Leverage Dynamics and Liquidity Management without Commitment

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Abstract

We study a continuous-time model of joint leverage and liquidity management without commitment. Our model highlights how cash reserves shape capital structure policy. Although shareholders typically resist leverage reductions in canonical models (i.e., the "leverage ratchet effect"), our model predicts leverage adjustments in both directions: (1) cash-rich firms with little debt tend to deleverage while accumulating cash, thereby reducing expected future debt obligations at low opportunity cost; (2) cash-strapped firms issue debt to replenish liquidity, even without corporate tax shields. Our equilibrium characterization yields financial policies, asset prices, return volatilities and credit spreads in closed form.

Keywords: Leverage Ratchet Effect, Lack of Commitment, Corporate Liquidity Management, Debt Repurchases, Joint Leverage and Liquidity Dynamics

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1 Introduction

Liquidity management is closely intertwined with leverage policy. In the survey conducted by Graham (2022), approximately 60% of small-firm CFOs and 37% of large-firm CFOs identified insufficient internal funds as an important or very important factor in their debt decisions. Consistent with the survey evidence, empirical studies document that firms often issue debt to meet near-term liquidity needs and use subsequent financial surpluses to repay debt (Denis and McKeon, 2012; Huang and Ritter, 2021). Consequently, understanding how leverage and liquidity evolve together over time is a central question in corporate finance theory.

This paper addresses this question in a continuous-time model in which shareholders cannot commit to future financing decisions. In line with DeMarzo and He (2021), we consider a risk-neutral economy in which a firm may adjust its long-term debt level through issuances or repurchases at competitive market prices. The key feature of our model is that the firm may use internal cash reserves as a source of financing for debt reduction and may issue new long-term debt to replenish cash reserves. Our goal is to examine the interdependent dynamics of leverage and liquidity within this framework.

Our model highlights the role of cash reserves in capital structure policy and generates leverage adjustments in both directions. Accordingly, our results help address an important concern raised by DeAngelo (2022) about capital structure models without commitment.[1] In canonical models by Admati et al. (2018) and DeMarzo and He (2021), shareholders resist debt reductions financed through recapitalization (i.e., the "leverage ratchet effect"), and cash reserves have little influence on capital structure decisions. Yet empirical evidence shows that many firms simultaneously delever and accumulate cash reserves (Dasgupta et al., 2011; DeAngelo et al., 2018), whereas cash-strapped ones issue debt to meet short-term liquidity needs (Denis and McKeon, 2012; Huang and Ritter, 2021). By jointly determining leverage and liquidity policies, our model can reconcile these empirical patterns even when shareholders cannot commit to future financing decisions.

The key distinction between these seminal works and ours lies in the source of funding for debt reduction. When shareholders draw down internal cash reserves to reduce debt, they give up cash that might otherwise never be paid out to them if the firm subsequently defaults. Therefore,

[1] Specifically, DeAngelo (2022) writes: "*real-world deleveraging behavior and cash-management activity are at strong odds with what the ratchet model predicts*" (p. 440).

shareholders incur a lower opportunity cost of debt reduction than when they provide additional funds through recapitalization. In turn, by lessening its future debt burdens, the firm may mitigate the risk of bankruptcy and increase shareholders' payoffs when the firm remains solvent. As a result, shareholders can find it optimal to reduce debt with cash.

We begin with a static analogue of our main model in which the firm can reduce its outstanding debt only once. To isolate the mechanism, we abstract from all frictions other than bankruptcy costs, which may reduce debt recovery. We compare equity value under three financial policies: (i) a "no-trade" benchmark without leverage adjustment; (ii) debt reduction through an external recapitalization, in which shareholders contribute new funds to repurchase debt; and (iii) debt reduction with cash, in which the firm uses its internal cash reserves to repurchase debt. The comparison illustrates how the source of funding determines whether a debt reduction destroys or creates shareholder value. We show that shareholders benefit from debt repurchases financed by internal cash, whereas they resist debt repurchases financed by external recapitalization—the "leverage ratchet effect" studied by Admati et al. (2018) and DeMarzo and He (2021).

We extend the analysis to our continuous-time model, in which cumulative operating cash flows follow an arithmetic Brownian motion. Shareholders can continuously issue new consol bonds (i.e., perpetuities), and repurchase outstanding ones in a competitive financial market. We incorporate two standard frictions from liquidity-management models (Décamps et al., 2011; Bolton et al., 2011): (i) agency costs of cash reserves and (ii) external financing costs. Specifically, dividend income is subject to personal taxation, thereby creating a wedge between the shareholders' benefit from dividend payouts and their cost of equity injections.[2] Together with bankruptcy costs, this wedge induces a precautionary demand for cash reserves inside the firm to buffer against operating losses, whereas agency costs render cash retention costly. Therefore, our analysis involves keeping track of both cash and debt levels as state variables.

We decompose the equilibrium leverage adjustment policy into three components: (i) debt tax shields; (ii) the required dollar return on debt per unit of face value; and (iii) the covariance between changes in debt prices and percentage changes in shareholders' marginal value of liquidity. The

[2]Importantly, we use personal taxation as a tractable way to capture external financing costs, not because personal taxation itself is essential to our mechanism. In Subsection 6, we show that the model remains tractable when personal taxation also applies to interest income. Therefore, our formulation can accommodate financing frictions in both debt and equity markets, including the case in which the same personal tax rate applies to both dividend and interest income.

first component captures the standard tax benefit of debt issuance in the trade-off theory. The second component is closely linked to the firm's future debt obligations because the debt price is the expected present value of future interest payments on one unit of face value until the firm goes bankrupt or the bond is repurchased, together with the price received upon repurchase. Hence, the second component constitutes both a benefit of debt repurchases and an additional cost of debt issuance. The third component can be interpreted as capturing shareholders' motive to replenish cash reserves through debt issuance.

Our decomposition of the equilibrium leverage policy produces two results. First, for a sufficiently low corporate tax rate, if the firm has little debt outstanding and sufficiently high cash reserves, it repurchases debt and accumulates cash reserves simultaneously. Second, the firm issues debt when its cash reserves are sufficiently low, even in the absence of corporate taxation.[3] These findings illustrate the role of the last two economic forces identified by our decomposition.

Our first finding is primarily driven by shareholders' incentive to reduce future debt obligations. Intuitively, when the corporate tax rate is low, the tax benefit of additional debt is limited. Moreover, when the firm also has ample cash reserves and little outstanding debt, bankruptcy risk is low. Consequently, the marginal benefit of issuing debt to accumulate additional cash is likewise small. Meanwhile, the low likelihood of bankruptcy implies that each unit of outstanding debt entails a relatively high expected present value of future debt obligations, and the opportunity cost of cash is low. Therefore, shareholders may find it optimal to draw down cash reserves to repurchase debt and thereby lower their future interest payments. Still, operating cash flows are expected to more than cover the cash spent on repurchases as well as interest payments, so the amount of outstanding debt falls while cash reserves are expected to rise.

Our second finding reflects shareholders' motive to strengthen their liquidity position through debt issuance, which is absent in existing leverage-ratchet models. When cash reserves are low, the firm faces a high risk of bankruptcy. Therefore, the shareholders' marginal value of additional cash is high, while the expected present value of future interest payments is relatively small. Thus, the firm tends to issue additional debt when its cash reserves become sufficiently low, even at the

[3] More precisely, both results only apply to circumstances with a positive amount of debt outstanding and positive cash reserves. Under our maintained parameter restrictions, the firm is immediately liquidated once it exhausts its cash reserves. When the firm is unlevered, equilibrium multiplicity arises. See Subsection 4.3.2 for a detailed discussion.

expense of additional future debt burdens.

From a technical perspective, equilibrium construction of the continuous-time model poses analytical challenges because it requires solving a system of partial differential equations (PDEs). Moreover, the equilibrium value and pricing functions are not homogeneous in the state variables, unlike the corresponding functions in many models in the continuous-time finance literature.[4] Therefore, we cannot invoke the standard homogeneity argument to reduce the equilibrium problem to ordinary differential equations (ODEs) involving a ratio of the two state variables.

To address this challenge, we develop an alternative approach based on the "no-trade equity valuation" method à la DeMarzo and He (2021).[5] Specifically, we formulate an auxiliary equity valuation problem in which the firm is prohibited from participating in the debt market but can still manage its cash reserves. Since debt never matures in our model, its aggregate face value is fixed in the auxiliary problem. Accordingly, cash is the only evolving state variable in the auxiliary problem, so it effectively reduces to a one-dimensional liquidity management problem. This reduction in dimensionality allows us to derive the equilibrium financial policy and asset prices in closed form. Hence, our innovation lies in combining the "no-trade equity valuation" method with the assumption of perpetual debt.

Beyond being of independent theoretical interest, our continuous-time model yields several empirical implications. First, the model's deleveraging prediction is in line with prior evidence that many firms reduce leverage while accumulating substantial cash reserves (Dasgupta et al., 2011; DeAngelo et al., 2018). Second, by showing that firms can issue debt to obtain or preserve internal liquidity when it is especially valuable, our model resonates with the "dash for cash" during financial crises (Campello et al., 2010; Ivashina and Scharfstein, 2010; Acharya and Steffen, 2020).[6] Third, more generally, the model predicts that liquidity needs are an important determinant of debt

[4]For instance, both DeMarzo and He (2021) and Bolton et al. (2011) invoke a homogeneity argument to reduce the dimensionality of their equilibrium problems.

[5]The equivalence between no-trade equity value and equilibrium equity value is analogous to the Coase conjecture (Coase, 1972). Namely, without commitment, a durable-goods monopolist cannot capture the gains from intertemporal price discrimination. See, for instance, Fuchs and Skrzypacz (2010) and Daley and Green (2020) for dynamic bargaining models that illustrate this point. See also DeMarzo and Urošević (2006) for an early work that features a related equivalence result.

[6]These empirical works provide suggestive evidence but are not exact institutional counterparts to our model. Specifically, our model speaks directly to issuance of publicly traded long-term corporate bonds, whereas the evidence reported in Campello et al. (2010) and Ivashina and Scharfstein (2010) concerns precautionary drawdowns of precommitted credit lines. Acharya and Steffen (2020) document public bond issuance associated with cash accumulation among highly rated firms but also show that firms across all rating classes drew down credit lines.

issuance, in line with prior empirical evidence (Denis and McKeon, 2012; Huang and Ritter, 2021). Finally, the model's asset-pricing implications resonate with evidence linking equity volatility to corporate bond yields and corporate bond returns to changes in equity value (Campbell and Taksler, 2003; Schaefer and Strebulaev, 2008).

1.1 Theoretical Literature Review

Our paper contributes to several strands of the theoretical literature. First, we contribute to the literature on dynamic capital structure without commitment. Seminal papers by Admati et al. (2018) and DeMarzo and He (2021) document the "leverage ratchet effect." Namely, when shareholders can adjust the debt level over time, they typically resist leverage reductions regardless of how much such reductions may increase firm value. Subsequent papers either identify mechanisms that help alleviate this commitment problem, or generate voluntary debt reductions even when shareholders cannot commit to future financial policies (Benzoni et al., 2022; Malenko and Tsoy, 2025; Hu et al., 2025; Hu, 2026).

We depart from this literature by introducing a tractable form of external financing costs into DeMarzo and He (2021)'s workhorse model. These external financing costs make internal cash reserves valuable to shareholders and generate a novel mechanism for debt repurchases financed with cash reserves. In so doing, our paper provides a potential explanation for the joint pattern of deleveraging and cash accumulation documented by DeAngelo et al. (2018) and helps address the concern raised by DeAngelo (2022).

Second, we contribute to the literature on dynamic liquidity management, including Bolton et al. (2011); Décamps et al. (2011). Our model is most closely related to Gryglewicz (2011) and Anderson and Carverhill (2012) in this literature. Like ours, both papers analyze capital structure and liquidity management within a unified framework. However, these papers hold the face value of long-term debt fixed after the initial financing decision. We extend these works and allow the firm to continually issue and repurchase long-term debt at competitive prices. As a result, we obtain an additional implication for the leverage ratchet effect. Specifically, we show that the firm can voluntarily draw down its cash reserves to reduce the face value of its outstanding long-term debt.

Relatedly, our paper contributes to the theoretical literature on debt repurchases (Gertner and Scharfstein, 1991; Julio, 2013; Crouzet, 2017). Our motivating example is most closely related to

Mao and Tserlukevich (2015), who interpret shareholders' gains from debt repurchases with cash reserves as wealth extraction from debtholders.[7] Our static model clarifies when this interpretation applies. When the recovery rate on debt is sufficiently high, shareholders tend to benefit at the expense of debtholders, consistent with shareholders' wealth extraction. However, for a sufficiently low recovery rate, debt repurchases can reduce expected bankruptcy costs and thus benefit all investors. Likewise, in our continuous-time model, debt repurchases tend to raise the price of the remaining debt, as debtholders recover nothing upon default. Finally, unlike Mao and Tserlukevich (2015), our model also allows debt issuance, dividends, and equity issuance.

The remainder of the paper is organized as follows. Section 2 uses a motivating example to illustrate how the source of funding affects shareholders' incentives to repurchase debt. Section 3 develops the continuous-time model. Sections 4 and 5 construct smooth equilibria and derive the model's implications for corporate financial policies and asset prices. Section 6 extends the analysis to incorporate personal taxation of interest income. Section 7 concludes.

2 Motivating Example

We present a numerical example to illustrate the following two points: 1) shareholders resist debt repurchases financed by shareholder recapitalization (i.e., the "leverage ratchet effect" as in Admati et al. (2018)), whereas 2) they may gain from debt repurchases financed by internal cash holdings. To elucidate the mechanism as transparently as possible, we consider a single class of pari passu debt and assume away all frictions except bankruptcy costs. In particular, debtholders may recover a fraction $\alpha \in [0, 1]$ of the terminal value available in default, with the remaining fraction destroyed. Henceforth, α is referred to as the *recovery rate on debt*. The full-recovery case $\alpha = 1$ corresponds to the Modigliani–Miller (MM) benchmark, whereas the case with $\alpha = 0$ parallels the assumption that debtholders recover nothing in our continuous-time model.[8]

For simplicity, the current section imposes the assumption that debtholders recover nothing upon bankruptcy, $\alpha = 0$. In the Internet Appendix, we show that both results in this section

[7] Similarly, Admati et al. (2018, Subsection IV.C) show that shareholders can extract wealth from debtholders by selling safe assets to reduce debt and interpret this gain as asset substitution. In particular, their corollary in Subsection IV.C establishes that this transaction is payoff-equivalent to recapitalizing the firm and substituting risky assets for safe assets.

[8] This assumption is also in line with the seminal work by DeMarzo and He (2021), as well as the dynamic model considered in Admati et al. (2018, Section III).

extend to a setting with a richer specification of cash flows, any feasible debt-repurchase amount, and an arbitrary recovery rate $\alpha \in [0, 1]$.

2.1 Setup

There are three dates $t \in \{0, 1, 2\}$. All agents are risk-neutral and the interest rate is normalized to zero. At the initial date $t = 0$, a firm is endowed with two assets: 1) cash reserves of \$100; 2) an intangible asset that yields an operating cash flow $\widetilde{X}$ at the terminal date $t = 2$, where

$$\widetilde{X} = \begin{cases} 100, & \text{with probability } \frac{1}{2}, \\ 0, & \text{with probability } \frac{1}{2}. \end{cases}$$

Moreover, the firm initially has 250 bonds outstanding, each with face value \$1 and zero coupons.

At the interim date $t = 1$, the firm can repurchase 150 bonds in a competitive financial market. Crucially, investors price the remaining debt based on the balance sheet after the repurchase. At the terminal date $t = 2$, the operating cash flow $\widetilde{X}$ is realized. If the firm is solvent, the terminal firm value consists of both cash holdings and operating cash flow (e.g., $\widetilde{X} + 100$ absent leverage adjustment). However, if the firm defaults, debtholders recover nothing and the remaining terminal value is destroyed.

To highlight the mechanism, we consider two distinct debt-reduction strategies against a "no-trade" benchmark without adjustment to the initial debt level. Specifically, we analyze 1) debt repurchases through recapitalization, where shareholders inject new funds to buy back debt; 2) debt repurchases with cash, where shareholders use the firm's existing cash reserves to buy back debt.

"No-Trade" Benchmark

Absent leverage adjustment, the terminal firm value before bankruptcy costs equals \$200 with probability $\frac{1}{2}$ and \$100 with probability $\frac{1}{2}$. Since both values are below the total face value of debt, the firm always defaults. Under zero recovery, the market values of debt and equity at the interim date $t = 1$ are both zero.

Case 1) Debt Repurchases Through Recapitalization

Suppose that the firm keeps the asset side of the balance sheet fixed and uses additional cash raised from its existing shareholders to repurchase 150 bonds. After the repurchase, 100 bonds remain outstanding. In the spirit of backward induction, let us first compute the market values of debt and equity immediately after the bond repurchase:

- Total market value of debt: $\frac{1}{2} \cdot 100 + \frac{1}{2} \cdot 100 = 100$
- Bond price: $\frac{100}{100} = 1$
- Total market value of equity: $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 100 = 50.$

We proceed by calculating shareholders' expected payoff from the leverage policy. To repurchase 150 bonds, shareholders must inject $150 \times 1 = 150$ into the firm. Therefore, shareholders' total expected payoff from bond repurchases via recapitalization is $-150 + 50 = -100$, which is less than 0 in the benchmark case. As a result, shareholders are worse off from bond repurchases through recapitalization.

Case 2) Debt Repurchases With Cash

Suppose instead that the firm uses part of its \$100 in cash holdings to repurchase 150 bonds. In this case, the terminal firm value before bankruptcy costs equals $\widetilde{X} + 100 - 150p$, where p denotes the bond price after the repurchase. If the firm repays its remaining debt when $\widetilde{X} = 100$ and defaults when $\widetilde{X} = 0$, the competitive bond price satisfies $p = \frac{1}{2}$. The firm therefore spends $150p = 75$ on the repurchase, leaving it with $100 - 75 = 25$ in cash.

Let us illustrate two channels through which shareholders can benefit from debt repurchases financed with cash reserves. First, such debt repurchases lower the threshold cash flow required for the firm to repay its remaining debt and tend to reduce the likelihood of bankruptcy. Formally, when the operating cash flow $\widetilde{X} = 100$, the terminal firm value is $125 > 100$, so the firm repays the remaining debt in full. When the operating cash flow $\widetilde{X} = 0$, the terminal firm value is $25 < 100$, so the firm defaults. Thus, the probability of default goes down from 1 to $\frac{1}{2}$.

Second, the shareholders can reduce their future debt obligations. As long as the remaining debt is risky after the repurchase, its price is below face value. The firm can therefore spend less than one dollar of cash to retire one dollar of future debt obligations. Thus, the reduction in promised debt payments exceeds the amount of cash spent on the repurchase, leaving a positive payoff to shareholders when the firm is solvent. Hence, shareholders' payoff is weakly higher in every state:

$$\max\{\widetilde{X} + 100 - 150p - 100, 0\} \geq \max\{\widetilde{X} + 100 - 250, 0\}.$$

In particular, the inequality is strict when $\widetilde{X} = 100$ and holds with equality when $\widetilde{X} = 0$. Accordingly,

$$\begin{aligned}
&\underbrace{\mathbb{E}\left[\max\{\widetilde{X} + 100 - 150p - 100, 0\}\right]}_{\text{Equity Value After Debt Repurchase With Cash}} = \frac{1}{2}\left(100 + 100 - 150 \cdot \frac{1}{2} - 100\right) = \frac{25}{2} \\
> \quad &\underbrace{\mathbb{E}\left[\max\{\widetilde{X} + 100 - 250, 0\}\right]}_{\text{Equity Value Without Leverage Adjustment}} = 0.
\end{aligned}$$

As a result, shareholders are strictly better off under the bond repurchase financed by cash holdings. In the Internet Appendix, we consider a richer specification of cash flows, any feasible debt-repurchase amount, and an arbitrary recovery rate $\alpha \in [0, 1]$, and establish both results in the richer setup.

Discussion

In the example above, the shareholders do not gain at the expense of the original debtholders because all investors benefit from the lower risk of bankruptcy. When the firm pays the original debtholders $150 \times \frac{1}{2} = 75$ to repurchase 150 bonds, their remaining bonds are worth $100 \times \frac{1}{2} = 50$. Hence, the original debtholders hold cash and bonds worth 125 in total, up from 0. Thus, the example shows that neither wealth extraction nor asset substitution is necessary for shareholders to benefit from debt repurchases with cash reserves.

By contrast, the Internet Appendix shows that shareholders tend to benefit at the expense of debtholders when the recovery rate α is sufficiently high. For instance, in the frictionless full-recovery case, $\alpha = 1$, any gain to shareholders must be exactly offset by an aggregate loss to the original debtholders. Accordingly, the original debtholders necessarily lose whenever shareholders gain from debt repurchases with cash. Thus, for sufficiently high recovery rates, shareholder gains can be interpreted as wealth extraction from debtholders, in line with interpretations given by Mao and Tserlukevich (2015) and Admati et al. (2018).

This difference reflects two opposing effects of debt repurchases with cash. On the one hand, a repurchase may raise debt prices by reducing bankruptcy risk. On the other hand, it may lower the value of debtholders' claims in default by depleting cash reserves. At zero recovery, original debtholders weakly benefit; at full recovery, they are weakly worse off. In the Internet Appendix, we show that the effect of debt repurchases on debt prices—and therefore on debtholders—crucially depends on the recovery rate.

Connections to the Continuous-Time Model

The motivating example elucidates why shareholders may voluntarily use cash reserves to reduce debt but lacks two features of our dynamic model. First, when a firm issues long-term debt, it raises cash immediately but meets debt obligations over time. This timing difference can induce debt issuance when the marginal value of additional cash is high. Yet, the static model cannot capture the motive because no liquidity need arises between debt issuance and debt repayment. Hence, the continuous-time model can generate leverage adjustments in both directions and characterize when the firm issues or repurchases debt. Second, our continuous-time model allows us to analyze how credit spreads and debt and equity return volatilities evolve with liquidity and leverage.

3 Continuous-Time Model

3.1 Agents and Intangible Asset

Time is continuous and indexed by $t \in [0, \infty)$. We study a firm owned by a coalition of shareholders and financed by competitive debtholders. All agents are risk-neutral and discount future cash flows at a constant rate $r > 0$.

The firm owns an intangible asset that generates cumulative operating cash flows $X := \{X_t\}_{t \geq 0}$. We assume X follows an arithmetic Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$:

$$dX_t = \mu \, dt + \sigma \, dZ_t, \qquad X_0 = 0,$$

where $Z := \{Z_t\}_{t \geq 0}$ is a standard Brownian motion with respect to the filtration $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$.[9] The parameters $\mu > 0$ and $\sigma > 0$ represent the drift and volatility of the cumulative operating cash

[9] More precisely, $\mathbb{F}$ is the $\mathbf{P}$-augmentation of the natural filtration $\{\sigma(Z_s : s \leq t) : t \geq 0\}$ generated by Z.

flow process X, respectively.

As is standard in the literature, we assume that when the shareholders allow the firm to enter bankruptcy, the intangible asset must be irreversibly liquidated and has zero scrap value upon liquidation.[10] Thus, bankruptcy is costly in our framework because liquidation destroys the firm's going-concern value. Since there is no liquidation value to distribute to debtholders, the recovery rate of debt is zero.[11] Consequently, debt seniority is irrelevant in this framework.

3.2 Capital Structure

The firm manages its capital structure using a single class of debt with aggregate face value $F_t \geq 0$ at time t. As in Leland (1994), the debt consists of consol bonds. Specifically, each consol bond pays coupons continuously at a constant rate $c > 0$ per unit of face value, until the firm either enters bankruptcy or repurchases it in the competitive financial market.

