}\varepsilon(x_L^{n*}) < 1 - \underline{\theta}.$$

Since $\varepsilon(x_1^*(\underline{\theta})) = 1 - \underline{\theta}$ and ε is strictly increasing,

$$x_L^{n*} < x_1^*(\underline{\theta}) = x_s(\underline{\theta}).$$

Thus the non-exclusive schedule begins below the exclusive schedule.

On $[\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\}]$, the non-exclusive schedule is flat at x_L^{n*} and the exclusive schedule is on its separating branch. Define

$$D(\theta) := \varepsilon(x_L^{n*}) - \varepsilon(x_s(\theta)).$$

Using the non-exclusive debt-limit first-order condition and the exclusive first-order condition on the separating region,

$$D(\theta) = \theta - \theta_L^{n*} + \frac{x_0^{n*}}{x_L^{n*}}\varepsilon(x_L^{n*}) + (1 - \delta^{s*})\frac{G(\theta)}{g(\theta)}.$$

The right-hand side is strictly increasing in θ because $G(\theta)/g(\theta)$ is increasing under Assumption 1.

The previous paragraph gives $D(\underline{\theta}) < 0$. If $\theta_L^{n*} \le \theta_H^{s*}$, then

$$D(\theta_L^{n*}) = \frac{x_0^{n*}}{x_L^{n*}}\varepsilon(x_L^{n*}) + (1-\delta^{s*})\frac{G(\theta_L^{n*})}{g(\theta_L^{n*})} > 0.$$

If instead $\theta_H^{s*} < \theta_L^{n*}$, then the order established above gives

$$x_L^{n*} \ge x_0^{n*} > x_0^{\min} \ge x_0^s = x_s(\theta_H^{s*}),$$

so $D(\theta_H^{s*}) > 0$. Thus D changes sign exactly once on $[\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\}]$. Since ε is strictly increasing, there is a unique

$$\theta^\dagger \in (\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\})$$

such that $x_n(\theta^\dagger) = x_s(\theta^\dagger)$. Moreover, $D(\theta) < 0$ before the crossing and $D(\theta) > 0$ after the crossing, so $x_n(\theta) < x_s(\theta)$ for $\theta < \theta^\dagger$ and $x_n(\theta) > x_s(\theta)$ while the non-exclusive schedule remains at the debt limit.

It remains to verify that the non-exclusive schedule stays above the exclusive schedule after the non-exclusive debt limit becomes slack. If $\theta_L^{n*} \le \theta < \theta_H^{s*}$, both schedules are on their separating branches and

$$\varepsilon(x_n(\theta)) - \varepsilon(x_s(\theta)) = \frac{x_0^{n*}}{x_n(\theta)}\varepsilon(x_n(\theta)) + (1-\delta^{s*})\frac{G(\theta)}{g(\theta)} > 0.$$

Since ε is strictly increasing, this implies $x_n(\theta) > x_s(\theta)$ on this region. If $\theta \ge \theta_H^{s*}$, then $x_s(\theta) = x_0^s$ while $x_n(\theta) \ge x_0^{n*} > x_0^s$. These cases imply

$$x_n(\theta) > x_s(\theta) \quad \text{for every } \theta > \theta^\dagger.$$

□

E Richer Contracting Mechanisms

E.1 Performance-Sensitive Existing Debt

Proof of Proposition 10

Proof. Let $\{\tau_0^{s*}, \tau_1^{s*}(\theta), x_1^{s*}(\theta)\}$ denote the exclusive-lending allocation from Section 4, and set $x_0^s = x_1^{s*}(\bar{\theta})$. The proof of Proposition 8 gives the stronger bound

$$0 \le \tau_1^{s*}(\theta) \le (x_1^{s*}(\theta) - x_0^s)\, q(x_1^{s*}(\theta)) \le x_1^{s*}(\theta) q(x_1^{s*}(\theta)) \quad \forall \theta \in \Theta.$$

Now define the report-contingent existing debt claim by

$$x_0^{\mathrm{ps}}(\theta) = x_1^{s*}(\theta) - \frac{\tau_1^{s*}(\theta)}{q(x_1^{s*}(\theta))}.$$