Before bankruptcy, shareholders can adjust the debt level through issuances or repurchases. Formally, we follow the prior literature and assume that the aggregate face value of outstanding debt evolves according to

$$dF_t = G_t dt, \tag{1}$$

at any time t before bankruptcy. Verbally, G_t represents the rate of leverage adjustment at time t, with $G_t > 0$ corresponding to debt issuance and $G_t < 0$ corresponding to debt repurchases. The law of motion (1) is referred to as the firm's "debt dynamics."

Formally, we require the leverage policy $\{G_t\}_{t\geq 0}$ to be $\mathbb{F}$-predictable and locally integrable with respect to time. We also impose the feasibility constraint that the aggregate face value of outstanding debt cannot become negative. Also, to rule out Ponzi schemes, we impose a sufficiently large finite debt limit $F^{\max} > 0$ on the aggregate face value of outstanding debt:

$$0 \leq F_t \leq F^{\max} \tag{2}$$

[10]For instance, see Décamps et al. (2011) for the use of this assumption in a liquidity management model, and DeMarzo and He (2021) for its use in a dynamic capital structure model.

[11]Precisely, we assume that, prior to bankruptcy, dividend payouts are unrestricted by creditor consent, dividend covenants, or clawback provisions. Thus, whenever shareholders liquidate the firm with positive cash reserves, they find it optimal to distribute the cash before bankruptcy, leaving debtholders with only the intangible asset that has zero scrap value upon bankruptcy.

at any time t before bankruptcy, where we choose $F^{\max} > \frac{\mu}{c}$ to be sufficiently large so that the upper debt limit does not bind in equilibrium characterized below.

3.3 Liquidity Management

Before bankruptcy, the firm holds cash reserves $M_t \geq 0$ to buffer against operating losses and mitigate bankruptcy risk at time t. If cash reserves are depleted, shareholders can either inject new equity or irreversibly liquidate the firm. The shareholders inject new equity to continue the firm's operation as a going concern only if doing so is optimal.[12] Accordingly, the firm's bankruptcy time $\widetilde{T}_b$ is the first time at which cash reserves reach zero and shareholders choose not to continue the firm's operations through an immediate equity injection, and is thus an $\mathbb{F}$-stopping time. Also, we require that the firm cannot hold a negative cash balance at any instant:

$$M_t \geq 0 \tag{3}$$

at any time t before bankruptcy.

Our model incorporates both corporate and personal taxation. Specifically, the firm faces a corporate tax rate $\tau_c \in [0, 1)$, while shareholders face a personal tax rate $\tau_p \in (0, 1)$ on dividend payouts. Importantly, personal taxation effectively acts as an external financing cost in our model. More precisely, shareholders receive only a fraction of dividend payouts due to personal taxation, but must contribute the full amount for each dollar subsequently injected. As a result, personal taxation makes it costly to distribute dividends out of cash reserves today and replace them later through equity issuance.[13] As a result, equity injections and dividend distributions occur only when cash reserves reach certain thresholds (if ever) in our model.

The firm can raise additional funds from shareholders through equity issuance or distribute part of its cash reserves as dividends at any time. Formally, we model the firm's payout policy as a process $\{L_t\}_{t\geq 0}$ that describes the cumulative dividends paid out to shareholders. Likewise, the firm's equity issuance policy is described by a process $\{I_t\}_{t\geq 0}$ that represents the cumulative amount of funds raised through equity issuance, with normalizations $L_{0^-} = I_{0^-} = 0$. Henceforth,

[12] More precisely, the shareholders can always distribute all cash reserves as dividends and let the firm enter bankruptcy without contributing additional funds, so the equity value must always be weakly positive.

[13] In the absence of leverage adjustment, our model is isomorphic to a dynamic liquidity management model with pure proportional issuance costs $\frac{\tau_p}{1-\tau_p}$ per dollar of equity issuance; see, e.g., Løkka and Zervos (2008).

we collectively refer to $\{(L_t, I_t)\}_{t\geq 0}$ as the firm's liquidity management policy.

In line with the prior literature on liquidity management, we assume that cash reserves earn zero return inside the firm. This assumption can be interpreted as a reduced-form way of capturing "agency costs of free cash flow" in the spirit of Jensen (1986). Namely, cash reserves may generate the risk-free return $r > 0$ inside the firm, but managers divert the interest income to pet projects or other wasteful activities for private benefit. Absent such agency costs, the firm would choose to postpone dividend payouts as much as possible to reduce bankruptcy costs, rendering the optimal dividend policy ill-defined.

Formally, cash reserves evolve as follows:

$$\begin{aligned} dM_t =& (1-\tau_c)\left\{ \underbrace{\mu\, dt + \sigma\, dZ_t}_{\text{Operating Cash Flows}} - \underbrace{cF_t\, dt}_{\text{Interest Payments}} \right\} \\ &+ \underbrace{p_t G_t\, dt}_{\substack{\text{Net Proceeds}\\ \text{From Leverage Adjustment}}} - \underbrace{dL_t}_{\text{Dividends}} + \underbrace{dI_t}_{\substack{\text{Proceeds}\\ \text{From Equity Issuances}}}, \end{aligned} \tag{4}$$

for any $t \leq \widetilde{T}_b$, where p_t denotes the price per unit of face value at time t, which we define in the next subsection.[14] We henceforth refer to the law of motion in (4) as the firm's "cash dynamics." Henceforth, we say that the firm *accumulates cash* at a financial position (M, F) when its cash reserves have positive instantaneous drift at that financial position.

In summary, bankruptcy costs induce a precautionary demand for internal liquidity, whereas personal taxation makes it costly to replenish liquidity through equity issuance. Moreover, agency costs of free cash flow make it costly to hold excessive cash reserves, inducing the firm to pay dividends once cash reserves reach or exceed some threshold. Taken together, these frictions generate a meaningful tradeoff between cash retention and dividend payouts.

3.4 Smooth Equilibrium

We focus on Markov perfect equilibria (MPEs) in which leverage adjustment is absolutely continuous with respect to time, as specified in (1) for all $t \leq \widetilde{T}_b$. We refer to such equilibria as *smooth equilibria*. The payoff-relevant state variables immediately before financial decisions at time t are

[14] Consistent with the prior literature (e.g., DeMarzo and He (2021)), we adopt an idealized version of a linear tax scheme under which operating losses are subsidized.

cash reserves M_{t^-} and the aggregate face value of outstanding debt F_{t^-}.[15] Henceforth, we refer to the pair (M_{t^-}, F_{t^-}) as the firm's financial position at time t and denote a generic financial position by (M, F).

In a smooth equilibrium, the current financial decisions prescribed by the equilibrium policy and equilibrium asset prices depend on the firm's history only through its current financial position. Formally, a financial policy $\{L_s, I_s, G_s\}_{s\geq t}$ starting at a financial position $(M_{t^-}, F_{t^-}) = (M, F)$ consists of a liquidity management policy and a leverage policy from time t onward. Moreover, the equity value and debt price can be expressed as functions of the financial position (M, F). We denote the equity value function by $V : \mathbb{R}_+ \times [0, F^{\max}] \to \mathbb{R}_+$ and the debt-pricing function by $p : \mathbb{R}_+ \times [0, F^{\max}] \to \mathbb{R}_+$, respectively. At the beginning of the Appendix, we formally define the class of admissible financial policies and impose suitable regularity assumptions on the equilibrium value and pricing functions.

A smooth equilibrium satisfies the following requirements. First, shareholders take the debt-pricing function p as given and choose an admissible financial policy to maximize the expected present value of after-tax dividends net of equity injections up to bankruptcy. Formally, given a financial position (M, F) at time t, the equity value at time t is defined by

$$V(M,F) = \sup_{\{L_s,I_s,G_s\}_{s\geq t}} \mathbb{E}\left[\int_t^{\widetilde{T}_b} e^{-r(s-t)}\Big\{(1-\tau_p)\, dL_s - dI_s\Big\} \,\Bigg|\, M_{t^-} = M,\ F_{t^-} = F\right]. \tag{5}$$

For each financial position (M, F) , an equilibrium financial policy $\{L^\star_s, I^\star_s, G^\star_s\}_{s\geq t}$ is an admissible policy starting from the financial position that attains the supremum in (5).[16] Observe that equation (5) is required to hold at every time $t < \widetilde{T}_b$ and at every feasible financial position (M, F). This requirement captures the shareholders' lack of commitment, as they can reoptimize their financial policy at a later time.

Second, debtholders must break even at every instant, whenever the firm has outstanding debt with face value $F > 0$. If the firm adjusts the face value of debt, outside investors purchase

[15] We incorporate any potential lump-sum dividend or any potential equity issuance at time t. We therefore condition on the pre-decision cash-reserve level M_{t^-}. Since leverage adjustment is absolutely continuous with respect to time, $F_{t^-} = F_t$ for all $t < \widetilde{T}_b$; we use left-limit notation for both components of the state vector for consistency.

[16] Throughout the manuscript, all expectations are evaluated over both the operating cash flow shock and the public randomization device introduced below, under the state dynamics induced by the financial policy under consideration. The Stieltjes time integral should be understood to include any jumps of L and I at the initial time t and at the bankruptcy time $\widetilde{T}_b$.

or sell the corresponding amount at the prevailing market price. More specifically, when debt is repurchased, the units sold to the firm are selected randomly and proportionally from among the outstanding units of debt.[17] Since debt repurchases occur on the equilibrium path in our model, we adopt this trading protocol to rule out the possibility of a debtholder withholding a particular unit of debt in order to become the final creditor. Allowing such behavior would lead to a "holdout game" between that particular debtholder and the coalition of shareholders and would require a separate analysis of the resulting continuation bargaining or tender equilibrium. We deliberately shut down such game-theoretic interactions.

Accordingly, the price per unit of face value satisfies

$$p(M,F) = \mathbb{E}\left[\int_t^{\widetilde{T}_b \wedge \widetilde{T}_S} e^{-r(s-t)} c \, ds + e^{-r(\widetilde{T}_S - t)} p(M_{\widetilde{T}_S^-}, F_{\widetilde{T}_S^-}) \mathbf{1}_{\{\widetilde{T}_S < \widetilde{T}_b\}} \,\middle|\, M_{t^-} = M,\ F_{t^-} = F \right] \quad (6)$$

for any financial position (M,F) with outstanding debt $F > 0$, where $\widetilde{T}_S$ denotes the sale time of a representative unit of debt outstanding at time t.[18] We use $p_t = p(M_{t^-}, F_{t^-})$ in the cash dynamics (4) at every time $t < \widetilde{T}_b$ for which $F_{t^-} > 0$.[19]

We maintain the following parametric assumptions throughout our analysis of the continuous-time model in the main text:

Assumption 1.

$$\mu < \frac{\sqrt{r}\,\sigma}{2}, \qquad \tau_p > 1 - \left(\frac{\sqrt{\mu^2 + 2r\sigma^2} - \mu}{\sqrt{\mu^2 + 2r\sigma^2} + \mu} \right)^{\frac{\mu}{\sqrt{\mu^2+2r\sigma^2}}}.$$

The first restriction is used below to show that the marginal price impact from infinitesimal debt issuance, $\mathcal{I}(M,F) := p(M,F)p_M(M,F) + p_F(M,F)$, is strictly negative throughout the relevant region.[20] We use this property to construct a well-defined leverage policy $\{G_t\}_{t\geq 0}$ induced by

[17]Formally, we enlarge the probability space to support a public randomization device independent of the operating cash flow filtration $\mathbb{F}$. For a representative unit of debt outstanding at time t, let $\widetilde{T}_0 := \inf\{u \geq t : F_u = 0\}$, and let $\xi \sim \text{Exp}(1)$ denote an exponential random variable generated by the public randomization device. Define $\widetilde{T}_S := \inf\{u \in [t, \widetilde{T}_0) : \int_t^u \frac{\max(-G_s^\star, 0)}{F_s} ds \geq \xi\}$, with the convention $\inf \varnothing := \infty$. The public randomization device is used only to select the units sold to the firm, so aggregate financial policies and state dynamics remain adapted to $\mathbb{F}$.

[18]Henceforth, we adopt the convention that $t_1 \wedge t_2 := \min\{t_1, t_2\}$.

[19]When the firm has zero outstanding debt, $F = 0$, there is no outstanding unit to which the break-even condition (6) can be applied. Instead, we define $p(M,0) := \lim_{F \downarrow 0} p(M,F)$, which we later show to be well-defined for every $M \geq 0$. This limit should be interpreted as the price at which the firm can begin issuing debt from zero leverage, given cash reserves M. We use this issuance price in the cash dynamics (4) when the firm issues debt from zero leverage. If the firm instead remains at zero leverage, then $G = 0$, and the value of $p(M,0)$ has no effect on the cash dynamics.

[20]To see the interpretation of this price impact, given the debt pricing function $p(M,F)$, an infinitesimal

smooth equilibria. The second assumption requires the personal tax rate τ_p to be sufficiently high. As we show below, this is sufficient to guarantee that shareholders never find it optimal to inject additional funds into the firm and instead allow the firm to enter bankruptcy when its cash reserves M_t are exhausted. While stronger than necessary, this assumption allows us to simplify the exposition.

3.5 Model Discussion

Several features of our framework warrant emphasis. First, personal taxation $\tau_p \in (0, 1)$ creates a wedge between the shareholders' benefit from dividend payouts and their cost of equity injections and thereby makes external financing costly. Together with bankruptcy costs, this wedge induces a precautionary demand for internal liquidity. Consequently, the current level of cash reserves becomes a payoff-relevant state variable, as in the dynamic liquidity management literature (Bolton et al., 2011; Løkka and Zervos, 2008; Décamps et al., 2011).

Second, we model cumulative operating cash flows as an arithmetic Brownian motion, which has independent and stationary increments. Hence, we need not track the current level of cumulative operating cash flows, unlike the literature on the leverage ratchet effect (DeMarzo and He, 2021; Benzoni et al., 2022; Hu et al., 2025).

Third, for simplicity, the baseline model abstracts from personal taxation of interest income; the personal tax rate τ_p applies only to dividend payouts. In Subsection 6, we show that personal taxation of interest income shifts the equilibrium leverage policy toward debt repurchases, and our main results continue to hold qualitatively as long as the personal tax rate on interest income is sufficiently small.

The equilibrium feedback embedded in (4) and (6) reflects the central difficulty in our analysis. Specifically, leverage adjustments affect current liquidity through their net proceeds and alter the future value of debt through their effect on default risk. Debt prices respond to both channels and, in turn, determine the proceeds generated by current leverage adjustments.

---

debt issuance of dF increases cash reserves by the proceeds $dM = p(M, F)\,dF$. Let $\mathcal{P}(dF) := p\big(M + p(M, F)dF,\ F + dF\big)$ denote the approximate debt price induced by the debt issuance. The chain rule implies that $\mathcal{P}'(0) = p_M(M, F)\,p(M, F) + p_F(M, F)$. Hence, the marginal price change induced by an infinitesimal debt issuance dF can be linearly approximated as $(p(M, F)p_M(M, F) + p_F(M, F))\,dF$.

4 Equilibrium Construction

We now solve for the equilibrium financial policy and asset prices induced by smooth equilibria. Throughout the section, we maintain the admissibility and regularity requirements stated at the beginning of the Appendix. These requirements allow us to apply the Itô calculus and dynamic programming arguments below. We later verify that the candidate equilibrium objects satisfy all the maintained requirements.

Our analysis proceeds as follows. In Subsection 4.1, we derive the shareholders' indifference conditions for smooth leverage adjustments. In Subsection 4.2, we compute equilibrium asset prices and derive the optimal liquidity management based on the "no-trade valuation method." In Subsection 4.3, we derive the equilibrium leverage policy. In Subsection 4.4, we then combine these results to characterize the equity value at every feasible financial position and the asset prices and financial policy at positions with positive debt, and establish existence of a smooth equilibrium.

4.1 Indifference Condition

We derive the indifference condition needed to apply the no-trade equity valuation method. Fix a feasible financial position (M, F) with strictly positive cash reserves $M > 0$, and a feasible leverage-adjustment rate G at (M, F). Consider the following deviation. Over the time interval $[0, \widetilde{T}_b \wedge \Delta)$ for a sufficiently small $\Delta > 0$, the firm adjusts its leverage at the constant rate G and abstains from issuing equity and distributing dividends. If the firm survives until time Δ, it switches back to the equilibrium financial policy associated with the resulting financial position. If the firm runs out of cash before time Δ, the firm irreversibly enters into bankruptcy.

Since the equilibrium financial policy must be sequentially optimal, we have

$$
\begin{aligned}
0 &\geq \mathbb{E}\left[e^{-r(\widetilde{T}_b \wedge \Delta)} V\left(M_{\widetilde{T}_b \wedge \Delta}, F_{\widetilde{T}_b \wedge \Delta}\right) \Big| M_{0^-} = M, F_{0^-} = F\right] - V(M, F) \\
&= \mathbb{E}\Bigg[\int_0^{\widetilde{T}_b \wedge \Delta} e^{-rs}\Bigg\{ -rV(M_s, F_s) + [(1-\tau_c)(\mu - cF_s) + p(M_s, F_s)G]\, V_M(M_s, F_s) \\
&\qquad\qquad + GV_F(M_s, F_s) + \frac{1}{2}(1-\tau_c)^2\sigma^2 V_{MM}(M_s, F_s)\Bigg\} ds \Big| M_{0^-} = M, F_{0^-} = F\Bigg],
\end{aligned}
$$

where the equality follows from Itô's formula. Dividing both sides by $\Delta > 0$ and letting $\Delta \to 0$

yields

$$
\begin{aligned}
0 \geq & -rV(M,F) + (1-\tau_c)(\mu - cF)V_M(M,F) + \frac{1}{2}(1-\tau_c)^2\sigma^2 V_{MM}(M,F) \\
& + G\left[p(M,F)V_M(M,F) + V_F(M,F)\right]
\end{aligned}
\tag{7}
$$

for every feasible G.

The right-hand side of (7) is affine in G, with coefficient $p(M,F)V_M(M,F) + V_F(M,F)$. For debt issuance ($G \geq 0$), this coefficient captures the marginal benefit of the proceeds from debt issuance, net of the marginal cost of the future debt obligations. For debt repurchases ($G < 0$), the negative of this coefficient captures the marginal benefit from reduced debt obligations, net of the marginal cost of the debt repurchase.

When the firm has outstanding debt with $F \in (0, F^{\max})$, it can both issue new debt ($G > 0$) and repurchase outstanding debt ($G < 0$) at any finite rate over a sufficiently short time interval. Hence, at any financial position (M,F) with $(M,F) \in (0,\infty) \times (0, F^{\max})$, the inequality in (7) can hold for every $G \in \mathbb{R}$ only if

$$
p(M,F)V_M(M,F) + V_F(M,F) = 0. \tag{8}
$$

In the Appendix, we show that the indifference condition in (8) extends to every financial position $(M,F) \in \mathbb{R}_+ \times (0, F^{\max}]$, with a corresponding one-sided inequality at $F = 0$.

The indifference condition in (8) has a natural interpretation. As first stressed by DeMarzo and He (2021), when shareholders cannot commit to a future debt policy, they earn no marginal gains from their equilibrium smooth leverage adjustments. As we show in the next subsection, this implies that the equity value is the same as if shareholders never adjusted leverage, $G \equiv 0$. In subsequent analyses, we rely on the indifference condition to compute asset prices based on the no-trade equity valuation method.

4.2 No-Trade Valuation

4.2.1 Equity Value

Let us derive the variational inequalities and the boundary condition satisfied by the equity value in any smooth equilibrium. For any function $f : \mathbb{R}_+ \times [0, F^{\max}] \to \mathbb{R}$ twice continuously differentiable in the first argument, define the second-order linear differential operator $\mathcal{L}$ in the absence of leverage adjustments by

$$\mathcal{L}f(M,F) := (1-\tau_c)(\mu - cF)f_M(M,F) + \frac{1}{2}(1-\tau_c)^2\sigma^2 f_{MM}(M,F).$$

Fix a financial position (M,F) with $M>0$. Evaluating (7) at $G=0$ yields

$$rV(M,F) \geq \mathcal{L}V(M,F). \tag{9}$$

Suppose that the firm distributes an immediate dividend $\ell \in (0,M)$ to shareholders and then follows the equilibrium policy. Since this deviation cannot improve upon the equilibrium policy,

$$V(M,F) \geq V(M-\ell,F) + (1-\tau_p)\ell.$$

Dividing by $\ell > 0$ and letting $\ell \downarrow 0$ gives

$$V_M(M,F) \geq 1-\tau_p. \tag{10}$$

Similarly, suppose that the firm raises $\iota > 0$ through an immediate equity injection by incumbent shareholders and then follows the equilibrium policy. This injection increases cash reserves by ι at a cost of ι to shareholders. Therefore, shareholder optimality implies

$$V(M,F) \geq V(M+\iota,F) - \iota.$$

Dividing by $\iota > 0$ and letting $\iota \downarrow 0$ gives

$$V_M(M,F) \leq 1. \tag{11}$$

We next provide a heuristic derivation of the boundary condition at zero cash reserves. When the firm runs out of cash, shareholders can take one of the two actions: 1) they can allow it to enter bankruptcy immediately, in which case $V(0,F)=0$; 2) alternatively, they can inject equity to keep it operating, in which case the marginal value of cash equals the marginal cost of an equity injection, so that $V_M(0^+, F) = 1$. Moreover, since shareholders can always choose bankruptcy, $V(0, F) \geq 0$. Together with (11), these optimality conditions at the boundary imply:

$$\max\left\{-V(0, F),\, V_M(0^+, F) - 1\right\} = 0 \tag{12}$$

for every feasible face value F.

Observe that neither the variational inequalities (9)-(11) nor the boundary condition (12) explicitly involves leverage adjustments or the debt price. In particular, the equity value in any smooth equilibrium satisfies the same variational inequalities and boundary condition as the equity value of the auxiliary problem in which the firm does not adjust its outstanding debt over time, $G \equiv 0$.

To move forward, we solve an auxiliary no-trade equity valuation problem, where "no trade" means that the firm does not participate in the debt market, so that $G \equiv 0$. Let us formulate a variational system to pin down the no-trade equity value $V^0 : \mathbb{R}_+ \times [0, F^{\max}] \to \mathbb{R}_+$ for face values $F \in (0, \bar{F})$, where $\bar{F} := \frac{\mu}{c}$ is the threshold face value of debt below which the firm expects to earn strictly positive operating cash flow net of interest payments per unit time (i.e., $\mu - cF > 0$). In line with the literature on liquidity management, we conjecture that the optimal liquidity management policy follows a barrier strategy. In particular, we postulate that there exists a function $M^\star : [0, \bar{F}) \to (0, \infty)$ such that for each face value $F \in [0, \bar{F})$, the firm distributes dividends whenever its cash reserves reach or exceed the dividend threshold $M^\star(F)$. Thus, for each face value $F \in [0, \bar{F})$, the firm neither raises equity nor distributes dividends when cash reserves lie in $(0, M^\star(F))$. Finally, when the firm exhausts its cash reserves, it either ceases operating as a going concern or raises new equity from shareholders, as prescribed by the boundary condition (12).

This barrier policy implies the following gradient constraint: for each face value $F \in [0, \bar{F})$,

$$V_M^0(M, F) = 1 - \tau_p$$

for all cash reserves $M \geq M^\star(F)$. Intuitively, each dollar of cash above the dividend threshold is immediately distributed and is therefore worth $1 - \tau_p$ dollars to shareholders.

Under the assumption that V^0 is twice continuously differentiable with respect to M, the following super-contact condition holds at the dividend threshold:

$$V^0_{MM}(M^\star(F), F) = 0$$

for each $F \in [0, \bar{F})$.

Define the retention region by

$$\mathcal{R} := \left\{(M, F) : 0 \le F < \bar{F}, \quad 0 < M < M^\star(F)\right\}.$$

Since the firm neither raises equity nor distributes dividends in the retention region $\mathcal{R}$, the no-trade equity value satisfies the continuation HJB equation $-rV^0(M, F) + \mathcal{L}V^0(M, F) = 0$ for any $(M, F) \in \mathcal{R}$. Taking the limit of the resulting equation as $M \uparrow M^\star(F)$ and using the gradient constraint and the super-contact condition gives

$$V^0(M^\star(F), F) = \frac{(1-\tau_p)(1-\tau_c)(\mu - cF)}{r} \tag{13}$$

for each $F \in [0, \bar{F})$.

Consequently, for each face value $F \in [0, \bar{F})$, we must solve the following variational system for the no-trade equity value V^0 and the dividend threshold $M^\star(F)$:

$$\max\left\{-V^0(0, F),\, V^0_M(0^+, F) - 1\right\} = 0. \tag{14}$$

$$-rV^0(M, F) + \mathcal{L}V^0(M, F) = 0, \qquad (M, F) \in \mathcal{R}. \tag{15}$$

$$V^0(M, F) = (1-\tau_p)\big(M - M^\star(F)\big) + \frac{(1-\tau_p)(1-\tau_c)(\mu - cF)}{r}, \qquad M \ge M^\star(F). \tag{16}$$

In the Appendix, we prove that for each face value $F \in [0, \bar{F})$, the variational system (14)–(16) uniquely determines the no-trade equity value function $V^0(\cdot, F)$ and its associated dividend threshold $M^\star(F)$. Moreover, since the continuation HJB equation (15) is effectively a linear ordinary differential equation in the cash reserves M, we can solve for the unique solution in closed form after imposing the lower boundary condition (14), as well as the super-contact and the smooth-pasting conditions at the dividend boundary $M = M^\star(F)$.

To complete the no-trade equity valuation, let us consider the remaining feasible face values $F \geq \bar{F} := \frac{\mu}{c}$. Since $\mu - cF \leq 0$, the firm expects to earn weakly negative operating cash flow net of interest payments per unit time. We therefore conjecture that the optimal policy is to distribute all existing cash reserves to shareholders and then allow the firm to enter bankruptcy. The corresponding candidate no-trade equity value is

$$V^0(M, F) = (1 - \tau_p)M \tag{17}$$

for every cash reserve $M \geq 0$ and every face value $F \in [\bar{F}, F^{\max}]$. Combining the preceding two cases, V^0 satisfies the variational inequalities (9)–(11) with V replaced by V^0, at every feasible financial position with $M > 0$.

The next proposition summarizes our findings thus far:

Proposition 1. *(i) For each face value $F \in [0, \bar{F})$, the variational system* (14)–(16) *admits a unique solution pair $(V^0(\cdot, F), M^\star(F))$. The function $V^0(\cdot, F)$ is twice continuously differentiable with respect to the cash reserve M on* $(0, \infty)$.

(ii) Define the "no-trade" equity value function V^0 based on the solution for $F \in [0, \bar{F})$ and (17) *for $F \geq \bar{F}$. Then, at every feasible financial position (M, F) with $M > 0$, the no-trade equity value function V^0 satisfies the variational inequalities* (9)–(11)*, with V^0 in place of V.*

Next, we solve for the no-trade equity value function V^0 in its closed form under Assumption 1, and report its properties to be invoked in later sections:

Corollary 1. *Under Assumption 1, the no-trade equity value V^0 satisfies the lower boundary condition $V^0(0, F) = 0$ for every face value $F \geq 0$, and admits a closed-form representation. Moreover, $V^0_M(M, F) > 0$ and $V^0_{MM}(M, F) < 0$ throughout the retention region $\mathcal{R}$.*

Heuristically, the second part of Assumption 1 states that the personal tax rate τ_p must be sufficiently high that the shareholders never find it optimal to inject funds into the firm's cash reserves. As a result, the firm is irreversibly liquidated when it runs out of cash, implying the lower boundary condition $V(0, F) = 0$ for $F \geq 0$.

The strict monotonicity and concavity of the no-trade equity value function V^0 in cash reserves M also have natural interpretations. Since the shareholders can always pay out any excessive cash reserves, they must be better off when the firm has more cash reserves. Additionally, the firm is irreversibly liquidated when it runs out of cash reserves M_t. Due to this bankruptcy risk, the marginal value of additional liquidity is high when cash reserves are low. Bankruptcy risk goes down as the firm accumulates cash reserves, so the marginal value of cash decreases with the level of cash reserves.[21]

For the remainder of the equilibrium construction, we take $V := V^0$ as the candidate equilibrium equity value function, where V^0 is the no-trade equity value function characterized by Proposition 1 and (17). We verify below that this candidate coincides with the equity value function in any smooth equilibrium.

4.2.2 Debt Price

We invoke the first-order conditions in Subsection 4.1 to construct a candidate debt-pricing function. By the variational inequality (10), the equity value satisfies $V_M(M, F) \geq 1-\tau_p > 0$ at every financial position (M, F) with the cash reserves $M > 0$. Hence, at every financial position $(M, F) \in \mathcal{R}$ with the face value $F > 0$, the shareholders' indifference condition (8) can be rearranged as

$$p(M, F) = -\frac{V_F(M, F)}{V_M(M, F)}. \tag{18}$$

The next lemma records several properties of the candidate debt-pricing function used below to interpret the equilibrium dynamics.

Lemma 1. *Under Assumption 1, the candidate debt-pricing function defined by* (18) *admits a continuous extension to* $\mathbb{R}_+ \times [0, F^{\max}]$ *and a closed-form representation. Moreover, the candidate debt pricing function* p *satisfies the following properties:*

$$0 < p(M, F) < \frac{c}{r},\ p_M(M, F) > 0,\ p_F(M, F) < 0 \tag{19}$$

throughout the retention region $\mathcal{R}$.

Intuitively, when the firm has more cash reserves M or less debt F outstanding, it is less likely

[21] See, for instance, Milne and Robertson (1996) and Moreno-Bromberg and Rochet (2018) for standard interpretations of this feature as an endogenous form of "corporate" risk aversion in cash reserves arising from bankruptcy risk.

to run out of cash reserves in the future. As a result, debtholders expect to receive coupons for longer, so the market value of unit debt is subsequently higher.

4.3 Leverage Policy

4.3.1 Positive Face Value $F > 0$

Let us begin with the leverage choice when the firm has outstanding debt with face value $F > 0$. Let $G^\star$ denote the equilibrium leverage adjustment rate at the financial position (M, F). By applying the Feynman–Kac formula to the debtholders' break-even condition in (6), we obtain the following debt-pricing equation throughout the retention region $\mathcal{R}$, except potentially at the zero-leverage boundary with $F = 0$:

$$\begin{aligned} r\, p(M,F) =& c + \big[(\mu - cF)(1-\tau_c) + p(M,F)\, G^\star\big] p_M(M,F) \\ &+ \frac{1}{2}\sigma^2(1-\tau_c)^2 p_{MM}(M,F) + G^\star\, p_F(M,F) \end{aligned} \tag{20}$$

for any financial position $(M, F) \in \mathcal{R}$ with $F > 0$. Rearranging (20) gives

$$G^\star = \frac{r\,p - c - (\mu - cF)(1-\tau_c)\, p_M - \frac{1}{2}\sigma^2(1-\tau_c)^2\, p_{MM}}{p\,p_M + p_F}, \tag{21}$$

whenever the firm has outstanding debt $F > 0$, and the price impact $p\,p_M + p_F \neq 0$ inside the retention region $\mathcal{R}$.

The denominator $p(M,F)\, p_M(M,F) + p_F(M,F)$ represents the instantaneous impact of infinitesimal debt issuance on the price of unit debt, given the current financial position (M, F). Its sign is a priori ambiguous because it reflects two countervailing forces. On the one hand, debt issuance increases the firm's cash reserves, which in turn tends to reduce bankruptcy risk in the short run and raise debt prices. This "liquidity effect" is captured by the first component $pp_M > 0$. On the other hand, a higher face value implies additional future debt obligations, which tends to increase the firm's future default risk in the long run and depress debt prices. This "dilution effect" is captured by the second component $p_F < 0$.

The next lemma shows that under our parametric assumptions, the dilution effect dominates throughout the retention region:

Lemma 2. *Under Assumption 1, for every $(M, F) \in \mathcal{R}$,*

$$p(M,F)p_M(M,F) + p_F(M,F) < 0 \tag{22}$$

This property has both an economic interpretation and a technical implication. Economically, it parallels the negative price impact of debt issuance in many leverage-ratchet models as in DeMarzo and He (2021). Technically, it ensures that the denominator in (21) never vanishes in the retention region $\mathcal{R}$. Hence, the expression for the equilibrium leverage adjustment rate $G^\star$ is well-defined at every financial position $(M, F) \in \mathcal{R}$ with $F > 0$.

Since the equity value $V(M, F)$ can be solved in closed form by Proposition 1, the bond price $p(M, F)$ can also be obtained in closed form by the debt pricing formula (18). Substituting the resulting expression for $p(M, F)$ into (21) then yields a closed-form expression for the equilibrium leverage-adjustment rate $G^\star$ when the firm has outstanding debt $F > 0$.

Remark 1. *The debtholders' break-even condition pins down the equilibrium leverage-adjustment rate $G^\star$, whereas shareholders are marginally indifferent among smooth leverage policies in the continuous-time model. Section IA.B instead allows shareholders to commit to a debt adjustment over a short interval $[t, t + dt)$. For any fixed bound on finite adjustment rates that contains $G^\star$, shareholders have a unique optimal leverage choice when $dt > 0$ is sufficiently small, and the optimal adjustment rate converges to $G^\star$ as $dt \downarrow 0$.*

4.3.2 Zero Face Value $F = 0$

We next analyze the leverage choice for an unlevered firm with $F = 0$. As first pointed out by DeMarzo and He (2021), equilibrium multiplicity arises whenever the leverage adjustment rate prescribed by (21) tends to a positive limit as $F \downarrow 0$.[22]

Let us consider two possibilities. If the firm begins issuing debt ($G^\star > 0$), the face value of its outstanding debt becomes positive immediately thereafter. Therefore, the debt-pricing equation applies to the ensuing financial positions with positive debt. Taking the limit of this equation as $F \downarrow 0$ and rearranging yields (21) at the zero-leverage boundary as well whenever $G^\star > 0$. By

[22] This multiplicity is specific to the zero leverage boundary $F = 0$ because the debtholders' break-even condition (6) does not apply to an unlevered firm with $F = 0$. Once the firm has a positive amount of outstanding debt, the break-even condition uniquely determines its leverage adjustment rate by (21).

Lemma 2, the denominator in (21) remains strictly negative at $F = 0$, so this leverage adjustment rate is well-defined.

Alternatively, the firm may remain unlevered by setting $G^\star = 0$. Since no debt is outstanding, it can issue new debt at any finite rate but cannot repurchase debt. Therefore,

$$p(M,0)V_M(M,0) + V_F(M,0) \leq 0$$

for every $M > 0$, which must bind whenever $G^\star > 0$. Hence, we have:

$$G^\star\Big(p(M,0)V_M(M,0) + V_F(M,0)\Big) = 0$$

Consequently, shareholders cannot obtain a positive marginal gain from smooth leverage adjustments at the zero-leverage boundary. Hence, setting $G^\star(M,0) = 0$ is sequentially optimal and constitutes an absorbing equilibrium selection for an unlevered firm. We make use of this continuation leverage policy in the equilibrium constructed below.

4.4 Smooth Equilibria

The next proposition summarizes the discussion thus far and records the properties of the equilibrium financial policy and equilibrium asset prices for a levered firm with $F > 0$. It further pins down the equity value at the zero-leverage boundary with $F = 0$:

Proposition 2. *Under Assumption 1, every smooth equilibrium satisfies the following properties:*
Financial Policy of a Levered Firm with $F > 0$

1. *(High Leverage) Suppose that the firm has outstanding debt with face value $F \geq \bar{F} := \frac{\mu}{c}$. The firm immediately distributes all its cash reserves as dividends and then enters bankruptcy. Shareholders receive the after-tax dividend $(1-\tau_p)M$, and their value is therefore given by* (17)*, while debtholders receive no recovery.*
2. *(Low Leverage) Suppose that the firm has outstanding debt with face value $F \in (0, \bar{F})$. Then:*
   (a) If cash reserves $M \in (0, M^\star(F))$, the firm adjusts leverage at the rate $G^\star$ given by (21)*, without distributing dividends ($dL_t = 0$) or issuing equity ($dI_t = 0$).*
   (b) If cash reserves $M \geq M^\star(F)$, the firm immediately distributes any cash in excess of the

dividend threshold $M^\star(F)$. Thereafter, whenever $F_t \in (0, \bar{F})$ and $M_t = M^\star(F_t)$, the firm adjusts debt at the limiting rate $\lim_{M \uparrow M^\star(F_t)} G^\star(M, F_t)$ and pays dividends only as needed to maintain $M_t \leq M^\star(F_t)$.

(c) *If the firm runs out of cash reserves, $M = 0$, it enters bankruptcy immediately without equity issuance. Equity has zero continuation value, and debtholders receive no recovery.*

Asset Prices

In every smooth equilibrium, the equity value is uniquely pinned down at every feasible financial position, including $F = 0$, and coincides with the no-trade equity value: $V(M, F) = V^0(M, F)$. At every feasible financial position (M, F) with $F > 0$, the debt price is pinned down by the relation $p(M, F) = -\frac{V_F^0(M,F)}{V_M^0(M,F)}$, where derivatives at a boundary are interpreted as the appropriate one-sided derivatives.

The preceding proposition characterizes smooth equilibria but does not explicitly establish the existence of a smooth equilibrium. Therefore, we construct a candidate equilibrium by combining the financial policy of a levered firm and the asset prices characterized in Proposition 2 with the continuation policy under which an unlevered firm never subsequently issues debt, as described in Subsection 4.3.2. In the Appendix, we establish that the induced state dynamics are well posed, verify the sequential optimality of the candidate financial policy, and establish that debtholders break even whenever the firm has outstanding debt, $F > 0$. These arguments yield the following existence result:

Proposition 3 (Existence of a Smooth Equilibrium)**.** *Under Assumption 1, there exists at least one smooth equilibrium. In this equilibrium, when the firm is levered with $F > 0$, the financial policy and asset prices are as characterized in Proposition 2. When the firm is unlevered with $F = 0$, it never issues debt; that is, the equilibrium leverage-adjustment rate satisfies $G^\star(M, 0) = 0$ for every $M > 0$. The firm therefore remains unlevered until bankruptcy.*

In Proposition 3, we construct a smooth equilibrium based on a particular continuation policy under which an unlevered firm remains unlevered until bankruptcy. As explained in Subsubsection 4.3.2, alternative smooth equilibria may instead prescribe debt issuance by an unlevered firm if $\lim_{F \downarrow 0} G^\star(M, F) > 0$. This multiplicity of equilibrium leverage choices arises only at the zero-leverage boundary $F = 0$. As shown in Proposition 2, all smooth equilibria prescribe the same financial policy whenever $F > 0$. In particular, the debtholders' break-even condition (6) uniquely determines the leverage-adjustment rate of a levered firm through (21) throughout the retention region $\mathcal{R}$.

5 Equilibrium Characterization

This section derives various implications of our model. In Subsection 5.1, we first decompose the equilibrium leverage policy to identify the forces that determine whether the firm issues or repurchases debt, and then present the two main results of the paper. In Subsection 5.2, we characterize how cash reserves affect stock and bond return volatility and how outstanding debt affects stock return volatility and credit spreads.

5.1 Joint Dynamics of Leverage and Liquidity

Let us decompose the numerator of the equilibrium leverage adjustment rate $G^\star$ given by (21) as follows:

Lemma 3. *Under Assumption 1, in any smooth equilibrium, the equilibrium leverage adjustment rate $G^\star$ satisfies*

$$G^\star \underbrace{[pp_M + p_F]}_{<0 \text{ by Lemma 2}} = -\tau_c c + rp + \underbrace{(1-\tau_c)^2\sigma^2 p_M \left(\frac{V_{MM}}{V_M}\right)}_{=\frac{1}{dt}\operatorname{Cov}\left(dp, \frac{dV_M}{V_M}\right)}. \tag{23}$$

inside the retention region $\mathcal{R}$, as long as the firm has outstanding debt, $F > 0$.

The left-hand side of (23) represents the change in the debt price induced by equilibrium leverage adjustment. Heuristically, when the firm adjusts its face value by an infinitesimal amount dF, the resulting transaction changes cash reserves by $p\,dF$. Therefore, the resulting change in the debt price is $(pp_M + p_F)\,dF$. Hence, in equilibrium, leverage adjustment contributes $(pp_M + p_F)\,G^\star dt$ to the instantaneous rate of change in the debt price per unit time. Since the marginal price impact $pp_M + p_F < 0$ by Lemma 2, debt issuance lowers the debt price, whereas debt repurchases raise it.

The right-hand side of (23) identifies three economic forces governing the equilibrium leverage policy. The first term, $-\tau_c c$, is the negative of the corporate tax shield generated by interest payments and captures a common motive for debt issuance in the leverage-ratchet literature. For instance, in the canonical model of DeMarzo and He (2021), the firm issues debt at a rate such that the shareholders' marginal gain from additional tax shields is equated with the decline in debt prices caused by additional debt issuance. Accordingly, the firm does not issue additional debt in their model when the corporate tax rate is zero. By contrast, our model generates two additional mechanisms that can induce the firm to adjust long-term debt even in the absence of corporate taxation.

The second term, rp, represents the instantaneous dollar return on the debt price per unit of face value. By the debtholders' break-even condition, the debt price p is the expected present value of future coupon payments received until bankruptcy or repurchase, together with any resale proceeds received upon repurchase. Accordingly, rp can be interpreted as the required dollar return on the expected present value of the firm's future debt obligations per unit of face value. Recall from Lemma 1 that the debt price increases with cash reserves ($p_M > 0$) and decreases with the face value of outstanding debt ($p_F < 0$). Hence, since the equilibrium leverage adjustment $G^\star$ has the opposite sign from the right-hand side of (23) by Lemma 2, when a firm has high cash reserves or a low face value of debt, it is more likely to repurchase existing debt than to issue new consols, ceteris paribus.

The intuition behind this is as follows. If a firm has larger cash reserves or less debt outstanding, the firm faces lower bankruptcy risk and is therefore expected to make interest payments for longer. Consequently, shareholders expect the firm to make greater future interest payments per unit of face value and face a stronger incentive to reduce its outstanding debt.

The last term, $\frac{(1-\tau_c)^2\sigma^2}{V_M} p_M V_{MM}$, is the instantaneous covariance between debt prices and percentage changes in the shareholders' marginal value of liquidity. Given that asset prices rise with cash reserves ($p_M, V_M > 0$) while the marginal value of cash falls with them ($V_{MM} < 0$), this term is strictly negative and thus contributes to debt issuance by Lemma 2. Intuitively, when the shareholders face a high risk of bankruptcy, they issue additional debt to accumulate additional cash to avoid the risk of bankruptcy in the short run.

Figure 1 illustrates how these economic forces vary with the firm's financial position. When cash reserves are low, the expected discounted value of future debt payments—and hence rp—is small, while the liquidity motive captured by the negative covariance term is large in magnitude. Hence, cash-strapped firms issue debt to replenish their cash reserves. As cash reserves increase and outstanding debt declines, rp rises and the liquidity motive weakens, making debt repurchases

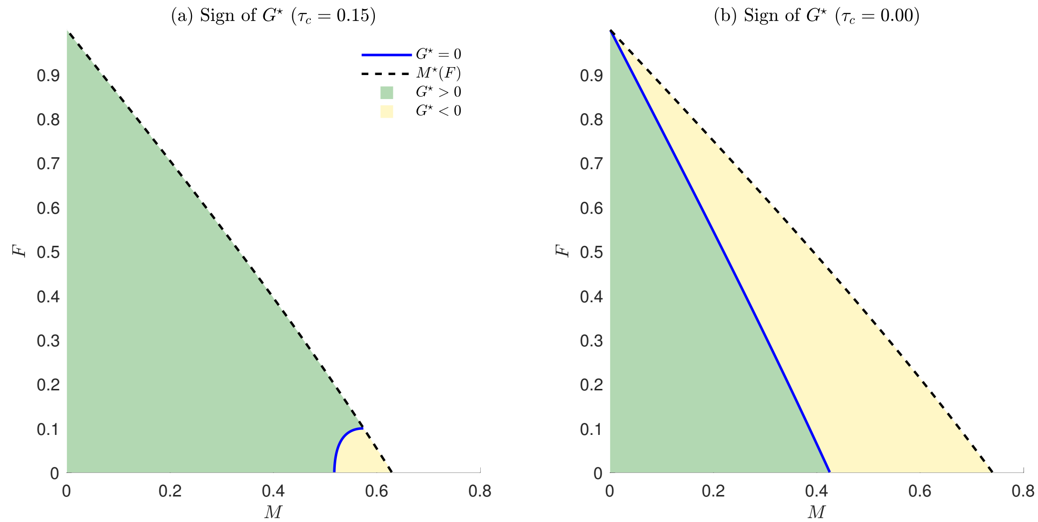

Figure 1: **Debt Issuance, Debt Repurchases, and Cash Accumulation.** This figure plots the sign of the equilibrium leverage adjustment $G^\star(M, F)$ over the retention region $\mathcal{R}$. Panel (a) corresponds to an economy with corporate taxation ($\tau_c = 0.15$), while Panel (b) corresponds to an economy without corporate taxation ($\tau_c = 0$). Green regions indicate debt issuance ($G^\star > 0$). Yellow regions indicate debt repurchases ($G^\star < 0$). The blue solid line marks the boundary at which no leverage adjustment occurs, $G^\star = 0$. The black dashed line denotes the equilibrium payout boundary $M^\star(F)$. Parameter values are $r = 0.05$, $\mu = 0.04$, $\sigma = 0.36$, $c = 0.04$, and $\tau_p = 0.25$.

more attractive.

5.1.1 Simultaneous Deleveraging and Cash Accumulation

The next proposition is the first main result of our paper. In particular, it identifies conditions under which debt reduction and cash accumulation occur simultaneously:

Proposition 4 (Debt reduction and cash accumulation)**.** *Under Assumption 1, there exists a threshold $\tau_c^* \in (0, 1)$ with the following properties:*

1. *For each corporate tax rate $\tau_c \in (0, \tau_c^*)$, there exists a unique cutoff $F^* \in (0, \bar{F})$ such that, for each face value $F \in (0, F^*)$, there exists a unique cash reserve cutoff $M^{\star\star}(F) \in (0, M^\star(F))$ satisfying $G^\star(M^{\star\star}(F), F) = 0$. For every $M \in (M^{\star\star}(F), M^\star(F))$,*

$$G^\star(M, F) < 0 < \underbrace{(1 - \tau_c)(\mu - cF) + p(M, F)G^\star(M, F)}_{\substack{\textit{Drift of cash reserves} \\ \textit{in the retention region } \mathcal{R}}}. \tag{24}$$

Consequently, the firm reduces outstanding debt while accumulating its cash reserves. Every-

where else in the retention region $\mathcal{R}$, the firm does not repurchase debt.