By the bounds above, $x_0^s \le x_0^{\mathrm{ps}}(\theta) \le x_1^{s*}(\theta)$ for every θ. Thus the performance-sensitive adjustment $x_0^{\mathrm{ps}}(\theta) - x_0^s$ is nonnegative and feasible. The face value sold to new lenders is

$$\Delta x_1^{\mathrm{ps}}(\theta) \coloneqq x_1^{s*}(\theta) - x_0^{\mathrm{ps}}(\theta) = \frac{\tau_1^{s*}(\theta)}{q(x_1^{s*}(\theta))}.$$

We next show that this new-issuance schedule is invertible on the separating region. Write $x(\theta) = x_1^{s*}(\theta)$ and $\tau(\theta) = \tau_1^{s*}(\theta)$. On the separating region, the envelope condition gives

$$\tau'(\theta) = \theta q(x(\theta))x'(\theta).$$

Therefore

$$\frac{d}{d\theta}\Delta x_1^{\mathrm{ps}}(\theta) = \frac{\tau'(\theta)q(x(\theta)) - \tau(\theta)q'(x(\theta))x'(\theta)}{q(x(\theta))^2} = \frac{x'(\theta)}{q(x(\theta))^2}\left[\theta q(x(\theta))^2 - \tau(\theta)q'(x(\theta))\right].$$

Since $x'(\theta) < 0$ on the separating region, $q'(x) < 0$, $\theta > 0$, and $\tau(\theta) \ge 0$, the derivative is strictly negative. Hence $\Delta x_1^{\mathrm{ps}}(\theta)$ is invertible on the separating region. On the pooled region, $\tau_1^{s*}(\theta) = 0$, so $\Delta x_1^{\mathrm{ps}}(\theta) = 0$ and there is no active issuance margin to invert.

Let $\theta^{\mathrm{ps}}(\Delta x_1) \coloneqq (\Delta x_1^{\mathrm{ps}})^{-1}(\Delta x_1)$ and define

$$x_0^{\mathrm{ps}}(\Delta x_1) \coloneqq x_1^{s*}(\theta^{\mathrm{ps}}(\Delta x_1)) - \Delta x_1.$$

After an issue of face value Δx_1, set the existing debt claim equal to $x_0^{\mathrm{ps}}(\Delta x_1)$. Consider the resulting performance-sensitive-debt implementation with baseline existing debt x_0^s, the issuance-indexed existing debt claim above, and initial transfer $\tau_0 = \tau_0^{s*}$. If type θ issues $\Delta x_1^{\mathrm{ps}}(\theta)$ to new lenders, the existing debt claim becomes $x_0^{\mathrm{ps}}(\theta)$ and total debt is $x_1^{s*}(\theta)$. New lenders buy the residual claim $\Delta x_1^{\mathrm{ps}}(\theta)$, so the borrower receives

$$\Delta x_1^{\mathrm{ps}}(\theta)q(x_1^{s*}(\theta)) = \tau_1^{s*}(\theta).$$

For any issuance level in the image of Δx_1^{ps} and its associated report $\hat{\theta}$, the same calculation gives the borrower continuation payoff

$$\tau_1^{s*}(\hat{\theta}) + \theta V_2(x_1^{s*}(\hat{\theta})),$$

which is exactly the single-lender reporting payoff. Among issuance levels in the constructed image, incentive compatibility therefore follows from the exclusive-lending mechanism.

Finally, the repayment promised to existing lenders is

$$\mathbb{E}\left[x_0^{\mathrm{ps}}(\theta)q(x_1^{s*}(\theta))\right] = \mathbb{E}\left[x_1^{s*}(\theta)q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta)\right] = \tau_0^{s*},$$

where the last equality uses the binding participation constraint in the exclusive-lending benchmark. Since $\tau_0^{s*} \geq I$, the financing requirement is satisfied. Hence, performance-sensitive debt replicates the exclusive-lending allocation along the prescribed issuance path. □

E.2 Contingent Prepayment

Proof of Proposition 11

Proof. Write

$$x(\theta) := x_1^{s*}(\theta), \qquad \tau(\theta) := \tau_1^{s*}(\theta), \qquad x_0 := x_0^s.$$

For an issuance level in the image of Δx_1^{ds} and its associated report $\hat{\theta}$, the sweep requires prepayment $b(\hat{\theta})$ and leaves post-prepayment debt $x(\hat{\theta})$. Gross new issuance is therefore $x(\hat{\theta}) - x_0 + b(\hat{\theta})$, and the borrower's net proceeds are

$$\left(x(\hat{\theta}) - x_0 + b(\hat{\theta})\right) q(x(\hat{\theta})) - b(\hat{\theta}).$$

By the definition of b in (29), these net proceeds equal $\tau(\hat{\theta})$. The reporting payoff of type θ is therefore

$$\tau(\hat{\theta}) + \theta V_2(x(\hat{\theta})),$$

which is exactly the reporting payoff in the exclusive-lending mechanism. Among issuance levels in the constructed image, incentive compatibility follows from the exclusive-lending benchmark.