2. *If $\tau_c = 0$, then, for each $F \in (0, \bar{F})$, there exists a unique $M^{\star\star}(F) \in (0, M^\star(F))$ satisfying $G^\star(M^{\star\star}(F), F) = 0$. Inequality* (24) *holds for every $M \in (M^{\star\star}(F), M^\star(F))$, whereas the firm does not repurchase debt at any other financial position in $\mathcal{R}$.*

The intuition behind the proposition is as follows. Suppose that a firm faces a low corporate tax rate and has sufficiently large cash reserves with little debt outstanding. Then, shareholders gain little from issuing additional debt to exploit tax shields or from using the proceeds from debt sales to accumulate additional cash to avert bankruptcy. At the same time, the firm faces a relatively low risk of bankruptcy, so the expected present value of its future interest payments is relatively high. Therefore, shareholders draw down cash reserves to repurchase debt and reduce their future interest burden. Nevertheless, expected operating cash flow net of interest payments more than covers the cash spent on repurchases, so the firm's cash reserves tend to drift upwards.

This result accords with evidence that many firms deleverage from historical peaks toward near-zero market leverage while markedly increasing cash balances (DeAngelo et al., 2018). Moreover, it is broadly consistent with prior evidence that firms use favorable cash-flow realizations both to reduce leverage and to accumulate cash over shorter horizons. For instance, Dasgupta et al. (2011) show that cash savings and debt reduction account for substantial shares of cash flow use, and Chang et al. (2014) similarly find that firms allocate transitory cash flow both to cash holdings and to reductions in external financing.

5.1.2 Debt Issuance in Response to Cash Depletion

Our second main result predicts that a levered firm issues debt whenever it has sufficiently small positive cash reserves:

Proposition 5 (Debt issuance when cash reserves are low)**.** *Suppose Assumption 1 holds. For every corporate tax rate $\tau_c \in [0, 1)$ and every face value $F \in (0, \bar{F})$, there exists $\varepsilon > 0$ such that $G^\star(M, F) > 0$ for every $M \in (0, \varepsilon)$.*

This result identifies a novel mechanism for debt issuance relative to DeMarzo and He (2021), where no debt issuance occurs in the absence of corporate taxation. When cash reserves are low, the firm faces a high risk of bankruptcy. Shareholders therefore issue additional debt to replenish the firm's cash reserves and reduce its bankruptcy risk. This motive arises for every corporate tax rate and continues to operate even when debt provides no tax benefits.

These predictions are in line with evidence that substantial leverage increases primarily finance operating needs, that subsequent debt reductions depend on the generation of financial surpluses, and that most net debt issuers would otherwise face immediate cash depletion (Denis and McKeon, 2012; Huang and Ritter, 2021). They also accord with the finding that debt repayment and earnings retention account for most of the decline in leverage during firms' peak-to-trough deleveraging episodes (DeAngelo et al., 2018).

This result also echoes the "dash for cash" during financial crises. Campello et al. (2010) show that constrained firms draw on credit lines to meet current liquidity needs and build cash reserves for future needs. Ivashina and Scharfstein (2010) document precautionary credit-line drawdowns during the 2008 financial crisis. Acharya and Steffen (2020) find a similar pattern during the initial phase of the COVID-19 crisis and show that U.S. firms drew down credit lines and accumulated cash. Admittedly, since most of this evidence concerns bank credit lines, it does not directly test our model, which speaks most directly to publicly traded corporate bonds. Still, it broadly supports the model's mechanism whereby corporations may increase debt when the marginal value of their internal liquidity is high.

5.2 Asset Pricing Implications

We next derive implications for stock and bond return volatility and credit spreads. Fix a financial position $(M, F) \in \mathcal{R}$ with $F > 0$. Since leverage adjustments are smooth, changes in face value have finite variation and do not directly contribute to instantaneous return volatility. Combining the laws of motion in (1) and (4) with the shareholders' indifference condition (8) and the HJB equations (15) and (20), Itô's lemma yields

$$\begin{aligned}
\underbrace{\frac{dV(M_t,F_t)}{V(M_t,F_t)}}_{\text{Instantaneous stock return}} &= \underbrace{r}_{\substack{\text{Expected return}\\ \text{in the risk-neutral economy}}} dt + \underbrace{(1-\tau_c)\sigma\frac{V_M(M_t,F_t)}{V(M_t,F_t)}}_{\text{Stock return volatility}} dZ_t, \\
\underbrace{\frac{dp(M_t,F_t)+c\,dt}{p(M_t,F_t)}}_{\text{Instantaneous bond return}} &= \underbrace{r}_{\substack{\text{Expected return}\\ \text{in the risk-neutral economy}}} dt + \underbrace{(1-\tau_c)\sigma\frac{p_M(M_t,F_t)}{p(M_t,F_t)}}_{\text{Bond return volatility}} dZ_t.
\end{aligned} \tag{25}$$

Accordingly, define stock and bond return volatility by

$$\sigma^S(M,F) := (1-\tau_c)\sigma\left(\frac{V_M(M,F)}{V(M,F)}\right), \qquad \sigma^B(M,F) := (1-\tau_c)\sigma\left(\frac{p_M(M,F)}{p(M,F)}\right). \tag{26}$$

These expressions show that a cash-flow shock affects the return on each security through its effect on cash reserves. The security's exposure to this shock is determined by its marginal value of cash relative to its market value.

Since the debt is a consol bond, define its current yield by

$$R^*(M,F) := \frac{c}{p(M,F)}. \tag{27}$$

The corresponding credit spread is $R^*(M,F) - r$. Since $p(M,F) < \frac{c}{r}$ throughout the retention region, this credit spread is strictly positive.

The following corollary summarizes the comparative statics of return volatility and credit spreads.

Corollary 2 (Asset-pricing implications)**.** *Under Assumption 1, for every* $(M,F) \in \mathcal{R}$ *with* $F > 0$*,*

$$\frac{\partial \sigma^S(M,F)}{\partial M} < 0, \qquad \frac{\partial \sigma^S(M,F)}{\partial F} > 0, \tag{28}$$

$$\frac{\partial \sigma^B(M,F)}{\partial M} < 0, \tag{29}$$

and

$$\frac{\partial R^*(M,F)}{\partial M} < 0, \qquad \frac{\partial R^*(M,F)}{\partial F} > 0. \tag{30}$$

The comparative statics with respect to cash reserves reflect the amplification of cash flow shocks when cash reserves are low. When the firm has low cash reserves, the marginal values of cash to both shareholders and debtholders are high relative to their respective security values. Consequently, a given cash flow shock produces larger percentage changes in both equity and debt values. Stock and bond return volatility therefore decline as the firm accumulates cash reserves. Similarly, since the debt price increases with cash reserves, the credit spread decreases with cash reserves.

Outstanding debt has the following effects on equity risk and credit spreads. Since debt promises a fixed stream of payments, an increase in its face value reduces the residual value accruing to shareholders without proportionally reducing their exposure to cash-flow shocks. The same shock therefore produces a larger percentage change in equity value, increasing stock return volatility. For debtholders, additional face value increases the firm's promised payments but also makes full repayment less likely. The resulting increase in bankruptcy risk reduces the market price per unit of debt. Since the credit spread is inversely related to the debt price, it increases with outstanding debt.

The leverage prediction for stock return volatility accords with evidence that financial leverage affects equity volatility (Choi and Richardson, 2016). The credit-spread predictions are also consistent with evidence that more highly levered firms have wider credit spreads and that instrumented increases in liquid-asset reserves reduce corporate bond spreads (Acharya et al., 2012). Finally, these predictions are broadly consistent with existing evidence that equity volatility is positively associated with corporate bond yield spreads and that corporate bond returns respond to changes in issuer equity values (Campbell and Taksler, 2003; Schaefer and Strebulaev, 2008).

6 Personal Taxation of Interest Income

Thus far, we have assumed away personal taxation of interest income. We now let $\tau_i \in [0,1)$ denote a separate personal tax rate on interest income. The next proposition shows that personal taxation of interest income shifts the leverage policy toward debt repurchases. In the Internet Appendix, we show that for each $F \in (0,\bar{F})$, the main results also continue to hold whenever τ_i is sufficiently small:

Proposition 6 (Personal taxation of interest income)**.** *Suppose Assumption 1 holds, and fix $F \in (0,\bar{F})$. For $\tau_i \in [0,1)$, let $G_i^\star$ denote the equilibrium leverage-adjustment rate when interest income is taxed at rate τ_i, and let $G^\star := G_0^\star$. The equilibrium debt price remains $p(M,F)$, and*

$$G_i^\star(M,F) = G^\star(M,F) + \frac{\tau_i c}{p(M,F)p_M(M,F) + p_F(M,F)} < G^\star(M,F) \tag{31}$$

for every $M \in (0, M^\star(F))$ whenever $\tau_i > 0$. Consequently, if the firm repurchases debt at (M,F) in the absence of interest-income taxation, it also repurchases debt at (M,F) when $\tau_i > 0$.

Interest taxation does not affect the firm's cash dynamics or shareholders' no-trade equity value. Shareholders' indifference therefore continues to pin down the same equilibrium debt price, $p = -V_F/V_M$. Instead, the leverage-adjustment rate changes so that debtholders break even despite the tax on coupon income. Because the marginal price impact from an infinitesimal debt issuance satisfies $pp_M + p_F < 0$, this adjustment tilts the equilibrium policy toward debt repurchases.

Remark 2. *Our use of personal taxation differs from that in recent work by Hu (2026). The key mechanism in his model operates through the difference between the corporate tax rate and the personal tax rate on interest income, whereas our main results continue to hold even when these two tax rates are zero. More importantly, while his analysis abstracts from corporate cash holdings, we endogenize cash reserves and study their interaction with leverage.*

7 Conclusion

Our paper studies joint leverage and liquidity management when shareholders cannot commit to future financial policies. To highlight the role of the funding source in debt repurchases, we present a tractable setting where shareholders can reduce leverage only once. We show that shareholders can gain from debt repurchases financed with cash reserves, even when the remaining debt is competitively priced. When the recovery rate is sufficiently low, the original debtholders may also gain because the repurchase lowers bankruptcy risk and reduces expected bankruptcy losses. Thus, shareholders can benefit from debt repurchases without extracting wealth from debtholders.

We then analyze our continuous-time model, in which the firm adjusts both cash and debt levels over time. We characterize the equilibrium policies and associated asset prices in closed form. Our model generates leverage adjustments in both directions. When the corporate tax rate is low, a cash-rich firm with little debt repurchases debt while accumulating its cash reserves. By contrast, when cash reserves become sufficiently low, a firm issues new debt to replenish liquidity even in the absence of corporate tax shields, as long as it remains in operation and has a positive amount of outstanding debt. Our model also yields implications for stock and bond return volatility and credit spreads.

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Appendix

This Appendix is organized as follows. Section A states the admissibility and regularity conditions used in the verification arguments. Section B constructs and uniquely identifies the no-trade equity value. Section C derives the candidate debt price and proves its key properties, including an upper bound on the marginal price impact from an infinitesimal debt issuance. Section D constructs the candidate equilibrium, verifies its existence, identifies the equity value at every feasible financial position, and characterizes the debt price and financial policy whenever the firm has positive debt. Finally, Section E proves the results related to the joint dynamics of liquidity and leverage in Section 5.

A Admissibility and Regularity Assumptions

This section has two parts. The first specifies the admissible financial policies, feasibility constraints, and integrability requirements. The second states the smoothness, boundary, dynamic-programming, and state-process conditions imposed on a smooth equilibrium and later verified for the candidate equilibrium.

Admissible Financial Policies We require both the cumulative dividend process $\{L_t\}_{t\geq 0}$ and the cumulative equity issuance process $\{I_t\}_{t\geq 0}$ to be non-decreasing, $\mathbb{F}$-adapted, and right-continuous with left limits. We impose the feasibility constraint that the firm cannot pay out more dividends than the sum of the cash reserves on its balance sheet and equity injection amounts before bankruptcy:

$$\Delta L_t \leq M_{t^-} + \Delta I_t, \tag{A.1}$$

for every $t \leq \widetilde{T}_b$, where $\Delta L_t := L_t - L_{t^-}$ and $\Delta I_t := I_t - I_{t^-}$.

We restrict attention to financial policies for which the debt and cash dynamics in (1) and (4) are well posed, the feasibility constraints (2), (3), and (A.1) are satisfied at every time before bankruptcy, and the expected present values of both dividend payouts and equity injections are finite. We adopt the convention that all controls are stopped at bankruptcy. Thus, no dividend payouts, equity injections, or leverage adjustments occur after $\widetilde{T}_b$.

Regularity Assumptions on V and p Throughout this Appendix, a smooth equilibrium is understood to satisfy the following regularity requirements. The equity value V is continuous on $\mathbb{R}_+ \times [0, F^{\max}]$ and, at every interior financial position, is twice continuously differentiable with respect to M and continuously differentiable with respect to F. The derivatives V_M, V_{MM}, V_F are locally bounded. The debt price p is continuous on $\mathbb{R}_+ \times [0, F^{\max}]$. At every interior financial position at which the debt-pricing equation (20) is used, p is twice continuously differentiable with respect to M and continuously differentiable with respect to F, and p_M, p_{MM}, p_F are locally bounded. At the boundary of the state space $\mathbb{R}_+ \times [0, F^{\max}]$, any derivative used below is interpreted as the appropriate continuous one-sided derivative.

The equilibrium financial policy induces a pathwise unique strong solution to the state equations up to bankruptcy, and the resulting state process is nonexplosive and strong Markov. The shareholder and debtholder values satisfy their respective dynamic programming principles, and an equilibrium financial policy attains the supremum in (5). We verify below that the candidate equilibrium satisfies these requirements.

B No-Trade Equity Value V^0

This section solves the no-trade cash-management problem used in the equilibrium construction. Subsection B.1 constructs a family of candidate equity values indexed by the payout boundary. Subsection B.2 constructs the bankruptcy and equity-issuance candidates. Subsection B.3 selects the no-trade equity value and establishes uniqueness, after which we prove Proposition 1. The final subsection imposes Assumption 1, derives the closed form, proves Corollary 1, and establishes the no-trade equity valuation result used to identify the equity value in every smooth equilibrium. Fix $F \in [0, \bar{F})$ and write $b(F) := (1 - \tau_c)(\mu - cF) > 0, \sigma_M := (1 - \tau_c)\sigma$. For a twice continuously differentiable function φ, define

$$(\mathcal{L}\varphi)(M, F) := b(F)\varphi_M(M, F) + \frac{1}{2}\sigma_M^2 \varphi_{MM}(M, F). \tag{A.2}$$

The construction below allows either bankruptcy or equity issuance at $M = 0$ and therefore retains the general case $\tau_p \in (0, 1)$.

B.1 Candidate Equity Values Indexed by the Payout Boundary

This subsection constructs the candidate values associated with an arbitrary payout boundary and establishes the curvature properties used to compare those candidates. It first solves the ordinary differential equation subject to the value-matching and smooth-pasting conditions at the candidate payout boundary and then proves strict monotonicity and concavity. The next subsection uses the resulting values and marginal values at $M = 0$ to construct the two possible lower-boundary selections.

For each candidate payout boundary $m > 0$, let $V^m(\cdot, F)$ solve

$$\begin{aligned} &\frac{1}{2}\sigma_M^2 V^m_{MM} + b(F)V^m_M - rV^m = 0, && 0 < M < m, \\ &V^m(m, F) = \frac{b(F)(1-\tau_p)}{r}, \quad V^m_M(m, F) = 1 - \tau_p, && \end{aligned} \tag{A.3}$$

and extend it above m by

$$V^m(M, F) := \frac{b(F)(1-\tau_p)}{r} + (1-\tau_p)(M - m), \qquad M \geq m. \tag{A.4}$$

Define

$$\lambda_\pm(F) := \frac{-(\mu - cF) \pm \sqrt{(\mu - cF)^2 + 2r\sigma^2}}{(1-\tau_c)\sigma^2}, \qquad \lambda_-(F) < 0 < \lambda_+(F).$$

Lemma A.1 (Candidate equity values)**.** *For $0 \leq M \leq m$,*

$$V^m(M, F) = \frac{1-\tau_p}{\lambda_+(F) - \lambda_-(F)} \left[-\frac{\lambda_-(F)}{\lambda_+(F)} e^{\lambda_+(F)(M-m)} + \frac{\lambda_+(F)}{\lambda_-(F)} e^{\lambda_-(F)(M-m)} \right]. \tag{A.5}$$

The function $M \mapsto V^m(M, F)$ is strictly increasing and strictly concave on $[0, m]$. In particular, $V^m_M(M, F) > 1 - \tau_p$ and $V^m_{MM}(M, F) < 0$ for $0 < M < m$.

Proof. Equation (A.5) is the unique solution of (A.3). Differentiating twice gives

$$V^m_{MM}(M, F) = \frac{(1-\tau_p)\lambda_+(F)\lambda_-(F)}{\lambda_+(F) - \lambda_-(F)} \left[e^{\lambda_-(F)(M-m)} - e^{\lambda_+(F)(M-m)} \right] < 0$$

for $M < m$. Hence $V^m_M(M, F) > V^m_M(m, F) = 1 - \tau_p > 0$, which proves both claims. □

B.2 Lower-Boundary Selection

This subsection determines whether the no-trade cash-management problem terminates when $M = 0$ or continues through equity issuance. We characterize the two alternatives using the value and marginal value at $M = 0$ and then verify the corresponding auxiliary equity value functions. The next subsection compares these two values and applies the lower-boundary condition to select the no-trade solution.

Lemma A.2 (Value and marginal value at $M = 0$)**.** *For fixed F, the function $m \mapsto V^m(0, F)$ is continuous and strictly decreasing from $b(F)(1-\tau_p)/r$ to $-\infty$, whereas $m \mapsto V^m_M(0, F)$ is continuous and strictly increasing from $1-\tau_p$ to ∞. Consequently, there are unique boundaries $M^d(F), M^c(F) > 0$ such that*

$$V^{M^d(F)}(0, F) = 0, \qquad V_M^{M^c(F)}(0, F) = 1.$$

Proof. Suppress the argument F. Since $\lambda_+ - \lambda_- > 0$, Equation (A.5) gives

$$V^m(0, F) = \frac{1-\tau_p}{\lambda_+ - \lambda_-}\left[-\frac{\lambda_-}{\lambda_+}e^{-\lambda_+ m} + \frac{\lambda_+}{\lambda_-}e^{-\lambda_- m}\right], \quad V^m_M(0, F) = \frac{1-\tau_p}{\lambda_+ - \lambda_-}\left[\lambda_+ e^{-\lambda_- m} - \lambda_- e^{-\lambda_+ m}\right].$$

The characteristic equations that determine λ_+ and λ_- imply $V^m(0, F)|_{m=0} = b(F)(1-\tau_p)/r$ and $V^m_M(0, F)|_{m=0} = 1-\tau_p$. Moreover,

$$\frac{\partial V^m(0, F)}{\partial m} = \frac{1-\tau_p}{\lambda_+ - \lambda_-}\left[\lambda_- e^{-\lambda_+ m} - \lambda_+ e^{-\lambda_- m}\right] < 0,$$

whereas

$$\frac{\partial V^m_M(0, F)}{\partial m} = \frac{(1-\tau_p)\lambda_+\lambda_-}{\lambda_+ - \lambda_-}\left[e^{-\lambda_+ m} - e^{-\lambda_- m}\right] > 0.$$

Finally, because $\lambda_- < 0 < \lambda_+$, $V^m(0, F) \to -\infty$ and $V^m_M(0, F) \to \infty$ as $m \to \infty$. The conclusions follow from the intermediate value theorem and strict monotonicity. □

For any $m > 0$, let $\mathcal{V}^m(\cdot, F)$ denote the function obtained by combining (A.5) below m with the linear extension (A.4). Define

$$\widehat{V}^d := \mathcal{V}^{M^d(F)}, \qquad \widehat{V}^c := \mathcal{V}^{M^c(F)}.$$

The first function corresponds to bankruptcy at zero; the second corresponds to minimal equity issuance at zero.

Proposition A.3 (The two auxiliary equity values)**.** *The function $\widehat{V}^d$ is increasing, concave, C^2 on $(0,\infty)$, satisfies $\widehat{V}^d(0,F)=0$, and solves*

$$\max\left\{\mathcal{L}\widehat{V}^d - r\widehat{V}^d,\ 1-\tau_p-\widehat{V}^d_M\right\}=0.$$

The function $\widehat{V}^c$ is also increasing, concave, C^2 on $(0,\infty)$, satisfies $\widehat{V}^c_M(0,F)=1$, and solves

$$\max\left\{\mathcal{L}\widehat{V}^c - r\widehat{V}^c,\ 1-\tau_p-\widehat{V}^c_M,\ \widehat{V}^c_M-1\right\}=0.$$

Both functions satisfy smooth pasting and super-contact at their respective payout boundaries.

Proof. Lemma A.1 gives the asserted properties below each boundary. At a boundary m, (A.3) gives $V^m_M(m,F)=1-\tau_p$ and $V^m_{MM}(m-,F)=0$, so the value and its first two derivatives match the linear extension. Above m,

$$\mathcal{L}\mathcal{V}^m(M,F)-r\mathcal{V}^m(M,F)=-r(1-\tau_p)(M-m)\le 0.$$

This proves the first variational inequality. For $\widehat{V}^c$, strict concavity and $\widehat{V}^c_M(0,F)=1$ additionally imply $1-\tau_p<\widehat{V}^c_M<1$ below $M^c(F)$, which proves the second. □

B.3 The No-Trade Equity Value and Its Uniqueness

This subsection compares the two auxiliary equity values, identifies the no-trade equity value selected by the lower boundary, and verifies that it is unique among the candidate equity values. The following subsection uses this selection to prove existence, uniqueness, and attainability in Proposition 1.

Lemma A.4 (Comparison and lower-boundary selection)**.** *For any $m_a<m_b$, $\mathcal{V}^{m_a}(M,F)>\mathcal{V}^{m_b}(M,F)$ for all $M\ge 0$. Hence the candidate satisfying* $\max\{-V^0(0,F),\,V^0_M(0^+,F)-1\}=0$ *is*

$$\big(V^0(\cdot,F),M^\star(F)\big)=\begin{cases}\big(\widehat{V}^d(\cdot,F),M^d(F)\big), & M^d(F)\le M^c(F),\\ \big(\widehat{V}^c(\cdot,F),M^c(F)\big), & M^c(F)<M^d(F).\end{cases}\tag{A.6}$$

Proof. For $M<m_a$, direct differentiation of (A.5) with respect to m is strictly negative. For $m_a\le M<m_b$, concavity places $\mathcal{V}^{m_b}$ below its tangent at m_b, while $\mathcal{V}^{m_a}$ is already on the linear

segment defined in (A.4); for $M \geq m_b$, the difference equals $(1-\tau_p)(m_b - m_a) > 0$. This proves the first claim.

If $M^d(F) \leq M^c(F)$, then $\widehat{V}^d(0,F) = 0$ and $\widehat{V}^d_M(0,F) \leq 1$ because $m \mapsto V^m_M(0,F)$ is increasing. If $M^c(F) < M^d(F)$, then $\widehat{V}^c_M(0,F) = 1$ and $\widehat{V}^c(0,F) = V^{M^c(F)}(0,F) > 0$ because $m \mapsto V^m(0,F)$ is decreasing. Thus (A.6) satisfies the lower-boundary condition. The strict comparison shows that no other choice can do so. □

B.4 Proof of Proposition 1

Proof of Proposition 1. Fix $F \in [0, \bar{F})$. The selected function in (A.6) is C^2 on $(0,\infty)$ and, by Proposition A.3, satisfies

$$\mathcal{L}V^0 - rV^0 \leq 0, \qquad 1-\tau_p \leq V^0_M \leq 1.$$

It solves $\mathcal{L}V^0 - rV^0 = 0$ on $(0, M^\star(F))$ and has the payout branch

$$V^0(M,F) = \frac{(1-\tau_p)(1-\tau_c)(\mu - cF)}{r} + (1-\tau_p)\big(M - M^\star(F)\big)$$

for $M \geq M^\star(F)$. Thus it satisfies (14)–(16).

Conversely, any solution of that system with boundary $m > 0$ must solve the ordinary differential equation and boundary conditions in (A.3) below m and hence equals $\mathcal{V}^m$. If its value at zero is zero, strict monotonicity of $m \mapsto V^m(0,F)$ gives $m = M^d(F)$; the remaining inequality at zero then requires $M^d(F) \leq M^c(F)$. If its marginal value at zero is one, strict monotonicity of $m \mapsto V^m_M(0,F)$ gives $m = M^c(F)$; the remaining inequality requires $M^c(F) \leq M^d(F)$. Therefore, (A.6) is the unique solution pair.

For $F \geq \bar{F}$, define $V^0(M,F) = (1-\tau_p)M$. Then $V^0_M = 1-\tau_p$, $V^0_{MM} = 0$, and

$$\mathcal{L}V^0 - rV^0 = (1-\tau_p)\big[(1-\tau_c)(\mu - cF) - rM\big] \leq 0.$$

This proves both parts of Proposition 1.

For completeness, the same function is the value of the no-trade stochastic-control problem. Applying Itô's formula to the stopped process $e^{-rt}V^0(M_t, F)$ and using the preceding HJB and gradient inequalities bounds the payoff from every admissible dividend and equity-issuance policy by $V^0(M,F)$. If $M^\star(F) = M^d(F)$, downward reflection at $M^d(F)$ followed by bankruptcy at zero attains the bound. If $M^\star(F) = M^c(F)$, the doubly reflected policy on $[0, M^c(F)]$ attains it, with minimal equity issuance at zero and minimal dividends at $M^c(F)$. Standard localization and the imposed integrability conditions remove the stopped terminal term. $\square$

B.5 Closed-Form Representation Under Assumption 1

This subsection imposes Assumption 1 to show that bankruptcy, rather than equity issuance, is selected at zero cash. We first derive the auxiliary boundaries, then obtain the closed-form no-trade equity value and prove Corollary 1. We conclude with the no-trade equity valuation lemma used in the characterization of all smooth equilibria.

Corollary A.5 (Closed-form auxiliary boundaries)**.** *For every $\tau_p \in (0,1)$ and $F \in [0, \bar{F})$,*

$$M^d(F) = \frac{2}{\lambda_+(F) - \lambda_-(F)} \log\left(-\frac{\lambda_-(F)}{\lambda_+(F)}\right),$$

whereas $M^c(F)$ is the unique positive solution of

$$\frac{1-\tau_p}{\lambda_+(F) - \lambda_-(F)} \left[\lambda_+(F) e^{-\lambda_-(F) M^c(F)} - \lambda_-(F) e^{-\lambda_+(F) M^c(F)}\right] = 1.$$

The associated auxiliary equity values are obtained by setting $m = M^d(F)$ or $m = M^c(F)$ in (A.5) *and* (A.4).

Proof. The second equation follows from $V_M^{M^c(F)}(0, F) = 1$. Substituting (A.5) into $V^{M^d(F)}(0, F) = 0$ and rearranging gives the first. $\square$

Lemma A.6 (Zero boundary value and closed-form representation)**.** *Under Assumption 1, for every $F \in [0, \bar{F})$,*

$$V^0(0, F) = 0, \qquad V_M^0(0^+, F) < 1, \qquad M^\star(F) = M^d(F) < M^c(F).$$

Moreover,

$$V^0(M, F) = \begin{cases} \dfrac{1-\tau_p}{\lambda_+(F) e^{\lambda_+(F) M^\star(F)} - \lambda_-(F) e^{\lambda_-(F) M^\star(F)}} \left[e^{\lambda_+(F) M} - e^{\lambda_-(F) M}\right], & 0 \le M < M^\star(F), \\[2ex] (1-\tau_p)\big(M - M^\star(F)\big) + \dfrac{(1-\tau_p)(1-\tau_c)(\mu - cF)}{r}, & M \ge M^\star(F). \end{cases}$$

For $F \geq \bar{F}$, $V^0(M,F) = (1-\tau_p)M$. In particular, $1-\tau_p \leq V^0_M(M,F) < 1$ wherever the derivative is defined.

Proof. Set

$$y(F) := \frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} \in (0,1).$$

Corollary A.5 and (A.5) give

$$\widehat{V}^d_M(0,F) = (1-\tau_p)\left(\frac{1+y(F)}{1-y(F)}\right)^{y(F)}.$$

The map $y \mapsto y\log\bigl((1+y)/(1-y)\bigr)$ is strictly increasing on $(0,1)$ because its derivative is $\log((1+y)/(1-y)) + 2y/(1-y^2) > 0$. Since $y(F) \leq y(0)$, the second restriction in Assumption 1 implies $\widehat{V}^d_M(0,F) < 1$. The strict monotonicity of $m \mapsto V^m_M(0,F)$ therefore yields $M^d(F) < M^c(F)$, so (A.6) selects $\widehat{V}^d$. The stated formulas follow from Corollary A.5; the derivative bound follows from concavity and the linear segment in (A.4). □

Proof of Corollary 1. For $F \in [0,\bar{F})$, Lemma A.6 gives $V^0(0,F) = 0$ and the stated closed-form representation. For $F \in [\bar{F}, F^{\max}]$, equation (17) gives $V^0(M,F) = (1-\tau_p)M$, so the same lower boundary condition holds and the closed form is immediate. This proves the first part of the corollary for every feasible face value.

It remains to establish the strict derivative inequalities. If $(M,F) \in \mathcal{R}$, then $0 < M < M^\star(F)$ and, by Lemma A.6, the no-trade equity value coincides with the candidate equity value $V^{M^\star(F)}$. Lemma A.1 therefore gives $V^0_M(M,F) > 1-\tau_p > 0, V^0_{MM}(M,F) < 0$, which proves the remaining claims. □

Lemma A.7 (No-trade equity valuation)**.** *Fix a feasible F. Let $W(\cdot,F)$ be a nonnegative C^2 value function satisfying*

$$\max\{\mathcal{L}W - rW,\, 1-\tau_p - W_M,\, W_M - 1\} = 0 \tag{A.7}$$

on $(0,\infty)$ and $\max\{-W(0,F), W_M(0^+,F)-1\} = 0$. If the value is attained by an admissible policy, then $W(\cdot,F) = V^0(\cdot,F)$.

Proof. Write $q := 1-\tau_p$. Equation (A.7) implies $q \leq W_M \leq 1$. At least one gradient boundary must be reached at a finite cash level: otherwise the continuation ODE holds on $(0,\infty)$, and its exponential solutions cannot have derivative bounded between the two strictly positive constants q and 1.

The continuation set cannot begin at a strictly positive cash level. Indeed, if a continuation interval had a positive left endpoint a, smooth contact would give $W_{MM}(a, F) = 0$. Contact with $W_M = 1$ makes the differentiated continuation ODE imply that W_M immediately exceeds one. Following contact with $W_M = q$, the unique continuation solution is

$$W_M(a+s, F) = \frac{q}{\lambda_+(F) - \lambda_-(F)} \left[-\lambda_-(F) e^{\lambda_+(F)s} + \lambda_+(F) e^{\lambda_-(F)s} \right].$$

It is strictly increasing and eventually exceeds one; it therefore cannot remain a continuation solution forever or meet the upper gradient boundary with smooth contact. Both possibilities contradict (A.7).

Thus there is a single continuation interval $(0, m)$, followed by an affine tail with slope either 1 or q. The slope-one tail is incompatible with attainment. At sufficiently large M its continuation residual is strictly negative, every finite equity injection is value-neutral and leaves the firm on the same tail, while dividends and bankruptcy are strictly value reducing. Attaining the value would require an unbounded initial injection, contrary to admissibility. Hence the tail has slope q.

If $F < \bar{F}$, then $m > 0$, because $W = qM$ near zero would make $\mathcal{L}W - rW > 0$. Smooth contact gives $W(m, F) = b(F)q/r$, so (W, m) satisfies the unique free-boundary system proved above and hence equals $(V^0, M^\star(F))$. If $F \geq \bar{F}$, a positive m would imply both $W(m, F) = b(F)q/r \leq 0$ and $W(m, F) \geq W(0, F) + qm > 0$, a contradiction. Therefore $m = 0$ and $W(M, F) = qM = V^0(M, F)$. □

C Debt Price

This section derives the candidate debt price from shareholder indifference and establishes the price properties used later. The first subsection obtains the closed form and its boundary extensions. The second proves an upper bound on the marginal price impact from an infinitesimal debt issuance. The final subsection proves the price, monotonicity, derivative-sign, and continuity claims in Lemma 1.

C.1 Closed-form expression for the candidate debt-pricing function

This subsection differentiates the no-trade equity value with respect to face value and cash reserves to obtain the candidate price $p = -V_F/V_M$. It first derives the debt-pricing formula in the retention region, then its payout-region counterpart, and finally verifies agreement and continuity at the payout, zero-leverage, and high-leverage boundaries.

Lemma A.8 (Candidate debt price)**.** *Suppose that Assumption 1 holds, and let $V := V^0$ be the candidate equity value characterized by Lemma A.6. Define the candidate debt-pricing function as*

$$p(M,F) := -\frac{V_F(M,F)}{V_M(M,F)},$$

for each position $(M,F) \in \mathbb{R}_+ \times [0, F^{\max}]$, with right derivatives used at $M = 0$ and $F = 0$ and left derivatives used at $F = F^{\max}$.

For $0 \le F < \bar{F}$ and $0 \le M < M^\star(F)$, the candidate debt price is

$$\begin{aligned} p(M,F) = \frac{c}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} \Bigg[&\frac{\left(\frac{\lambda_+(F)+\lambda_-(F)+2\lambda_+(F)\lambda_-(F)M^\star(F)}{\lambda_+(F)-\lambda_-(F)}\right)\left[e^{\lambda_+(F)M} - e^{\lambda_-(F)M}\right]}{\lambda_-(F)e^{\lambda_-(F)M} - \lambda_+(F)e^{\lambda_+(F)M}} \\ &+ \frac{M\left[\lambda_+(F)e^{\lambda_+(F)M} + \lambda_-(F)e^{\lambda_-(F)M}\right]}{\lambda_-(F)e^{\lambda_-(F)M} - \lambda_+(F)e^{\lambda_+(F)M}}\Bigg]. \end{aligned} \tag{A.8}$$

For $0 \le F < \bar{F}$ and $M \ge M^\star(F)$, it is given by

$$p(M,F) = (1-\tau_c)c\left[\frac{(\mu - cF)^2}{r\left[(\mu-cF)^2 + 2r\sigma^2\right]} + \frac{\sigma^2(\mu - cF)}{\left[(\mu-cF)^2+2r\sigma^2\right]^{3/2}}\log\left(\frac{\sqrt{(\mu-cF)^2+2r\sigma^2} + \mu - cF}{\sqrt{(\mu-cF)^2+2r\sigma^2} - (\mu - cF)}\right)\right]. \tag{A.9}$$

The expressions in (A.8) *and* (A.9) *agree at $M = M^\star(F)$. At the zero leverage boundary, the candidate debt price is defined by $p(M,0) := \lim_{F\downarrow 0} p(M,F)$, and is obtained by evaluating* (A.8) *or* (A.9) *at $F = 0$, as applicable. Finally, $p(M,F) = 0$ whenever $M = 0$ or $F \in [\bar{F}, F^{\max}]$.*

Proof of Lemma A.8. Under Assumption 1, Lemma A.6 gives $V(0,F) = 0$ for every $F \in [0,\bar{F})$ and $V_M(M,F) \ge 1 - \tau_p > 0$ for every $M \ge 0$ and $F \in [0, F^{\max}]$, where the derivative at $M = 0$ is understood as a right derivative. Thus, the ratio $-V_F/V_M$ is well defined whenever the relevant derivative with respect to F exists.

Fix $F \in (0, \bar{F})$. For $0 < M < M^\star(F)$, Lemma A.6 gives

$$V(M,F) = \frac{1-\tau_p}{\lambda_+(F)e^{\lambda_+(F)M^\star(F)} - \lambda_-(F)e^{\lambda_-(F)M^\star(F)}} \left[e^{\lambda_+(F)M} - e^{\lambda_-(F)M}\right].$$

Differentiating this expression with respect to M and F gives

$$\begin{aligned} -\frac{V_F(M,F)}{V_M(M,F)} &= \frac{\frac{d}{dF}\log\left[\lambda_+(F)e^{\lambda_+(F)M^\star(F)} - \lambda_-(F)e^{\lambda_-(F)M^\star(F)}\right]\left[e^{\lambda_+(F)M} - e^{\lambda_-(F)M}\right]}{\lambda_+(F)e^{\lambda_+(F)M} - \lambda_-(F)e^{\lambda_-(F)M}} \\ &\quad - \frac{M\left[\lambda_+'(F)e^{\lambda_+(F)M} - \lambda_-'(F)e^{\lambda_-(F)M}\right]}{\lambda_+(F)e^{\lambda_+(F)M} - \lambda_-(F)e^{\lambda_-(F)M}}. \end{aligned}$$

The definitions of $\lambda_+(F)$ and $\lambda_-(F)$ imply $\lambda_+'(F) = \frac{c\lambda_+(F)}{\sqrt{(\mu-cF)^2+2r\sigma^2}}$ and $\lambda_-'(F) = -\frac{c\lambda_-(F)}{\sqrt{(\mu-cF)^2+2r\sigma^2}}$. Moreover, the super-contact condition at the dividend boundary gives

$$\lambda_-^2(F)e^{\lambda_-(F)M^\star(F)} = \lambda_+^2(F)e^{\lambda_+(F)M^\star(F)}.$$

Using this identity when differentiating the normalization term yields

$$\begin{aligned} &\frac{d}{dF}\log\left[\lambda_+(F)e^{\lambda_+(F)M^\star(F)} - \lambda_-(F)e^{\lambda_-(F)M^\star(F)}\right] \\ &\qquad = -\frac{c}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \frac{\lambda_+(F)+\lambda_-(F)+2\lambda_+(F)\lambda_-(F)M^\star(F)}{\lambda_+(F)-\lambda_-(F)}. \end{aligned}$$

Substitution of these identities into the preceding expression for $-V_F/V_M$ gives (A.8).

We next consider $M \geq M^\star(F)$. In this region,

$$V(M,F) = (1-\tau_p)\big(M - M^\star(F)\big) + \frac{(1-\tau_p)(1-\tau_c)(\mu - cF)}{r}.$$

Consequently, $V_M(M,F) = 1-\tau_p$ and

$$V_F(M,F) = -(1-\tau_p)\left[M^{\star\prime}(F) + \frac{(1-\tau_c)c}{r}\right].$$

It follows that

$$p(M,F) = M^{\star\prime}(F) + \frac{(1-\tau_c)c}{r}. \tag{A.10}$$

The closed-form expression for the dividend threshold can equivalently be written as

$$M^\star(F) = \frac{(1-\tau_c)\sigma^2}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \log\left(\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}+\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}-(\mu-cF)}\right).$$

Differentiating gives

$$M^{\star\prime}(F) = (1-\tau_c)c\sigma^2\left[\frac{\mu - cF}{[(\mu-cF)^2+2r\sigma^2]^{3/2}}\log\left(\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}+\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}-(\mu-cF)}\right)-\frac{2}{(\mu-cF)^2+2r\sigma^2}\right].$$

Substituting this derivative into (A.10) and using

$$\frac{1}{r}-\frac{2\sigma^2}{(\mu-cF)^2+2r\sigma^2}=\frac{(\mu-cF)^2}{r\,[(\mu-cF)^2+2r\sigma^2]}$$

gives (A.9).

The two expressions agree at $M = M^\star(F)$. Indeed, differentiating the value-matching condition $V(M^\star(F),F) = \frac{(1-\tau_p)(1-\tau_c)(\mu-cF)}{r}$ with respect to F and using $V_M(M^\star(F),F) = 1-\tau_p$ gives

$$V_F(M^\star(F),F) = -(1-\tau_p)\left[M^{\star\prime}(F)+\frac{(1-\tau_c)c}{r}\right].$$

Thus, the ratio $-V_F/V_M$ has the same value when the dividend boundary is approached from either region.

Because $V(0,F) = 0$ for every $F \in [0,\bar{F})$, we have $V_F(0,F) = 0$. Together with $V_M(0^+,F) > 0$, this gives $p(0,F) = 0$. The same conclusion follows directly by setting $M = 0$ in (A.8).

The functions $\lambda_\pm(F)$ and $M^\star(F)$ are right-continuous at $F = 0$, and the two debt-price expressions agree at the dividend boundary. Therefore, $p(M,0) = \lim_{F\downarrow 0} p(M,F)$ is well defined and is obtained by evaluating the applicable closed-form expression at $F = 0$.

Finally, as $F \uparrow \bar{F}$, $M^\star(F) \longrightarrow 0$ and $M^{\star\prime}(F) \longrightarrow -\frac{(1-\tau_c)c}{r}$. Hence, $\lim_{F\uparrow\bar{F}} p(M,F) = 0$ for every fixed $M \geq 0$. For $F \in [\bar{F}, F^{\max}]$, (17) gives $V(M,F) = (1-\tau_p)M$. Thus, $V_F(M,F) = 0$, and therefore $p(M,F) = -\frac{V_F(M,F)}{V_M(M,F)} = 0$. This completes the proof. $\square$

C.2 Upper Bound on Marginal Price Impact

This subsection proves that the dilution effect of debt issuance strictly dominates its liquidity effect in the marginal price impact from an infinitesimal debt issuance. The quantitative upper bound obtained here is also used to establish cash accumulation during debt repurchases.

Lemma A.9 (Upper Bound on Marginal Price Impact)**.** *Under Assumption 1, for every* $0 < F < \bar{F}$

and $0 < M \le M^\star(F)$,

$$p(M,F)p_M(M,F) + p_F(M,F) < -\frac{r\,p(M,F)^2}{(1-\tau_c)(\mu - cF)} < 0. \tag{A.11}$$

At $M = M^\star(F)$*, the derivatives are understood as one-sided limits from the retention region.*

Proof. Fix $0 < F < \bar{F}$ and $0 < M \le M^\star(F)$, and write

$$\alpha := 1-\tau_c, \qquad k := \mu - cF, \qquad S := \sqrt{k^2 + 2r\sigma^2}, \qquad x := \frac{k}{S},$$

and define

$$z := \frac{SM}{\alpha\sigma^2}, \qquad L := \log\left(\frac{1+x}{1-x}\right), \qquad t := L - z, \qquad u := 1 - x^2.$$

Because $0 < M \le M^\star(F)$, we have $0 < z \le L$ and $0 \le t < L$. Assumption 1 gives $0 < x < 1/3$. Finally, let

$$R := \frac{\sinh z}{\cosh z - x\sinh z}, \qquad A := x + ut.$$

Let $H(z) := \cosh z - x\sinh z$. Since $H(z) = \sqrt{u}\cosh(z - \operatorname{arctanh}(x))$, we have $H(z) \ge \sqrt{u}$ for $0 \le z \le L$. Moreover, $R'(z) = 1/H(z)^2 \le 1/u$ and $R(0) = 0$. Hence,

$$0 < R \le \frac{z}{u}. \tag{A.12}$$

The closed-form debt price in (A.8) can be written as

$$p(M,F) = \frac{\alpha c}{2r} f(x,z), \qquad f(x,z) := u\big[xz + RA\big]. \tag{A.13}$$

Since $x_k = u/S$, $z_k = xz/S$, and $\partial_F = -c\partial_k$, direct differentiation gives

$$pp_M + p_F = \frac{\alpha c^2}{2rS} d(x,z), \qquad d := \frac{ff_z}{u} - uf_x - xzf_z. \tag{A.14}$$

Equations (A.13) and (A.14) imply that the desired inequality is equivalent to

$$2xd + f^2 < 0. \tag{A.15}$$

To verify (A.15), direct simplification yields

$$2xd + f^2 = u\left(C_0 + C_1R + C_2R^2 + C_3R^3\right), \tag{A.16}$$

where

$$\begin{aligned}
C_0 &:= xz\left[xuz + 6x^2 - 2\right], \\
C_1 &:= 2x\left[LuA + 6txu + 7x^2 - 3\right], \\
C_2 &:= A\left[tu(1+3x^2) - x(3-7x^2)\right], \\
C_3 &:= -2xuA^2.
\end{aligned}$$

We now show that the right-hand side of (A.16) is strictly negative. Let $y := x^2 \in (0, 1/9)$ and $\ell := uL/x$. The power-series expansion of $\operatorname{arctanh}(x)$ gives

$$\ell \le 2 - \frac{4}{3}y =: \bar{\ell}. \tag{A.17}$$

Indeed,

$$L = 2\operatorname{arctanh}(x) \le 2x + \frac{2x^3}{3(1-x^2)}.$$

First, $xuz \le x^2\ell \le 2y$, so $C_0 < xz(8y-2) < 0$. Also, $C_3 < 0$. Because $t \le L$ and $A \le x + uL$, we have

$$\begin{aligned}
\frac{C_1}{2x} &\le (uL)^2 + 7xuL + 7x^2 - 3 \\
&= y(\ell^2 + 7\ell + 7) - 3 \\
&\le \frac{16y^3 - 132y^2 + 225y - 27}{9} < 0.
\end{aligned} \tag{A.18}$$

The final polynomial in (A.18) is increasing on $[0, 1/9]$ and equals $-2630/6561$ at $y = 1/9$.

If $C_2 \le 0$, all four terms in (A.16) are strictly negative. Suppose instead that $C_2 > 0$. By (A.12),

$$\frac{C_2 R}{2x} < \frac{Atz(1+3y)}{2x} \le \frac{(1+\bar{\ell})y\bar{\ell}^2(1+3y)}{8(1-y)^2},$$

where the second inequality uses $A \le x(1+\bar{\ell})$ and $tz \le L^2/4$. Combining this bound with (A.18) gives

$$\frac{C_1 + C_2 R}{2x} < \frac{P(y)}{54(1-y)^2}, \tag{A.19}$$

where

$$P(y) := 48y^5 - 748y^4 + 2682y^3 - 3555y^2 + 1755y - 162.$$

For $y \in [0, 1/9]$,

$$P'(y) \geq 1755 - \frac{7110}{9} - \frac{2992}{729} = \frac{700493}{729} > 0.$$

Moreover, $P(1/9) = -144140/19683 < 0$. Hence, $P(y) < 0$ on $[0, 1/9]$, and (A.19) implies $C_1 + C_2 R < 0$. Together with $C_0 < 0$, $C_3 < 0$, and $R > 0$, this proves (A.15). Therefore,

$$pp_M + p_F < -\frac{rp^2}{(1-\tau_c)(\mu - cF)},$$

which completes the proof. $\square$

Proof of Lemma 2. The result follows immediately from Lemma A.9. $\square$

C.3 Proof of Lemma 1

This subsection derives an alternative closed-form expression for the debt price and uses it to prove the price's sign, monotonicity, curvature, and limiting properties.

Proof of Lemma 1. Equations (A.8) and (A.9) agree at $M = M^\star(F)$ and give the asserted closed form. To establish its sign properties, write

$$\alpha := 1 - \tau_c, \qquad S := \sqrt{(\mu - cF)^2 + 2r\sigma^2}, \qquad x := \frac{\mu - cF}{S},$$

$$\ell := \log\left(\frac{1+x}{1-x}\right), \qquad z := \frac{SM}{\alpha\sigma^2}.$$

Thus $0 \leq z \leq \ell$ in the closure of the retention region and, under Assumption 1, $0 < x \leq 1/3$. Define

$$H(z) := \cosh z - x \sinh z, \quad R(z) := \frac{\sinh z}{H(z)}, \quad A(z) := x + (1 - x^2)(\ell - z).$$

The retention-region price simplifies to

$$p(M, F) = \frac{\alpha c \sigma^2}{S^2} B(z), \qquad B(z) := xz + A(z)R(z). \tag{A.20}$$

Because $H(z) = \sqrt{1 - x^2}\cosh(z - \operatorname{arctanh} x) > 0$, all these functions are regular on $[0, \ell]$.