We next verify feasibility. The denominator in (29) is positive because $q(x) < 1$. The proof of Proposition 8 gives

$$0 \leq \tau(\theta) \leq (x(\theta) - x_0)\, q(x(\theta)),$$

so $b(\theta) \geq 0$. The upper bound $b(\theta) \leq x_0$ is equivalent to

$$H(\theta) := x(\theta)q(x(\theta)) - \tau(\theta) \leq x_0.$$

On the separating region, the envelope condition for the exclusive-lending mechanism gives

$$\tau(\theta) = \theta_H V_2(x_0) - \theta V_2(x(\theta)) - \int_{\theta}^{\theta_H} V_2(x(\tilde{\theta}))d\tilde{\theta}.$$

Thus

$$H(\theta) = W_1(x(\theta), \theta) - \theta_H V_2(x_0) + \int_{\theta}^{\theta_H} V_2(x(\tilde{\theta}))d\tilde{\theta}.$$

Differentiating this whole object gives the envelope cancellation

$$H'(\theta) = \frac{\partial W_1}{\partial x_1}(x(\theta), \theta)x'(\theta) + V_2(x(\theta)) - V_2(x(\theta)).$$

Hence

$$H'(\theta) = \frac{\partial W_1}{\partial x_1}(x(\theta), \theta)x'(\theta).$$

On the separating region,

$$\frac{\partial W_1}{\partial x_1}(x(\theta), \theta) = (1 - \delta^{s*})\frac{G(\theta)}{g(\theta)}q(x(\theta)) \geq 0,$$

and $x'(\theta) < 0$. Therefore $H'(\theta) \leq 0$, so the upper-prepayment constraint is tightest at $\underline{\theta}$. Condition (30) gives $H(\underline{\theta}) \leq x_0$ and therefore $b(\theta) \leq x_0$ for all θ.

It remains to verify that gross issuance is invertible on the separating region. Since $\Delta x_1^{\text{ds}}(\theta) = x(\theta) + b(\theta) - x_0$, differentiating (29) gives

$$\frac{d}{d\theta}\Delta x_1^{\text{ds}}(\theta) = \frac{x'(\theta)}{1 - q(x(\theta))}\left[1 - \theta q(x(\theta)) + q'(x(\theta))\Delta x_1^{\text{ds}}(\theta)\right].$$

Because $b(\theta) \leq x_0$ and $q'(x) < 0$, the bracketed term is bounded below by

$$1 - \theta q(x(\theta)) + q'(x(\theta))x(\theta) \geq (1 - \theta)q(x(\theta)) + q'(x(\theta))x(\theta) = \frac{\partial W_1}{\partial x_1}(x(\theta), \theta) \geq 0.$$

The inequality is strict because $q(x) < 1$. Since $x'(\theta) < 0$ on the separating region, $d\Delta x_1^{\text{ds}}(\theta)/d\theta < 0$ there. Hence, define

$$\theta^{\text{ds}}(\Delta x_1) := (\Delta x_1^{\text{ds}})^{-1}(\Delta x_1).$$

The required prepayment after face-value issue Δx_1 is

$$b^{\text{ds}}(\Delta x_1) := b(\theta^{\text{ds}}(\Delta x_1)).$$

The rest of the implementation specifies this prepayment as a share of the cash proceeds from new issuance. In particular, defining $\alpha^{\mathrm{ds}}(\Delta x_1)$ as the solution to

$$b^{\mathrm{ds}}(\Delta x_1) = \alpha^{\mathrm{ds}}(\Delta x_1) q \left(x_0^s + \Delta x_1 - b^{\mathrm{ds}}(\Delta x_1) \right) \Delta x_1$$

gives the nonlinear debt sweep

$$\alpha^{\mathrm{ds}}(\Delta x_1) = \frac{b^{\mathrm{ds}}(\Delta x_1)}{q\left(x_0^s + \Delta x_1 - b^{\mathrm{ds}}(\Delta x_1)\right)\Delta x_1}.$$