Let $Q := H'/H$. Direct differentiation gives

$$R' = \frac{1}{H^2}, \qquad B' = x - (1-x^2)R + \frac{A}{H^2}, \qquad B'' = -\frac{2\{(1-x^2) + AQ\}}{H^2}.$$

Moreover, $Q' = (1-x^2)/H^2 > 0$, $Q(0) = -x$, and $0 < A \leq x + (1-x^2)\ell < 3x$, where the last inequality follows from $\operatorname{arctanh}(x) < x/(1-x^2)$. Therefore $(1-x^2) + AQ > 1 - 4x^2 \geq 5/9$, so $B'' < 0$. At $z = \ell$, $H = 1$, $R = 2x/(1-x^2)$, $A = x$, and $B'(\ell) = 0$. Consequently,

$$p_M(M, F) = \frac{c}{S} B'(z) > 0 \quad \text{for } 0 < M < M^\star(F), \qquad p_M(M^\star(F)-, F) = 0,$$

and $p_{MM} < 0$ throughout the retention region. Since $B(0) = 0$, $p(0, F) = 0$ and hence $p(M, F) > 0$ for $M > 0$.

At the payout boundary,

$$p(M^\star(F), F) = \frac{\alpha c}{r}\left[x^2 + x(1-x^2)\operatorname{arctanh}(x)\right] < \frac{2\alpha c}{r} x^2 \leq \frac{2c}{9r} < \frac{c}{r}.$$

Because p is increasing in M below the boundary and constant above it, the same upper bound holds throughout the retention region. Lemma 2 gives $pp_M + p_F < 0$; since $p, p_M > 0$, this implies $p_F < 0$, including the right derivative at $F = 0$.

Finally, the closed forms give well-defined right limits at $F = 0$. As $F \uparrow \bar{F}$, $x \downarrow 0$ and the preceding expression for $p(M^\star(F), F)$ converges to zero. Hence $0 \leq p(M, F) \leq p(M^\star(F), F) \to 0$ uniformly in M, which matches the extension $p = 0$ for $F \geq \bar{F}$ and proves continuity on the entire feasible state space. □

D Equilibrium Construction

The proofs of Propositions 2 and 3 proceed in four subsections. Subsection D.1 establishes the regularity and boundary behavior of the candidate value, price, and leverage rate. Subsection D.2 verifies shareholder optimality, debtholder break-even, and existence. Subsection D.3 identifies the equity value at every feasible position and the debt price and financial policy at every position with positive debt in an arbitrary smooth equilibrium. Subsection D.4 then gives separate proofs of the two propositions.

Throughout this section, coupon income is untaxed, so the candidate debt price and leverage-adjustment rate satisfy (20) and (21). Personal taxation of coupon income is introduced only in Section 6.

D.1 Construction of the Candidate Equilibrium Objects

Proposition 1 and Lemmas A.6, A.8, and 2 construct the candidate equity value V^0, the payout boundary $M^\star$, and the candidate debt-pricing function p, and establish that $p(M,F)p_M(M,F) + p_F(M,F) < 0$ throughout the retention region. The following lemmas establish the additional properties needed for verification. The first establishes the joint regularity and boundary behavior of V^0, $M^\star$, and p; the second proves that the candidate leverage rate is well defined and derives the boundary bounds used to construct its state process.

Lemma A.10 (Regularity and boundary behavior of the candidate objects)**.** *Suppose Assumption 1 holds. The boundary $M^\star$ is infinitely differentiable on $[0,\bar{F})$, and V^0 and p possess the derivatives used in* (20) *and* (21) *in the retention region, with finite one-sided extensions to $F = 0$ and $M = M^\star(F)$. Moreover,*

$$M^\star(F) \to 0, \qquad M^{\star\prime}(F) \to -\frac{(1-\tau_c)c}{r} \quad \text{as } F \uparrow \bar{F}.$$

For $0 < M < M^\star(F)$ and $0 \le F < \bar{F}$, $0 < p < c/r$, $p_M > 0$, and $p_F < 0$. At $M = 0$ and at the payout boundary,

$$p(0,F) = 0, \qquad p_M(M^\star(F)-,F) = 0, \qquad p_F(M^\star(F)-,F) < 0$$

for $F \in [0,\bar{F})$, and

$$\lim_{F\uparrow\bar{F}} \sup_{0\le M\le M^\star(F)} p(M,F) = 0.$$

The same limit holds at every fixed M when the price is extended constantly over the payout region.

Proof. The closed-form boundary is

$$M^\star(F) = \frac{(1-\tau_c)\sigma^2}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \log\left(\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}+\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}-\mu+cF}\right),$$

which is infinitely differentiable for $F < \bar{F}$. Since the terms in this expression are infinitely differentiable for $F < \bar{F}$, direct differentiation gives

$$M^{\star\prime}(F) = \frac{c(1-\tau_c)\sigma^2}{(\mu - cF)^2 + 2r\sigma^2}\left[\frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} \log\left(\frac{\sqrt{(\mu - cF)^2 + 2r\sigma^2} + \mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2} - \mu + cF}\right) - 2\right].$$

Letting $F \uparrow \bar{F}$ gives the two asserted limits.

The closed forms for V^0, $M^\star$, and p are analytic in the interior and have finite one-sided derivatives at $F = 0$ and at the payout boundary. The proof of Lemma 1 establishes $0 < p < c/r$, $p_M > 0$, $p_F < 0$, $p(0, F) = 0$, and $p_M(M^\star(F)-, F) = 0$, as well as the required finite extension of p_{MM}.

It remains only to record the face-value derivative at the payout boundary. Direct differentiation of its price gives

$$\begin{aligned} \frac{d}{dF} p(M^\star(F), F) = -&\frac{(1-\tau_c)c^2}{r\sqrt{(\mu - cF)^2 + 2r\sigma^2}}\left(1 - \frac{(\mu - cF)^2}{(\mu - cF)^2 + 2r\sigma^2}\right) \\ &\times \left[\frac{3(\mu - cF)}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} + \left(1 - \frac{3(\mu - cF)^2}{(\mu - cF)^2 + 2r\sigma^2}\right)\right. \\ &\qquad \left. \times \operatorname{arctanh}\left(\frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}}\right)\right] < 0. \end{aligned}$$

Because $p_M(M^\star(F)-, F) = 0$, the chain rule identifies this derivative with $p_F(M^\star(F)-, F)$. Finally, monotonicity in M implies

$$\sup_{0 \le M \le M^\star(F)} p(M, F) = p(M^\star(F), F),$$

and the formula for $p(M^\star(F), F)$ in the proof of Lemma 1 tends to zero as $F \uparrow \bar{F}$. This proves the remaining claims. □

Lemma A.11 (Regularity of the candidate leverage-adjustment rate)**.** *Suppose that Assumption 1 holds. For* $0 < M < M^\star(F)$ *and* $0 < F < \bar{F}$*, the candidate leverage-adjustment rate* $G^\star$ *is well defined and is given by* (21)*.*

For every $M \in (0, M^\star(0))$*, the limit*

$$\lim_{F \downarrow 0} G^\star(M, F)$$

exists. Moreover, for every $F \in (0, \bar{F})$,

$$\lim_{M \uparrow M^\star(F)} G^\star(M, F) = \frac{rp(M^\star(F), F) - \tau_c c}{p_F(M^\star(F)-, F)},$$

where derivatives at the payout boundary are understood as limits from the retention region. The corresponding limit at the zero-leverage corner also exists and satisfies

$$\lim_{\substack{(M,F)\to(M^\star(0),0) \\ F>0,\ 0<M<M^\star(F)}} G^\star(M, F) = \frac{rp(M^\star(0), 0) - \tau_c c}{p_F(M^\star(0)-, 0)}.$$

Finally, for every compact set $K \subset [0, \bar{F})$, *uniformly over* $F \in K \cap (0, \bar{F})$,

$$G^\star(M, F) = O\left(\frac{1}{M}\right)$$

and

$$p(M, F)G^\star(M, F) = O(1)$$

as $M \downarrow 0$.

Proof. Define $D(M, F) := p(M, F)p_M(M, F) + p_F(M, F)$ and

$$N(M, F) := rp(M, F) - c - (1 - \tau_c)(\mu - cF)p_M(M, F) - \frac{1}{2}(1 - \tau_c)^2\sigma^2 p_{MM}(M, F).$$

By construction, the candidate leverage-adjustment rate solves (20). By Lemma 2, $D(M, F) < 0$ throughout the retention region. Therefore, (20) has the unique solution $G^\star(M, F) = \frac{N(M,F)}{D(M,F)}$, which is precisely (21).

We first derive an equivalent expression for the numerator. In the retention region, the candidate equity value satisfies $rV^0 = (1 - \tau_c)(\mu - cF)V^0_M + \frac{1}{2}(1 - \tau_c)^2\sigma^2 V^0_{MM}$. Differentiating this equation with respect to F, using $V^0_F = -pV^0_M$, and then using the derivative of the same equation with respect to M gives

$$(1 - \tau_c)(\mu - cF)p_M + \frac{1}{2}(1 - \tau_c)^2\sigma^2 p_{MM} = -(1 - \tau_c)c - (1 - \tau_c)^2\sigma^2 p_M \frac{V^0_{MM}}{V^0_M}. \tag{A.21}$$

Consequently,

$$N(M, F) = rp(M, F) - \tau_c c + (1 - \tau_c)^2\sigma^2 p_M(M, F)\frac{V^0_{MM}(M, F)}{V^0_M(M, F)},$$

and hence

$$G^\star(M,F) = \frac{rp(M,F) - \tau_c c + (1-\tau_c)^2\sigma^2 p_M(M,F)\frac{V^0_{MM}(M,F)}{V^0_M(M,F)}}{p(M,F)p_M(M,F) + p_F(M,F)}. \tag{A.22}$$

Fix $M \in (0, M^\star(0))$. Since $M^\star$ is continuous at zero, $M < M^\star(F)$ for all sufficiently small $F > 0$. By Lemma A.10, every term on the right-hand side of (A.22) admits a finite right-hand limit as $F \downarrow 0$. Moreover, Lemma 2 gives $p(M,0)p_M(M,0) + p_F(M,0) < 0$, where $p_F(M,0)$ denotes the right derivative. Therefore,

$$\lim_{F\downarrow 0} G^\star(M,F) = \frac{rp(M,0) - \tau_c c + (1-\tau_c)^2\sigma^2 p_M(M,0)\frac{V^0_{MM}(M,0)}{V^0_M(M,0)}}{p(M,0)p_M(M,0) + p_F(M,0)},$$

which is finite.

Next, fix $F \in (0, \bar{F})$. The boundary conditions for the candidate equity value give $V^0_M(M^\star(F), F) = 1 - \tau_p$, $V^0_{MM}(M^\star(F)-, F) = 0$. Lemma A.10 also gives $p_M(M^\star(F)-, F) = 0$ and $p_F(M^\star(F)-, F) < 0$. Taking $M \uparrow M^\star(F)$ in (A.22) therefore yields

$$\lim_{M\uparrow M^\star(F)} G^\star(M,F) = \frac{rp(M^\star(F),F) - \tau_c c}{p_F(M^\star(F)-,F)}.$$

The closed-form expressions are jointly continuous as (M,F) approaches $(M^\star(0), 0)$ from within the retention region. Moreover, direct differentiation of the debt price at the payout boundary gives

$$p_F(M^\star(0)-,0) = -\frac{(1-\tau_c)c^2(1-x_0^2)}{r\sqrt{\mu^2+2r\sigma^2}}\left[3x_0 + (1-3x_0^2)\operatorname{arctanh}(x_0)\right] < 0,$$

where $x_0 := \frac{\mu}{\sqrt{\mu^2+2r\sigma^2}} < \frac{1}{3}$. Taking the joint limit in (A.22) consequently gives

$$\lim_{\substack{(M,F)\to(M^\star(0),0)\\ F>0,\ 0<M<M^\star(F)}} G^\star(M,F) = \frac{rp(M^\star(0),0) - \tau_c c}{p_F(M^\star(0)-,0)}.$$

It remains to establish the asymptotic bounds as $M \downarrow 0$. Fix a compact set $K \subset [0, \bar{F})$. Because $M^\star$ is continuous and strictly positive on K, $\inf_{F\in K} M^\star(F) > 0$. Thus, the closed-form expression for p is jointly analytic on a common neighborhood of $\{0\} \times K$. For $F \in K$, write $S(F) := \sqrt{(\mu - cF)^2 + 2r\sigma^2}$, $\quad x(F) := \frac{\mu - cF}{S(F)}$. Assumption 1 implies $0 \le x(F) \le x(0) < \frac{1}{3}$. Expanding (A.8) around $M = 0$ gives, uniformly over $F \in K$, $p(M,F) = a_1(F)M + O_K(M^2)$,

where

$$a_1(F) = \frac{2c}{S(F)} \left[x(F) + \big(1 - x(F)^2\big) \operatorname{arctanh}\big(x(F)\big) \right].$$

Joint analyticity also gives $p_M(M,F) = a_1(F) + O_K(M)$ and $p_F(M,F) = a_1'(F)M + O_K(M^2)$. Direct differentiation yields

$$a_1'(F) = \frac{c^2}{S(F)^2} \left[-4 + 6x(F)^2 + 6x(F)\big(1 - x(F)^2\big) \operatorname{arctanh}\big(x(F)\big) \right].$$

It follows that

$$\begin{aligned} D(M,F) &= \left[a_1(F)^2 + a_1'(F) \right] M + O_K(M^2), \\ a_1(F)^2 + a_1'(F) &= \frac{c^2}{S(F)^2} \Big[-4 + 10x(F)^2 \\ &\qquad + 14x(F)\big(1 - x(F)^2\big) \operatorname{arctanh}\big(x(F)\big) \\ &\qquad + 4\big(1 - x(F)^2\big)^2 \operatorname{arctanh}\big(x(F)\big)^2 \Big]. \end{aligned}$$

For $0 \leq x < 1$, the inequality $\operatorname{arctanh}(x) < \frac{x}{1-x^2}$ implies

$$-4 + 10x^2 + 14x(1 - x^2) \operatorname{arctanh}(x) + 4(1 - x^2)^2 \operatorname{arctanh}(x)^2 < -4 + 28x^2.$$

Since $x(F) \leq x(0) < 1/3$, $-4 + 28x(F)^2 \leq -4 + 28x(0)^2 < -\frac{8}{9}$. Therefore, there exist constants $d_K > 0$ and $\varepsilon_K > 0$ such that $|D(M,F)| \geq d_K M$ whenever $F \in K \cap (0, \bar{F})$ and $0 < M < \varepsilon_K$.

By Lemma A.10, $rp - \tau_c c + (1-\tau_c)^2 \sigma^2 p_M \frac{V^0_{MM}}{V^0_M}$ is uniformly bounded on the same neighborhood; here we use $V^0_M \geq 1 - \tau_p > 0$. Equation (A.22) thus implies $|G^\star(M,F)| \leq \frac{C_K}{M}$ for some constant $C_K < \infty$. Hence, $G^\star(M,F) = O\left(\frac{1}{M}\right)$ uniformly over $F \in K \cap (0, \bar{F})$.

Finally, the expansion of p gives $|p(M,F)| \leq \widetilde{C}_K M$ for all sufficiently small M, uniformly over $F \in K$. Therefore, $|p(M,F)G^\star(M,F)| \leq \widetilde{C}_K C_K$, which proves $p(M,F)G^\star(M,F) = O(1)$. □

D.2 Verification of a Candidate Smooth Equilibrium

We now verify the optimality of the candidate equity value and financial policy among the full set of admissible smooth financial policies. We first prove a shareholder upper bound, then construct the candidate reflected state process and establish its well-posedness. We next verify shareholder optimality and debtholder break-even, and finally combine those results to establish existence of a smooth equilibrium. Throughout this subsubsection, we fix the candidate equity value V^0 and the candidate debt-pricing function $p(M,F) = -\frac{V^0_F(M,F)}{V^0_M(M,F)}$ constructed above, with the one-sided and continuous extensions specified in Lemma A.8.

Lemma A.12 (Full-policy shareholder verification)**.** *Suppose that Assumption 1 holds. For every feasible initial financial position (M,F) and every admissible smooth financial policy $\{L_t, I_t, G_t\}_{t\geq 0}$,*

$$\mathbb{E}\left[\int_0^{\widetilde{T}_b} e^{-rt}\left\{(1-\tau_p)\,dL_t - dI_t\right\}\,\middle|\, M_{0^-} = M,\ F_{0^-} = F\right] \leq V^0(M,F), \tag{A.23}$$

where $\widetilde{T}_b$ is the bankruptcy time induced by the financial policy.

The bound is attained by an admissible financial policy. If $F \in [0,\bar{F})$, one such policy sets $G_t = 0$ and $I_t = 0$, pays the minimal dividends needed to reflect cash reserves at $M^\star(F)$, and enters bankruptcy when cash reserves first reach zero. If $F \geq \bar{F}$, it immediately distributes all cash reserves and enters bankruptcy. Consequently, the value of the shareholders' full financial policy problem under the candidate debt-pricing function equals $V^0(M,F)$.

Proof. Proposition 1 and Lemma A.6 imply

$$V^0 \geq 0, \qquad 1-\tau_p \leq V^0_M \leq 1, \qquad \mathcal{L}V^0 - rV^0 \leq 0 \tag{A.24}$$

at every feasible financial position with positive cash reserves. By the definition of the candidate debt-pricing function,

$$p(M,F)V^0_M(M,F) + V^0_F(M,F) = 0 \tag{A.25}$$

at every feasible financial position, with the appropriate one-sided derivative at a face-value boundary. For $F \geq \bar{F}$, this equality continues to hold because $p(M,F) = 0$ and $V^0(M,F) = (1-\tau_p)M$.

The value-matching, smooth-pasting, and super-contact conditions imply that V^0, V^0_M, and V^0_{MM} match their payout-region extensions at $M = M^\star(F)$. Differentiating value matching along the payout boundary shows that V^0_F also matches there. Moreover, Lemma A.10 gives $M^{\star\prime}(F) \longrightarrow -\frac{(1-\tau_c)c}{r}$ as $F \uparrow \bar{F}$, so the payout-region formula implies $V^0_F(M,F) \longrightarrow 0$ as $F \uparrow \bar{F}$, consistently with the high-leverage extension. Thus, the generalized Itô formula applies to V^0 along every localized admissible state process without a local-time term at either $M = M^\star(F)$ or $F = \bar{F}$.

Fix an arbitrary admissible financial policy. Let ρ_n be a sequence of stopping times increasing to $\widetilde{T}_b$ such that, before ρ_n, the state remains in a compact subset of the positive-cash region and the accumulated variation of the controls is bounded. For $T > 0$, set $\theta_{n,T} := \rho_n \wedge T$. Applying the generalized Itô formula to $e^{-rt}V^0(M_t, F_t)$, the absolutely continuous finite-variation term is

$$e^{-rt}\left\{\mathcal{L}V^0 - rV^0 + G_t\left[pV^0_M + V^0_F\right]\right\}dt,$$

which is nonpositive by (A.24) and (A.25). The continuous dividend and equity-issuance terms, after adding the shareholders' contemporaneous cash flows, equal

$$e^{-rt}\left\{\left[(1-\tau_p) - V^0_M\right]dL^c_t + \left[V^0_M - 1\right]dI^c_t\right\} \le 0.$$

At a jump time of the liquidity-management policy, F does not jump and $\Delta M_t = \Delta I_t - \Delta L_t$. The gradient bounds in (A.24) give

$$V^0(M_t, F_t) - V^0(M_{t^-}, F_t) + (1-\tau_p)\Delta L_t - \Delta I_t \le 0. \tag{A.26}$$

Indeed, if $\Delta M_t \ge 0$, the increase in V^0 is no greater than ΔM_t; if $\Delta M_t < 0$, the decrease in V^0 is no greater than $(1-\tau_p)\Delta M_t$. This proves (A.26) in both cases.

Combining these inequalities with the jump inequality (A.26), and noting that the stopped stochastic integral has zero expectation, gives

$$\begin{aligned} V^0(M, F) \ge\ & \mathbb{E}\left[e^{-r\theta_{n,T}}V^0(M_{\theta_{n,T}}, F_{\theta_{n,T}})\right] \\ & + \mathbb{E}\left[\int_0^{\theta_{n,T}} e^{-rt}\left\{(1-\tau_p)\,dL_t - dI_t\right\}\right]. \end{aligned} \tag{A.27}$$

Since $V^0 \ge 0$, it follows that

$$\mathbb{E}\left[\int_0^{\theta_{n,T}} e^{-rt}\left\{(1-\tau_p)\,dL_t - dI_t\right\}\right] \le V^0(M, F). \tag{A.28}$$

The admissibility conditions imply

$$\begin{aligned} \lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\int_0^{\theta_{n,T}} e^{-rt}\,dL_t\right] &= \mathbb{E}\left[\int_0^{\widetilde{T}_b} e^{-rt}\,dL_t\right], \\ \lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\int_0^{\theta_{n,T}} e^{-rt}\,dI_t\right] &= \mathbb{E}\left[\int_0^{\widetilde{T}_b} e^{-rt}\,dI_t\right]. \end{aligned}$$

Taking these limits in (A.28) proves (A.23).

It remains to establish attainment. Suppose first that $F \in [0, \bar{F})$. Under the stated no-trade barrier policy, the cash process is the standard one-dimensional diffusion reflected downward at $M^\star(F)$ and killed at zero. In the retention region, $\mathcal{L}V^0 - rV^0 = 0$; on the support of dividends, $V^0_M = 1 - \tau_p$; and at bankruptcy, $V^0(0, F) = 0$. Hence, every inequality used above binds. Applying Itô's formula along this policy and using the boundedness of V^0 on $[0, M^\star(F)] \times \{F\}$ gives

$$V^0(M, F) = \mathbb{E}\left[\int_0^{\widetilde{T}_b} e^{-rt}(1 - \tau_p)\, dL_t\right],$$

including the initial dividend $(M - M^\star(F))^+$ when necessary. If $F \geq \bar{F}$, immediate payout followed by bankruptcy yields $(1 - \tau_p)M = V^0(M, F)$. Thus, the upper bound is attained at every feasible initial financial position. □

Because the policy attaining the upper bound satisfies $G_t \equiv 0$, the same argument also verifies that V^0 coincides with the value of the auxiliary no-trade financial-policy problem. We next define the candidate equilibrium policy. From an initial position with $0 < F < \bar{F}$, the firm first distributes $(M - M^\star(F))^+$ and thereafter follows

$$dF_t = G^\star(M_t, F_t)\, dt, \tag{A.29}$$

$$dM_t = [(1 - \tau_c)(\mu - cF_t) + p(M_t, F_t)G^\star(M_t, F_t)]\, dt + (1 - \tau_c)\sigma\, dZ_t - dL^\star_t, \tag{A.30}$$

where $L^\star$ gives minimal downward reflection at $M = M^\star(F)$. The firm never issues equity. If cash reserves reach zero, the firm enters bankruptcy. If the face value reaches zero first, the firm sets $G^\star = 0$ thereafter and follows the optimal no-trade liquidity-management policy at $F = 0$. From an initial position with $F \geq \bar{F}$, the firm immediately distributes all cash reserves and enters bankruptcy. In particular, the bankruptcy time $\widetilde{T}^\star_b$ is induced by $\{L^\star, 0, G^\star\}$ and is not a separate control.