Finally, existing lenders receive the cash prepayment plus the value of the remaining existing claim. Their expected payoff is

$$\mathbb{E}\left[(x_0 - b(\theta))q(x(\theta)) + b(\theta)\right] = \mathbb{E}\left[x(\theta)q(x(\theta)) - \tau(\theta)\right] = \tau_0^{s*},$$

where the last equality is the binding participation constraint in the exclusive-lending benchmark. Since $\tau_0^{s*} \geq I$, the financing requirement is satisfied. Hence, the debt sweep replicates the exclusive-lending allocation along the prescribed issuance path. □

E.3 Short-Term Debt

Motivated by the solution in Section 3, we consider contracts (b, x_0, x_1^L), where b is the amount of short-term debt that matures at $t = 1$, x_0 is the initial amount of long-term debt that matures at $t = 2$, and x_1^L is the limit in total debt due at $t = 2$. We first consider contracts for which the debt limit lies in the range of the flexible-debt schedule, $x_1^L \in [x_0, x_1^f(\underline{\theta}, x_0)]$. We can then represent the limit by a marginal type θ_L such that $x_1^f(\theta_L) = x_1^L$. Because the borrower is financially constrained, the following roll-over constraint must be satisfied at $t = 1$:

$$q(x_1(\theta))(x_1(\theta) - x_0) \geq b.$$

This constraint specifies that the amount raised at $t = 1$ must be sufficient to pay back the maturing short-term debt. The cap can support rollover only if $q(x_1^L)(x_1^L - x_0) \geq b$. Moreover, if $\bar{\theta} = 1$ (Assumption 4), Lemma 2 establishes that $x_1^f(\bar{\theta}) - x_0 = 0$. Feasibility and continuity therefore imply that there is a marginal type $\theta_H \in [\theta_L, \bar{\theta}]$ such that

$$q(x_1^f(\theta_H))(x_1^f(\theta_H) - x_0) = b.$$

For this region, we can consider the following generalization of the problem in (16). If the rollover floor lies above $x_1^f(\underline{\theta}, x_0)$, rollover binds for every type, and the cutoff representation does not apply. We treat this fully pooled case directly below.

$$\begin{aligned}
&\max_{\substack{\theta_L,\theta_H\in\Theta\\ \theta_L\leq\theta_H}} \int_{\underline{\theta}}^{\theta_L} W_1(x_1^f(\theta_L),\theta)dG(\theta) + \int_{\theta_L}^{\theta_H} W_1(x_1^f(\theta),\theta)dG(\theta) + \int_{\theta_H}^{\bar{\theta}} W_1(x_1^f(\theta_H),\theta)dG(\theta)\\
&\text{subject to}\\
&x_0\left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H))\right)G(\theta_L) + \int_{\theta_L}^{\theta_H}\left(q(x_1^f(\theta)) - q(x_1^f(\theta_H))\right)dG(\theta)\right]\\
&+ x_1^f(\theta_H)q(x_1^f(\theta_H)) \geq I.
\end{aligned} \tag{50}$$

Let λ be the multiplier of the investment constraint. We can write

$$\begin{aligned}
\max_{\substack{\theta_L,\theta_H\in\Theta\\ \theta_L\leq\theta_H}} &\int_{\underline{\theta}}^{\theta_L}\left(W_1(x_1^f(\theta_L),\theta) + \lambda x_0 q(x_1^f(\theta_L))\right)dG(\theta) + \int_{\theta_L}^{\theta_H}\left(W_1(x_1^f(\theta),\theta) + \lambda x_0 q(x_1^f(\theta))\right)dG(\theta)\\
&+\int_{\theta_H}^{\bar{\theta}}\left(W_1(x_1^f(\theta_H),\theta) + \lambda x_0 q(x_1^f(\theta_H))\right)dG(\theta) + \lambda q(x_1^f(\theta_H))(x_1^f(\theta_H) - x_0).
\end{aligned}$$

Dividing by $1+\lambda$ and letting $\delta = 1/(1+\lambda)$, we get that the previous problem is equivalent to

$$\begin{aligned}
\max_{\substack{\theta_L,\theta_H\in\Theta\\ \theta_L\leq\theta_H}} &\int_{\underline{\theta}}^{\theta_L}\left((1-\delta)x_0 q(x_1^f(\theta_L)) + \delta W_1(x_1^f(\theta_L),\theta)\right)dG(\theta) + \int_{\theta_L}^{\theta_H}\left((1-\delta)x_0 q(x_1^f(\theta)) + \delta W_1(x_1^f(\theta),\theta)\right)dG(\theta)\\
&+\int_{\theta_H}^{\bar{\theta}}\left((1-\delta)x_0 q(x_1^f(\theta_H)) + \delta W_1(x_1^f(\theta_H),\theta)\right)dG(\theta) + (1-\delta)q(x_1^f(\theta_H))(x_1^f(\theta_H) - x_0).
\end{aligned}$$

We first establish how the value of the contract varies with long-term debt. Let $W_0(x_0)$ be the value after optimizing over (θ_L, θ_H), and write

$$D(\theta) \coloneqq \frac{\partial x_1^f(\theta, x_0)}{\partial x_0} > 0.$$

For later use, define

$$A_L \coloneqq \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L))\right] dG(\theta),$$
$$A_H \coloneqq \int_{\theta_H}^{\bar{\theta}} \left[x_0 q'(x_1^f(\theta_H)) + \delta(\theta_H - \theta) q(x_1^f(\theta_H))\right] dG(\theta) + (1-\delta)\theta_H q(x_1^f(\theta_H)).$$

For $\delta > 0$, the envelope derivative of the normalized Lagrangian has the same sign as $W_0'(x_0)$ and is

$$\begin{aligned}\mathcal{D}(x_0) = x_0 \int_{\theta_L}^{\theta_H} q'(x_1^f(\theta)) D(\theta) dG(\theta) + D(\theta_L) A_L + D(\theta_H) A_H \\ + (1-\delta)\left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H))\right) G(\theta_L) \right. \\ \left. + \int_{\theta_L}^{\theta_H} \left(q(x_1^f(\theta)) - q(x_1^f(\theta_H))\right) dG(\theta)\right]. \end{aligned} \tag{51}$$

Suppose first that $\theta_L < \theta_H$. The first-order conditions for the two thresholds give $A_L = A_H = 0$. The same conclusion applies at $\theta_L = \underline{\theta}$, because $G(\underline{\theta}) = 0$. An optimum with $\theta_H = \bar{\theta}$ must have $\delta = 1$, in which case $A_H = 0$ as well; otherwise, a small decrease in θ_H raises the normalized objective. Hence,

$$\begin{aligned}\mathcal{D}(x_0) = x_0 \int_{\theta_L}^{\theta_H} q'(x_1^f(\theta)) D(\theta) dG(\theta) \\ + (1-\delta)\left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H))\right) G(\theta_L) \right. \\ \left. + \int_{\theta_L}^{\theta_H} \left(q(x_1^f(\theta)) - q(x_1^f(\theta_H))\right) dG(\theta)\right]. \end{aligned} \tag{52}$$

Both terms are weakly negative. If $x_0 > 0$, the first term is strictly negative because $q' < 0$, $D > 0$, and $g > 0$.

Now suppose that $\theta_L = \theta_H$. The separate first-order conditions cannot be imposed. Let $\mu \geq 0$ be the multiplier on $\theta_H - \theta_L \geq 0$. At an interior pooled threshold, joint KKT stationarity implies

$$\frac{\partial \mathcal{L}}{\partial \theta_L} + \frac{\partial \mathcal{L}}{\partial \theta_H} = 0.$$

Since the two thresholds coincide, this condition implies $A_L + A_H = 0$. Their derivatives of flexible debt with respect to x_0 are also equal, so the two threshold terms in (51) cancel. The integral over

$[\theta_L, \theta_H]$ and the price-difference terms vanish. Therefore,

$$\mathcal{D}(x_0) = 0. \tag{53}$$

At a support endpoint, the same conclusion follows directly from the pooled contracts characterized below. At a kink between separating and pooled solutions, the one-sided envelope derivatives are the corresponding limits of (52) and (53), and are nonpositive. Thus W_0 is weakly decreasing in x_0, strictly decreasing along any segment with $x_0 > 0$ and $\theta_L < \theta_H$, and flat only when total debt is fully pooled.

We now characterize the allocation at $x_0 = 0$. In this case, $x_1^f(\theta, 0) = x_1^*(\theta)$. Since short-term debt is the only source of initial funds, the borrower sets $b = I$. Let x_1^b be the smaller solution to

$$x_1^b q(x_1^b) = I.$$

The rollover constraint then gives

$$x_1(\theta) = \max\{x_1^*(\theta), x_1^b\}.$$

The solution depends on whether x_1^b lies below $x_1^*(\underline{\theta})$.