Lemma A.13 (Well-posed candidate state dynamics)**.** *Under Assumption 1, for every feasible initial financial position, equations* (A.29)–(A.30)*, together with the stated boundary rules, admit a pathwise unique, nonexplosive strong solution. The solution is strong Markov. Before bankruptcy,*

$$0 \leq F_t < \bar{F}, \qquad 0 \leq M_t \leq M^\star(F_t),$$

and the dividend policy satisfies

$$\int_0^{\widetilde{T}_b^\star} \mathbf{1}_{\{M_t < M^\star(F_t)\}}\, dL_t^\star = 0. \tag{A.31}$$

Moreover,

$$\int_0^{T\wedge \widetilde{T}_b^\star} |G^\star(M_t, F_t)|\, dt < \infty \qquad \text{a.s. for every } T < \infty. \tag{A.32}$$

Consequently, F is absolutely continuous, the candidate policy is state-feasible, and the stated boundary continuations can be pasted to the positive-leverage solution.

Proof. We proceed in four steps.

Step 1: Uniform bounds on the candidate coefficients. Lemma A.11 already proves the required bounds uniformly when F belongs to a compact subset of $[0, \bar{F})$. It remains to control the collapsing corner $(M, F) = (0, \bar{F})$.

For $0 \le \theta \le 1$, substitute $M = \theta M^\star(F)$ into the closed-form formulas for $M^\star$, p, p_M, and p_{MM}. A Taylor expansion as $F \uparrow \bar{F}$, uniform in θ, gives

$$M^\star(F) = \frac{(1-\tau_c)(\mu - cF)}{r} + O\left((\mu - cF)^3\right), \tag{A.33}$$

$$p\big(\theta M^\star(F), F\big) = \frac{c(1-\tau_c)(\mu - cF)^2}{r^2\sigma^2}\left(2\theta - \theta^2\right) + O\left((\mu - cF)^3\right), \tag{A.34}$$

$$p_M\big(\theta M^\star(F), F\big) = \frac{2c(\mu - cF)}{r\sigma^2}(1-\theta) + O\left((\mu - cF)^2\right), \tag{A.35}$$

$$p_{MM}\big(\theta M^\star(F), F\big) = -\frac{2c}{(1-\tau_c)\sigma^2} + O(\mu - cF). \tag{A.36}$$

For $\theta > 0$, differentiating the same closed-form formula for p with respect to F, holding M fixed, yields

$$\frac{p\big(\theta M^\star(F), F\big) p_M\big(\theta M^\star(F), F\big) + p_F\big(\theta M^\star(F), F\big)}{\theta M^\star(F)} = -\frac{2c^2}{r\sigma^2} + O(\mu - cF). \tag{A.37}$$

The quotient in (A.37) has the same continuous extension at $\theta = 0$. Moreover, the numerator of (21) satisfies

$$rp - c - (1-\tau_c)(\mu - cF)p_M - \frac{1}{2}(1-\tau_c)^2\sigma^2 p_{MM} = -\tau_c c + O(\mu - cF) \tag{A.38}$$

uniformly in θ. Equations (A.37) and (A.38) therefore imply

$$MG^{\star}(M,F) \longrightarrow \frac{\tau_c r \sigma^2}{2c} \tag{A.39}$$

as $(M,F) \to (0,\bar{F})$ from the retention region, uniformly in the normalized cash position. Equation (A.34) also implies $\frac{p(M,F)}{M} \longrightarrow 0$ at the same corner.

Away from this corner, Lemmas A.10, A.11, and 2 imply that $MG^{\star}$ and p/M extend continuously to the compactified closure of the retention region. At $M = M^{\star}(F)$, we use $p_M(M^{\star}(F)-,F) = 0$ and $p_F(M^{\star}(F)-,F) < 0$. At $M = 0$, we use the expansion as $M \downarrow 0$ in the proof of Lemma A.11. Together with (A.39), these observations give a finite constant C such that

$$|G^{\star}(M,F)| \le \frac{C}{M}, \qquad 0 \le p(M,F) \le CM, \qquad |p(M,F)G^{\star}(M,F)| \le C \tag{A.40}$$

throughout the positive-leverage retention region.

Step 2: Local reflected solutions. Fix a compact set $K \subset (0,\bar{F})$ and $\varepsilon \in (0, \inf_{F\in K} M^{\star}(F))$. Direct differentiation of the closed-form expressions used in the proof of Lemma A.10 shows that the numerator and denominator of (21) admit continuously differentiable one-sided extensions to $\{(M,F) : F \in K,\ \varepsilon \le M \le M^{\star}(F)\}$. At the payout boundary, Lemma A.10 gives

$$\begin{aligned} &p(M^{\star}(F),F)p_M(M^{\star}(F)-,F) + p_F(M^{\star}(F)-,F) \\ &\quad = p_F(M^{\star}(F)-,F) < 0. \end{aligned}$$

The denominator in (21) is strictly negative in the interior by Lemma 2. Hence, by continuity and compactness, it is bounded away from zero on the displayed set. It follows that $G^{\star}$ and $pG^{\star}$, and therefore all coefficients in (A.29)–(A.30), are locally Lipschitz on every localization bounded away from $M = 0$, $F = 0$, and $F = \bar{F}$. Since F has finite variation on such a localization,

$$\begin{aligned} d\left[M_t - M^{\star}(F_t)\right] = &\Big[(1-\tau_c)(\mu - cF_t) + \big(p(M_t,F_t) - M^{\star\prime}(F_t)\big)G^{\star}(M_t,F_t)\Big]dt \\ &+ (1-\tau_c)\sigma\, dZ_t - dL_t^{\star}. \end{aligned}$$

The one-sided Skorokhod map on $(-\infty, 0]$ is Lipschitz. Applying this map to the preceding equation and using Picard iteration therefore gives a pathwise unique strong solution up to the first exit from each localized domain. The Skorokhod regulator is continuous after the possible initial dividend and satisfies (A.31).

Step 3: Occupation near the bankruptcy boundary. By (A.40), the drift $(1-\tau_c)(\mu-cF_t)+p(M_t,F_t)G^\star(M_t,F_t)$ in the cash dynamics is uniformly bounded before bankruptcy. Let τ_n denote localization before M falls below $1/n$ or F approaches either face-value boundary. Applying Itô's formula to $\sqrt{M_t}$ up to $T\wedge\tau_n$ gives

$$\begin{aligned} d\sqrt{M_t} = {} & \frac{(1-\tau_c)(\mu-cF_t)+p(M_t,F_t)G^\star(M_t,F_t)}{2\sqrt{M_t}}dt \\ & + \frac{(1-\tau_c)\sigma}{2\sqrt{M_t}}\,dZ_t - \frac{1}{2\sqrt{M_t}}\,dL_t^\star - \frac{(1-\tau_c)^2\sigma^2}{8M_t^{3/2}}dt. \end{aligned}$$

The reflection term is nonpositive. Taking expectations, using the bounded cash drift, and absorbing the resulting integral of $M_t^{-1/2}$ into the negative $M_t^{-3/2}$ term yield, for every $T<\infty$,

$$\sup_n \mathbb{E}\left[\int_0^{T\wedge\tau_n} \frac{dt}{M_t^{3/2}}\right] < \infty. \tag{A.41}$$

Indeed, the absorption follows from $x^{-1/2}\le \varepsilon x^{-3/2}+C_\varepsilon, \qquad x>0$, for a sufficiently small $\varepsilon>0$. By monotone convergence, (A.41) continues to hold up to the first time at which cash reaches zero or a face-value boundary is reached. Since cash reserves are bounded above by $M^\star(0)$, (A.41) implies

$$\int_0^{T\wedge\widetilde{T}_b^\star} \frac{dt}{M_t} < \infty \qquad \text{a.s.} \tag{A.42}$$

If F reaches zero while cash reserves remain positive, the same estimate applies to the subsequently pasted no-trade reflected diffusion at $F=0$; in any case, $G^\star=0$ after that time. Thus, (A.42) holds over the entire interval up to bankruptcy. Combining (A.40) and (A.42) proves (A.32).

Step 4: Passage to the boundary and pasting. Equation (A.32) implies that F has finite total variation on every finite interval before bankruptcy and therefore has a limit whenever a localized solution reaches the boundary of the retention region. Since $M^{\star\prime}$ has a finite extension at $\bar{F}$, the path $M^\star(F)$ also has finite variation on such an interval. The one-sided Skorokhod map then implies that $L^\star$ remains finite on every finite interval, so M has a limit as well. If the limit of F is zero while cash reserves remain positive, we paste the unique no-trade reflected cash process at $F=0$ and set $G^\star=0$ thereafter. If F approaches $\bar{F}$, then $M_t\le M^\star(F_t)$ and $M^\star(F_t)\to 0$, so cash reaches zero simultaneously and the firm enters bankruptcy. If cash reaches zero first, the firm likewise enters bankruptcy. These continuations preserve feasibility and do not permit a finite-time explosion.

Localized pathwise uniqueness, the Lipschitz property of the Skorokhod map, and uniqueness of each pasted boundary continuation imply global pathwise uniqueness.

It remains to verify the strong Markov property. Fix a bounded $\mathbb{F}$-stopping time $\tau < \widetilde{T}_b^\star$. Conditional on $\mathcal{F}_\tau$, the shifted process $\{Z_{\tau+s} - Z_\tau\}_{s\geq 0}$ is a Brownian motion independent of $\mathcal{F}_\tau$. After time τ, the state process and the increment $\{L^\star_{\tau+s} - L^\star_\tau\}_{s\geq 0}$ satisfy the same time-homogeneous reflected equations starting from (M_τ, F_τ). Pathwise uniqueness therefore implies that their conditional law depends on the pre-τ history only through (M_τ, F_τ). Localization extends this restart property to arbitrary stopping times. The zero-leverage and bankruptcy continuations likewise depend only on the financial position at which the relevant boundary is reached. Hence, the pasted state process is strong Markov. This completes the proof. □

Lemma A.14 (Shareholder optimality of the candidate policy)**.** *Under Assumption 1, the candidate financial policy $\{L^\star, 0, G^\star\}$ is admissible, attains $V^0(M,F)$ at every feasible initial financial position, and is optimal against every admissible smooth financial-policy deviation.*

Proof. Admissibility follows from Lemma A.13. Apply Itô's formula to $e^{-rt}V^0(M_t, F_t)$ along the candidate state process. In the retention region, $\mathcal{L}V^0 - rV^0 = 0$, and the leverage-adjustment term vanishes because $G^\star \left[pV^0_M + V^0_F\right] = 0$. On the support of $dL^\star_t$, $V^0_M = 1 - \tau_p$, and at bankruptcy, $V^0(0,F) = 0$. If F reaches zero before bankruptcy, the pasted no-trade policy at $F = 0$ attains $V^0(\cdot, 0)$ by Lemma A.12.

Before bankruptcy, the reflected state satisfies $0 \leq F_t < \bar{F}, 0 \leq M_t \leq M^\star(F_t) \leq M^\star(0)$. Thus, V^0 is bounded along the candidate path and

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-r(T\wedge \widetilde{T}_b^\star)} V^0\left(M_{T\wedge \widetilde{T}_b^\star}, F_{T\wedge \widetilde{T}_b^\star}\right)\right] = 0.$$

Consequently, all inequalities in the proof of Lemma A.12 bind along the candidate policy, and its payoff equals $V^0(M,F)$. The upper bound in that lemma then proves optimality against every admissible smooth financial-policy deviation. □

Lemma A.15 (Debtholder's break-even condition). *Suppose that Assumption 1 holds. Under the candidate financial policy, the candidate debt-pricing function satisfies the break-even condition* (6) *at every financial position with $F > 0$. If the aggregate face value of debt is smoothly reduced to zero before bankruptcy, every representative unit outstanding beforehand is sold before aggregate debt is exhausted almost surely. Hence, no separate final-redemption price is required.*

Proof. When $F_t > 0$ and the firm repurchases debt, the conditional instantaneous sale intensity of a representative outstanding unit is $\frac{\left(-G^\star(M_t,F_t)\right)^+}{F_t}$. By the definition of $G^\star$, the candidate debt price satisfies

$$\begin{aligned} rp = c &+ \left[(1-\tau_c)(\mu - cF) + pG^\star\right] p_M \\ &+ \frac{1}{2}(1-\tau_c)^2\sigma^2 p_{MM} + G^\star p_F \end{aligned} \tag{A.43}$$

throughout the positive-leverage retention region. Before sale or bankruptcy, the local break-even equation for a representative unit is

$$\begin{aligned} \left(r + \frac{(-G^\star)^+}{F}\right) p = c &+ \left[(1-\tau_c)(\mu - cF) + pG^\star\right] p_M \\ &+ \frac{1}{2}(1-\tau_c)^2\sigma^2 p_{MM} + G^\star p_F + \frac{(-G^\star)^+}{F} p. \end{aligned}$$

The two sale-intensity terms cancel because a selected unit is sold at its contemporaneous market price. The remaining equation is exactly (A.43).

The finite-variation term generated by dividend reflection vanishes because $p_M(M^\star(F)-, F) = 0$. The bankruptcy term vanishes because $p(0, F) = 0$, and the bound $0 \le p < c/r$ gives the required transversality. Applying Itô's formula up to sale or bankruptcy, localizing, and then using bounded convergence therefore yields

$$p(M_t, F_t) = \mathbb{E}_t\left[\int_t^{\widetilde{T}_b^\star \wedge \widetilde{T}_S} e^{-r(s-t)} c\, ds + e^{-r(\widetilde{T}_S - t)} p(M_{\widetilde{T}_S^-}, F_{\widetilde{T}_S^-}) \mathbf{1}_{\{\widetilde{T}_S < \widetilde{T}_b^\star\}}\right],$$

which is (6) under the candidate financial policy.

It remains to consider smooth exhaustion of aggregate debt. Let $\widetilde{T}_0$ be the first time after t at which F reaches zero. For $t < u < \widetilde{T}_0$,

$$\log\left(\frac{F_u}{F_t}\right) = \int_t^u \frac{\left(G^\star(M_s, F_s)\right)^+}{F_s} ds - \int_t^u \frac{\left(-G^\star(M_s, F_s)\right)^+}{F_s} ds.$$

Therefore,

$$\int_t^u \frac{\left(-G^\star(M_s, F_s)\right)^+}{F_s}\, ds \geq -\log\left(\frac{F_u}{F_t}\right).$$

As $u \uparrow \widetilde{T}_0$, the right-hand side tends to infinity. The conditional probability that the representative unit remains unsold until $\widetilde{T}_0$ is consequently

$$\exp\left(-\int_t^{\widetilde{T}_0} \frac{\left(-G^\star(M_s, F_s)\right)^+}{F_s}\, ds\right) = 0.$$

Thus, the unit is sold before aggregate debt is exhausted almost surely, which proves the final assertion. □

Lemma A.16 (Existence of a smooth equilibrium)**.** *Under Assumption 1, the candidate equity value V^0, the candidate debt-pricing function p, and the candidate financial policy described above constitute a smooth equilibrium.*

Proof. Lemma A.13 provides a well-posed admissible strong Markov state process. Lemma A.14 shows that the prescribed financial policy attains V^0 and is optimal against every admissible smooth financial-policy deviation. Lemma A.15 establishes debtholder break-even under the proportional trading protocol whenever $F > 0$.

At $F = 0$, no debt is outstanding, so the debtholders' break-even condition does not apply. Lemma A.12 shows that the absorbing selection $G^\star = 0$ is sequentially optimal even against deviations that begin issuing debt. When $F \geq \bar{F}$, immediate payout followed by bankruptcy attains $V^0(M, F) = (1 - \tau_p)M$. Lemma A.10 and the argument in Lemma A.13 verify the regularity and state-process requirements imposed on smooth equilibria. Hence, the candidate objects satisfy all the requirements of a smooth equilibrium. □

D.3 Characterization of Smooth Equilibria at Positive Debt

Having verified existence, we now identify the equity value at every feasible financial position and the debt price and financial policy at positions with positive debt in an arbitrary smooth equilibrium. The following identification lemma first invokes the no-trade equity valuation result to recover the equity value, then uses shareholder indifference and debtholder break-even to identify the price and leverage rate at positive debt, and finally pins down dividends, equity issuance, and bankruptcy there. The next subsection uses this lemma to prove Proposition 2.

Lemma A.17 (Equilibrium identification)**.** *Suppose Assumption 1 holds. In every smooth Markov perfect equilibrium,* $V = V^0$ *at every feasible financial position, and equity is never issued. At every position with* $F > 0$*,*

$$p(M, F) = -\frac{V_F^0(M, F)}{V_M^0(M, F)},$$

If $0 < F < \bar{F}$*, the leverage-adjustment rate equals* $G^\star$ *throughout the retention region, dividends minimally reflect cash reserves at* $M^\star(F)$*, and bankruptcy occurs when cash reaches zero. If* $F \geq \bar{F}$*, the debt price is zero and shareholders immediately distribute all cash and enter bankruptcy.*

Proof. Fix an arbitrary smooth equilibrium. The localized leverage deviation derived in Subsection 4.1 implies

$$pV_M + V_F = 0$$

at every interior face value: otherwise an arbitrarily large locally feasible positive or negative leverage rate would improve shareholder value. At $F = 0$ and $F = F^{\max}$, the corresponding one-sided inequalities hold. The dividend, equity-issuance, and continuation deviations derived in Subsection 4.2, together with the dynamic programming principle, give

$$\max\{\mathcal{L}V - rV,\, 1 - \tau_p - V_M,\, V_M - 1\} = 0$$

for $M > 0$, and $\max\{-V(0, F), V_M(0^+, F) - 1\} = 0$. The equilibrium policy attains its value by definition. Lemma A.7 therefore applies at every face value F and yields $V(M, F) = V^0(M, F)$. Lemma A.6 then gives $V_M^0 < 1$, so equity issuance is strictly suboptimal.

Suppose $0 < F < \bar{F}$. Since $V_M^0 > 0$, the indifference condition identifies $p = -V_F^0/V_M^0$. In the retention region, the debtholder pricing equation gives

$$G(M, F)\big(pp_M + p_F\big) = rp - c - (1 - \tau_c)(\mu - cF)p_M - \frac{1}{2}(1 - \tau_c)^2\sigma^2 p_{MM}.$$

Lemma 2 makes the coefficient on G strictly negative, so this equation has the unique solution $G = G^\star$ in (21).

Below $M^\star(F)$, $V_M^0 > 1 - \tau_p$, so dividends are strictly suboptimal. Above the boundary, $\mathcal{L}V^0 - rV^0 = -r(1 - \tau_p)(M - M^\star(F)) < 0$; hence any initial excess cash must be paid immediately.

Smooth contact and the same strict inequalities imply that subsequent dividends are precisely the minimal reflection at $M^\star(F)$. At zero cash, $V^0(0,F) = 0$ and $V^0_M(0^+,F) < 1$, so continuation through equity issuance is strictly suboptimal. Since the cash diffusion is nondegenerate, the firm must enter bankruptcy when cash first reaches zero. Earlier bankruptcy is suboptimal because $V^0(M,F) > (1-\tau_p)M$ for $M > 0$. Zero recovery gives $p(0,F) = 0$.

Finally, if $F \geq \bar{F}$, the no-trade equity valuation gives $V^0(M,F) = (1-\tau_p)M$. Immediate payout and bankruptcy attain this value, while any positive continuation time has a strictly negative continuation residual. The absence of future coupons and recovery then implies $p(M,F) = 0$. Since $V^0_F(M,F) = 0$ and $V^0_M(M,F) = 1-\tau_p$ in this region, the formula $p = -V^0_F/V^0_M$ also holds. This establishes every stated policy and price property. □

D.4 Proofs of Propositions 2 and 3

Proof of Proposition 2. Lemma A.17 establishes the result. □

Proof of Proposition 3. Lemma A.16 establishes the result. □

E Proofs for Section 5

This section proves the baseline leverage-policy decomposition and the two propositions in Section 5. The proof first establishes a monotonicity result for the leverage-policy numerator and then uses its boundary values to characterize the issuance and repurchase regions. The proofs of the interest-income-tax and asset-pricing results appear in the Internet Appendix.

E.1 Proofs of the Baseline Leverage-Policy Results

We first prove the decomposition in Lemma 3. We then establish a monotonicity result for its numerator and use the boundary values of that numerator to prove Propositions 4 and 5.

Proof of Lemma 3. Equation (A.22), established in the proof of Lemma A.11, implies

$$G^\star(pp_M + p_F) = -\tau_c c + rp + (1-\tau_c)^2\sigma^2 p_M \frac{V_{MM}}{V_M},$$

where $V = V^0$ by Lemma A.17. This is the algebraic identity in (23).

Inside the retention region, the Brownian innovations are

$$dp_t = (1-\tau_c)\sigma p_M \, dZ_t, \qquad d(V_M)_t = (1-\tau_c)\sigma V_{MM} \, dZ_t.$$

Therefore,

$$\frac{1}{V_M}\frac{d\langle p, V_M\rangle_t}{dt} = (1-\tau_c)^2\sigma^2 p_M \frac{V_{MM}}{V_M},$$

which gives the covariance interpretation stated in the lemma. □

For the two policy propositions, define

$$x(F) := \frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}}, \qquad \mathcal{G}(x) := 2x^2 + x(1-x^2)\log\left(\frac{1+x}{1-x}\right). \tag{A.44}$$

Assumption 1 implies $0 < x(F) \le x(0) < 1/3$ for $F \in [0, \bar{F})$.

Lemma A.18 (Monotonicity of the baseline policy numerator)**.** *For each $F \in (0, \bar{F})$, define*

$$N(M,F) := -\tau_c c + rp(M,F) + (1-\tau_c)^2\sigma^2 p_M(M,F)\frac{V_{MM}(M,F)}{V_M(M,F)}. \tag{A.45}$$

Then $M \mapsto N(M,F)$ extends continuously to $[0, M^\star(F)]$ and is strictly increasing. Its boundary values are

$$N(0,F) = c\left[-\tau_c - 2(1-\tau_c)\mathcal{G}(x(F))\right] < 0, \tag{A.46}$$

$$N(M^\star(F),F) = c\left[-\tau_c + \frac{1-\tau_c}{2}\mathcal{G}(x(F))\right]. \tag{A.47}$$

Moreover,

$$\operatorname{sgn}\big(G^\star(M,F)\big) = -\operatorname{sgn}\big(N(M,F)\big) \tag{A.48}$$

throughout the retention region.

Proof. Fix $F \in (0, \bar{F})$ and write

$$\phi(M) := \frac{V_{MM}(M,F)}{V_M(M,F)}, \qquad q(M) := p_M(M,F).$$

The no-trade HJB for V and its boundary conditions imply

$$\phi' = -\phi^2 - \frac{2(\mu - cF)}{(1-\tau_c)\sigma^2}\phi + \frac{2r}{(1-\tau_c)^2\sigma^2}, \tag{A.49}$$

with

$$\phi(0) = -\frac{2(\mu - cF)}{(1-\tau_c)\sigma^2}, \qquad \phi(M^\star(F)) = 0.$$

The closed-form solution gives

$$-\frac{2(\mu - cF)}{(1-\tau_c)\sigma^2} \le \phi(M) < 0 \qquad \text{for } 0 \le M < M^\star(F).$$

Equation (A.21) gives

$$q' + 2\left[\phi + \frac{\mu - cF}{(1-\tau_c)\sigma^2}\right] q = -\frac{2c}{(1-\tau_c)\sigma^2}, \qquad q(M^\star(F)) = 0. \tag{A.50}$$

Solving backward from the payout boundary yields $q(M) > 0$ for $0 \le M < M^\star(F)$.