Consider first $I < I^p$. Since $xq(x)$ is increasing below its maximizer,

$$I < I^p = x_1^*(\underline{\theta})q(x_1^*(\underline{\theta})) \quad \Longleftrightarrow \quad x_1^b < x_1^*(\underline{\theta}).$$

There is therefore a unique $\theta_H \in (\underline{\theta}, \bar{\theta})$ such that $x_1^*(\theta_H) = x_1^b$. The allocation has a separating region of positive measure. Moreover, the multiplier on the financing constraint is positive: relaxing the rollover requirement lowers debt for a positive measure of types whose debt is fixed at x_1^b and strictly raises surplus. Thus $\delta < 1$, and the second term in (52) makes the right derivative at $x_0 = 0$ strictly negative. Since W_0 cannot subsequently increase, no $x_0 > 0$ can attain the same value. It follows that $x_0 = 0$ and $b = I$ are uniquely optimal.

Finally, suppose that $I^p \leq I < I^{\max}$. Then

$$x_1^b \geq x_1^*(\underline{\theta}) \geq x_1^*(\theta)$$

for every $\theta \in \Theta$, so rollover binds for every type and total debt is fully pooled at x_1^b. We treat this case directly rather than assigning a cutoff outside the type support. For any $x_0 \in [0, x_1^b]$, set

$$b = (x_1^b - x_0)q(x_1^b), \qquad x_1^L = x_1^b.$$

Initial proceeds are

$$x_0 q(x_1^b) + b = x_1^b q(x_1^b) = I.$$

Moreover, for $z \in [x_0, x_1^b)$,

$$\frac{d}{dz}\left[(z - x_0)q(z)\right] = q(z) + zq'(z) - x_0 q'(z) > 0,$$

because repayment is increasing for $z < x_1^b$, while $-x_0 q'(z) \geq 0$. Thus no $z < x_1^b$ raises enough to repay b, while the debt limit rules out $z > x_1^b$. Every maturity mix above therefore implements the same pooled allocation.

To compare these contracts with separating contracts in the cutoff problem, define

$$x_0^p := x_1^b \left[1 - \frac{1 - \underline{\theta}}{\varepsilon(x_1^b)}\right] \geq 0.$$

The flexible-debt first-order condition implies that $x_1^f(\underline{\theta}, x_0^p) = x_1^b$. Thus $(\theta_L, \theta_H) = (\underline{\theta}, \underline{\theta})$ and $x_0 = x_0^p$ provide an in-domain representation of the same pooled allocation. If $x_0 < x_0^p$, then for every $\theta_H \in \Theta$,

$$x_1^f(\theta_H, x_0) \leq x_1^f(\underline{\theta}, x_0) < x_1^b.$$

Moreover, the bracketed term in the financing constraint (50) is nonpositive because $\theta_L \leq \theta_H$, x_1^f is decreasing in type, and q is decreasing. Financing capacity is therefore bounded above by

$$x_1^f(\theta_H, x_0) q(x_1^f(\theta_H, x_0)) < x_1^b q(x_1^b) = I,$$

where the strict inequality uses the fact that repayment is increasing up to the smaller root x_1^b. Hence no contract in the cutoff problem is feasible for $x_0 < x_0^p$. Since W_0 is weakly decreasing on its feasible domain, no separating contract with $x_0 \geq x_0^p$ improves on the pooled contract at x_0^p. The maturity mixes stated in Proposition 12 implement that same allocation and are therefore all optimal. At $I = I^p$, $x_0^p = 0$.

It remains to consider $I = I^{\max}$. For every feasible contract, the rollover constraint implies, state by state,

$$x_0 q(x_1(\theta)) + b \leq x_1(\theta) q(x_1(\theta)) \leq I^{\max}.$$

Initial proceeds equal $\mathbb{E}[x_0 q(x_1(\theta)) + b]$. Raising $I^{\max}$ therefore requires both inequalities to bind almost surely. Since $xq(x)$ has the unique maximizer x^m, this requires $x_1(\theta) = x^m$ for every type and binding rollover in every state. Thus total debt is fully pooled at $x_1^b = x^m$. For every

$x_0 \in [0, x^m]$, the contract

$$b = (x^m - x_0)q(x^m), \qquad x_1^L = x^m$$

implements this allocation, which gives the continuum stated in Proposition 12.