Differentiating (A.45) and using (A.49) and (A.50) yields

$$N_M = q\left[3r - 4(1-\tau_c)(\mu - cF)\phi - 3(1-\tau_c)^2\sigma^2\phi^2\right] - 2(1-\tau_c)c\phi. \tag{A.51}$$

The quadratic expression in brackets is concave in ϕ. At the two endpoints of the relevant interval, it equals, respectively,

$$3r \qquad \text{and} \qquad 3r - \frac{4(\mu - cF)^2}{\sigma^2}.$$

Both are strictly positive under Assumption 1. Hence, the bracketed expression is strictly positive throughout the interval. Since $q > 0$ and $\phi < 0$ in the interior, (A.51) gives $N_M > 0$.

The expansion as $M \downarrow 0$ in the proof of Lemma A.11 gives

$$p_M(0, F) = \frac{2c}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}}\left[x(F) + (1 - x(F)^2)\operatorname{arctanh}(x(F))\right].$$

Substituting this expression and the boundary value of $\phi(0)$ into (A.45) gives (A.46). At the payout boundary, $V_{MM} = 0$ and

$$p(M^\star(F), F) = \frac{(1-\tau_c)c}{2r}\mathcal{G}(x(F)),$$

which gives (A.47). Finally, Equation (A.22) and Lemma 2 give (A.48). □

Proof of Proposition 4. Let $x_0 := x(0)$ and define

$$\tau_c^* := \frac{\mathcal{G}(x_0)}{2 + \mathcal{G}(x_0)}. \tag{A.52}$$

Because $x_0 > 0$, we have $\tau_c^* \in (0, 1)$. Moreover,

$$\mathcal{G}'(x) = 6x + (1 - 3x^2)\log\left(\frac{1+x}{1-x}\right) > 0$$

for $x \in (0, x_0]$. Since $F \mapsto x(F)$ is continuous and strictly decreasing, (A.47) is continuous and strictly decreasing in F.

Fix $\tau_c \in (0, \tau_c^*)$. By (A.52), the payout-boundary value of N is strictly positive at $F = 0$ and converges to $-\tau_c c < 0$ as $F \uparrow \bar{F}$. Hence, there is a unique $F^* \in (0, \bar{F})$ such that

$$N(M^\star(F^*), F^*) = 0, \quad \text{or, equivalently,} \quad rp(M^\star(F^*), F^*) = \tau_c c.$$

For every $F \in (0, F^*)$, Lemma A.18 gives a unique $M^{\star\star}(F) \in (0, M^\star(F))$ at which $N = 0$. Equations (A.48) then imply issuance below this cash cutoff and repurchases above it. For $F \in [F^*, \bar{F})$, the payout-boundary value is nonpositive. Strict monotonicity and (A.46) imply $N < 0$ throughout the interior, so the firm issues debt at every such position. This proves the policy classification in part 1.

Whenever $G^\star < 0$, equations (A.45) and (A.48) give $N > 0$. Since $p_M > 0$, $V_M > 0$, and $V_{MM} < 0$, we also have $0 < N < rp$. Therefore,

$$0 < -pG^\star = \frac{pN}{-(pp_M + p_F)} < \frac{rp^2}{-(pp_M + p_F)} < (1 - \tau_c)(\mu - cF),$$

where the final inequality follows from (A.11). This proves positive cash drift throughout the repurchase region.

Finally, suppose $\tau_c = 0$. Equation (A.46) is strictly negative, whereas (A.47) is strictly positive for every $F \in (0, \bar{F})$. Strict monotonicity therefore gives a unique $M^{\star\star}(F)$ with issuance below it and repurchases above it. The preceding cash-drift argument applies unchanged. This proves part 2. □

Proof of Proposition 5. Fix $\tau_c \in [0, 1)$ and $F \in (0, \bar{F})$. Equation (A.46) gives $N(0, F) < 0$. By continuity, $N(M, F) < 0$ for all sufficiently small $M > 0$. Equation (A.48) then gives $G^\star(M, F) > 0$ on this interval. This proves the proposition. □

Internet Appendix for "Leverage Dynamics and Liquidity Management without Commitment"

This supplement is organized as follows. Section IA.A generalizes the motivating example in the main text. Section IA.B allows shareholders to commit to a debt adjustment over a short interval and shows that their uniquely optimal rate converges to $G^{\star}$ as the interval shrinks. Section IA.C proves the asset-pricing implications. Section IA.D concerns personal taxation of interest income.

IA.A A Three-Date Model of Debt Repurchases with Recovery in Default

This appendix generalizes the three-date benchmark by allowing the recovery rate on debt, α, to lie in $[0,1]$. The model highlights the contrast between two methods of financing debt reductions in the main model. Under the first method, shareholders use an external recapitalization to reduce face value. The repurchase price then reflects the higher value of the debt after the transaction, generating the familiar leverage-ratchet effect through a wealth transfer to existing debtholders. Under the second method, the firm uses cash reserves to repurchase risky debt. The repurchase then changes the firm's assets and can benefit shareholders even in a competitive debt market.

We fix a pre-transaction balance sheet (M, F) at date $t = 1$ and study a repurchase of $\Delta F \in (0, F)$ units of face value executed at the post-transaction competitive price. We compare three cases: (i) no leverage adjustment, with equity value $V(F)$; (ii) a repurchase funded by an external recapitalization, which lowers promised liabilities to $F - \Delta F$ while leaving the firm's terminal assets $\widetilde{X} + M$ unchanged; and (iii) a repurchase funded by cash reserves, which simultaneously reduces cash and face value to $M - p\Delta F$ and $F - \Delta F$. In the third case, the repurchase price p is determined by a fixed-point condition.

IA.A.1 Environment

There are three dates, $t \in \{0, 1, 2\}$: the initial date $t = 0$, the repurchase date $t = 1$, and the terminal date $t = 2$. The risk-free rate is normalized to zero, and all agents are risk neutral. Thus, the price at date $t = 1$ of any traded claim equals its expected payoff at date $t = 2$ under the pricing measure.

At date $t = 0$, the firm is endowed with cash reserves $M \geq 0$ and an asset that produces a nonnegative operating payoff $\widetilde{X} \geq 0$, with $\mathbb{E}[\widetilde{X}] < \infty$, at date $t = 2$. It also has outstanding zero-coupon debt with face value $F > 0$ that matures at date $t = 2$. No actions, cash flows, or information arrive between date $t = 0$ and the repurchase date $t = 1$, so (M, F) is also the firm's pre-transaction balance sheet at date $t = 1$. Thus, in the absence of a leverage adjustment at date $t = 1$, the firm's terminal assets equal $\widetilde{X} + M$. Equity has limited liability.

Let $\alpha \in [0, 1]$ denote the fraction of the firm's available assets that debtholders recover in default. In the absence of a leverage adjustment, the payoffs at date $t = 2$ are

$$\text{Debt payoff} = \begin{cases} F, & \widetilde{X} + M \geq F, \\ \alpha(\widetilde{X} + M), & \widetilde{X} + M < F, \end{cases} \qquad \text{Equity payoff} = (\widetilde{X} + M - F)^{+}.$$

After a leverage adjustment, the same payoff rule applies to the resulting cash reserves and face value. Thus, if the firm defaults, shareholders receive zero, debtholders receive the fraction α of the assets then inside the firm, and the fraction $1 - \alpha$ is lost in bankruptcy. The case $\alpha = 1$ is the frictionless Modigliani–Miller benchmark, whereas the case $\alpha = 0$ is the zero-recovery benchmark and matches the simplifying assumption in Section III of Admati et al. (2018, p. 165). All prices and values below depend on α, but we leave this dependence implicit.

Let $p(F)$ denote the competitive per-unit price of debt at date $t = 1$ in the absence of a leverage adjustment. The recovery rule implies

$$p(F) = \mathbb{P}(\widetilde{X} + M \geq F) + \frac{\alpha}{F}\,\mathbb{E}\Big[(\widetilde{X} + M)\mathbf{1}_{\{\widetilde{X}+M<F\}}\Big]. \tag{IA.1}$$

The market value of debt is $p(F)F$, and the market value of equity is

$$V(F) := \mathbb{E}\big[(\widetilde{X} + M - F)^{+}\big].$$

Debt and equity together satisfy

$$V(F) + p(F)F = \mathbb{E}[\widetilde{X} + M] - (1 - \alpha)\mathbb{E}\Big[(\widetilde{X} + M)\mathbf{1}_{\{\widetilde{X}+M<F\}}\Big]. \tag{IA.2}$$

When $\alpha = 1$, equation (IA.2) reduces to the usual Modigliani–Miller identity.

Throughout this appendix, the initial cash level M is held fixed until we specify the repurchase policy. To keep notation light, we write the per-unit debt price as $p(F)$ and the equity value as $V(F)$, suppressing their dependence on M and α. In a debt repurchase with cash, post-transaction cash equals $M - p\Delta F$ and therefore depends on the repurchase price. We denote the competitive price by p^* once the fixed point has been determined.

Policy A: debt reduction via external recapitalization. We first consider an externally financed repurchase. Shareholders inject funds at date $t = 1$ and use them immediately to repurchase and retire ΔF units of face value. Because the injected cash is paid to debtholders on impact, the firm's terminal assets $\widetilde{X} + M$ are unchanged; only promised liabilities fall from F to $F - \Delta F$. Competitive pricing therefore values the remaining debt at the post-transaction per-unit price $p(F - \Delta F)$, and the total payment required to repurchase ΔF units of face value is $p(F - \Delta F)\Delta F$.

The value to the existing shareholders, net of their contribution at date $t = 1$, is

$$V^{\text{recap}}(F; \Delta F) := V(F - \Delta F) - p(F - \Delta F)\Delta F. \tag{IA.3}$$

Proposition IA.1 (External recapitalization)**.** *For every $\alpha \in [0, 1]$, every $F > 0$, and every $\Delta F \in (0, F)$,*

$$V^{\text{recap}}(F; \Delta F) \le V(F),$$

with strict inequality if and only if $p(F - \Delta F) > p(F)$.

Proof. For any realization $x \ge 0$, the per-unit debt payoff in the absence of a leverage adjustment is

$$\begin{cases} 1, & x + M \ge F, \\ \alpha(x + M)/F, & x + M < F. \end{cases}$$

This payoff is weakly decreasing in F. Taking expectations therefore shows that $p(F)$ is weakly decreasing in F, and hence $p(F - \Delta F) \ge p(F)$.

Using equation (IA.1) and the definitions of $V(F)$ and $V^{\text{recap}}(F;\Delta F)$ gives

$$V^{\text{recap}}(F;\Delta F) - V(F) = \mathbb{E}\Big[(\widetilde{X} + M - F)\mathbf{1}_{\{F-\Delta F\leq \widetilde{X}+M<F\}}\Big] - \frac{\alpha\Delta F}{F-\Delta F}\mathbb{E}\Big[(\widetilde{X}+M)\mathbf{1}_{\{\widetilde{X}+M<F-\Delta F\}}\Big]. \tag{IA.4}$$

Both terms on the right-hand side are weakly negative. The first is strictly negative if and only if $\mathbb{P}(F-\Delta F \leq \widetilde{X} + M < F) > 0$, and the second is strictly negative if and only if

$$\alpha\mathbb{E}\Big[(\widetilde{X}+M)\mathbf{1}_{\{\widetilde{X}+M<F-\Delta F\}}\Big] > 0.$$

Moreover, direct subtraction of the two debt prices gives

$$p(F-\Delta F) - p(F) = \mathbb{E}\left[\left(1 - \frac{\alpha(\widetilde{X}+M)}{F}\right)\mathbf{1}_{\{F-\Delta F\leq \widetilde{X}+M<F\}}\right] + \frac{\alpha\Delta F}{F(F-\Delta F)}\mathbb{E}\Big[(\widetilde{X}+M)\mathbf{1}_{\{\widetilde{X}+M<F-\Delta F\}}\Big].$$

The first term in this expression is strictly positive if and only if the first term in equation (IA.4) is strictly negative, and the same equivalence holds for the second terms. Therefore, $V^{\text{recap}}(F;\Delta F) < V(F)$ if and only if $p(F-\Delta F) > p(F)$. □

Proposition IA.1 gives the leverage-ratchet result for every recovery rate. A recapitalized repurchase weakly reduces default risk and weakly raises the value of the debt. Because the repurchase occurs at the post-transaction price, existing debtholders receive any resulting increase in value, and existing shareholders cannot obtain a gain.

The key restriction behind Proposition IA.1 is that recapitalization leaves the assets backing the remaining debt unchanged. We now change only the source of funds: the firm uses its cash reserves to repurchase debt. This makes the post-transaction debt price endogenous, because the repurchase itself reduces the cash that remains inside the firm.

Policy B: debt reduction using cash reserves. Fix the same repurchase amount $\Delta F \in (0, F)$, but suppose the firm uses cash already inside the firm. If the repurchase occurs at a competitive per-unit price p, it requires a cash outflow $p\Delta F$ at date $t = 1$. Post-transaction cash therefore equals $M - p\Delta F$, and remaining face value equals $F - \Delta F$. A repurchase financed entirely with cash requires $p\Delta F \leq M$.

The price cannot be read off from $p(F - \Delta F)$ because the cash backing the remaining debt depends on the price itself. Competitive pricing requires

$$\begin{aligned} p = {} & \mathbb{P}(\widetilde{X} + M - p\Delta F \geq F - \Delta F) \\ & + \frac{\alpha}{F - \Delta F} \mathbb{E}\Big[(\widetilde{X} + M - p\Delta F)^{+} \mathbf{1}_{\{\widetilde{X} + M - p\Delta F < F - \Delta F\}}\Big] . \end{aligned} \tag{IA.5}$$

The positive-part operator makes the right-hand side well defined for every candidate price $p \in [0, 1]$. Whenever $p\Delta F \leq M$, it is redundant.

Definition IA.2 (Feasible repurchase amount)**.** *Fix $\alpha \in [0, 1]$ and a pre-transaction balance sheet (M, F). A repurchase amount $\Delta F \in (0, F)$ is* feasible *if equation* (IA.5) *admits a solution $p^* \in [0, 1]$ satisfying*

$$p^* \Delta F \leq M.$$

For a feasible repurchase amount, we call the unique solution characterized in Lemma IA.3 the associated competitive price.

Lemma IA.3 (Competitive price)**.** *Fix $\alpha \in [0, 1]$, $F > 0$, $M \geq 0$, and $\Delta F \in (0, F)$. Define*

$$\begin{aligned} \Phi(p) := {} & p - \mathbb{P}(\widetilde{X} + M - p\Delta F \geq F - \Delta F) \\ & - \frac{\alpha}{F - \Delta F} \mathbb{E}\Big[(\widetilde{X} + M - p\Delta F)^{+} \mathbf{1}_{\{\widetilde{X} + M - p\Delta F < F - \Delta F\}}\Big] , \qquad p \in [0, 1]. \end{aligned}$$

Then Φ is strictly increasing, $\Phi(0) \leq 0$, and $\Phi(1) \geq 0$. Consequently, equation (IA.5) *has at most one solution in $[0, 1]$. If $\alpha = 1$ or $\widetilde{X}$ has an atomless distribution, Φ is continuous and equation* (IA.5) *has a unique solution $p^* \in [0, 1]$.*

For any feasible repurchase amount ΔF, its associated competitive price p^* satisfies:

(i) $p^* > 0$ if and only if

$$\mathbb{P}(\widetilde{X} + M \geq F - \Delta F) + \frac{\alpha}{F - \Delta F} \mathbb{E}\Big[(\widetilde{X} + M) \mathbf{1}_{\{\widetilde{X} + M < F - \Delta F\}}\Big] > 0. \tag{IA.6}$$

When $\alpha = 1$, condition (IA.6) *is equivalent to $\mathbb{P}(\widetilde{X} + M > 0) > 0$.*

(ii) $p^* < 1$ if and only if the remaining debt is risky in the sense that

$$\mathbb{P}(\widetilde{X} + M - p^* \Delta F < F - \Delta F) > 0.$$

In particular, if both conditions hold, then $p^* \in (0,1)$.

Proof. For each realization of $\widetilde{X}$, the per-unit payoff to the remaining debt at a candidate price p is

$$\begin{cases} 1, & \widetilde{X} + M - p\Delta F \geq F - \Delta F, \\ \dfrac{\alpha(\widetilde{X} + M - p\Delta F)^+}{F - \Delta F}, & \widetilde{X} + M - p\Delta F < F - \Delta F. \end{cases}$$

The displayed payoff is weakly decreasing in p because the assets remaining inside the firm decline with p; this remains true at the boundary between full repayment and default because $\alpha \leq 1$. Thus, Φ is strictly increasing on $[0,1]$, although it need not be continuous. Moreover, $\Phi(0) \leq 0$ because debt payoffs are nonnegative, and $\Phi(1) \geq 0$ because the per-unit payoff is at most one. Hence, the fixed-point equation has at most one solution.

If $\alpha = 1$, the per-unit debt payoff can be written as

$$\min\left\{1, \frac{(\widetilde{X} + M - p\Delta F)^+}{F - \Delta F}\right\},$$

which is continuous in p and bounded by one. If $\widetilde{X}$ is atomless, the only possible discontinuity occurs when $\widetilde{X} + M - p\Delta F = F - \Delta F$, an event of probability zero for each p. Dominated convergence therefore makes Φ continuous in either case; the endpoint inequalities yield existence, and strict monotonicity yields uniqueness.

For part (i), the left-hand side of condition (IA.6) is the expected per-unit payoff at $p = 0$. It is positive exactly when $\Phi(0) < 0$, which by uniqueness is equivalent to $p^* > 0$.

For part (ii), the per-unit payoff equals one almost surely exactly when the remaining debt is not risky. The fixed-point condition therefore gives $p^* < 1$ exactly when the remaining debt is risky. $\square$

When $\alpha = 1$, Lemma IA.3 reduces to the frictionless competitive-price result. When $\alpha < 1$ and $\widetilde{X}$ has atoms, the right-hand side of equation (IA.5) can jump when the firm moves between full repayment and default. A solution therefore need not exist for every repurchase amount, so feasibility is a substantive restriction. In what follows, we restrict attention to feasible repurchase amounts and let p^* denote the associated competitive price.

The following lemma gives a sufficient condition under which the competitive price exists for every recovery rate and varies continuously with α.

Lemma IA.4 (Continuity in the recovery rate)**.** *Fix $F > 0$, $M \geq 0$, and $\Delta F \in (0, F)$. Make the dependence on the recovery rate explicit and write the competitive price as $p^*(\alpha)$. Suppose equation* (IA.5) *has a solution $p^*(0)$ at $\alpha = 0$, and let $p^*(1)$ denote its unique solution at $\alpha = 1$. Then $p^*(0) \leq p^*(1)$. Suppose, in addition, that*

$$\mathbb{P}(\widetilde{X} + M = x) = 0 \quad \textit{for every } x \in [F - \Delta F + p^*(0)\Delta F, F - \Delta F + p^*(1)\Delta F]\,. \tag{IA.7}$$

Then equation (IA.5) *has a unique solution $p^*(\alpha)$ for every $\alpha \in [0, 1]$, and $p^*(\alpha)$ is continuous and weakly increasing in α. If*

$$p^*(1)\Delta F \leq M,$$

then ΔF is feasible for every $\alpha \in [0, 1]$.

Condition (IA.7) *holds, in particular, if $\widetilde{X}$ has an atomless distribution.*

Proof. Let $h(p, \alpha)$ denote the right-hand side of equation (IA.5). At every fixed candidate price p, $h(p, \alpha)$ is weakly increasing in α. Therefore,

$$h(p^*(0), 1) \geq h(p^*(0), 0) = p^*(0).$$

Because the fixed-point residual $p - h(p, 1)$ is strictly increasing and vanishes at $p^*(1)$, this inequality implies $p^*(0) \leq p^*(1)$.

Under condition (IA.7), $h(p, \alpha)$ is jointly continuous on

$$[p^*(0), p^*(1)] \times [0, 1].$$

Indeed, for any convergent sequence of candidate prices and recovery rates, the state-by-state debt payoff converges except possibly when

$$\widetilde{X} + M - p\Delta F = F - \Delta F.$$

Condition (IA.7) assigns probability zero to this event for every $p \in [p^*(0), p^*(1)]$. Because the debt payoff is bounded by one, joint continuity follows from dominated convergence.

For every $\alpha \in [0,1]$,

$$h(p^*(0), \alpha) \geq h(p^*(0), 0) = p^*(0),$$

and

$$h(p^*(1), \alpha) \leq h(p^*(1), 1) = p^*(1).$$

Joint continuity and the intermediate value theorem therefore yield a solution $p^*(\alpha) \in [p^*(0), p^*(1)]$. Uniqueness follows from Lemma IA.3.

If $\alpha_2 > \alpha_1$, then

$$h(p^*(\alpha_1), \alpha_2) \geq h(p^*(\alpha_1), \alpha_1) = p^*(\alpha_1).$$

The strict monotonicity of the fixed-point residual implies $p^*(\alpha_2) \geq p^*(\alpha_1)$. Thus, $p^*(\alpha)$ is weakly increasing.

Finally, consider any sequence $\alpha_n \to \alpha$. Because $p^*(\alpha_n) \in [p^*(0), p^*(1)]$, every subsequence has a further subsequence converging to some $\bar{p}$ in this interval. Joint continuity gives $\bar{p} = h(\bar{p}, \alpha)$, so uniqueness implies $\bar{p} = p^*(\alpha)$. Hence, the entire sequence converges to $p^*(\alpha)$, proving continuity. Monotonicity also gives $p^*(\alpha)\Delta F \leq p^*(1)\Delta F$, which proves the final feasibility claim. $\square$

Given ΔF and the associated competitive price p^*, the equity payoff at date $t = 2$ is

$$(\widetilde{X} + M - p^*\Delta F - (F - \Delta F))^+ = (\widetilde{X} + M - F + (1 - p^*)\Delta F)^+.$$

Thus, the firm reduces face value by ΔF while using only $p^*\Delta F$ of cash. The difference $(1-p^*)\Delta F$ weakly raises the shareholders' residual payoff in every state in which their post-repurchase payoff is positive.

Proposition IA.5 (Shareholders' gain)**.** *Fix (M, F) and a feasible debt repurchase amount $\Delta F \in (0, F)$, and let $p^* \in [0,1]$ denote its associated competitive price. Let*

$$\begin{aligned} V^{\text{cash}}(F; \Delta F) &:= \mathbb{E}\big[(\widetilde{X} + M - p^*\Delta F - (F - \Delta F))^+\big] \\ &= \mathbb{E}\big[(\widetilde{X} + M - F + (1 - p^*)\Delta F)^+\big]. \end{aligned}$$

For every $\alpha \in [0,1]$,

$$V^{\text{cash}}(F;\Delta F) \geq V(F).$$

The gain is strict if and only if

$$\mathbb{P}\big(\widetilde{X} + M - p^*\Delta F > F - \Delta F\big) > 0 \quad \textit{and} \quad \mathbb{P}\big(\widetilde{X} + M - p^*\Delta F < F - \Delta F\big) > 0. \tag{IA.8}$$

Proof. The shareholder payoffs in the absence of a leverage adjustment and after the debt repurchase with cash are

$$(\widetilde{X} + M - F)^+ \qquad \text{and} \qquad (\widetilde{X} + M - F + (1-p^*)\Delta F)^+,$$

respectively. Because $(1-p^*)\Delta F \geq 0$, for every realization,

$$(\widetilde{X} + M - F + (1-p^*)\Delta F)^+ \geq (\widetilde{X} + M - F)^+. \tag{IA.9}$$

The inequality is strict with positive probability if and only if $p^* < 1$ and the first probability in condition (IA.8) is strictly positive. Lemma IA.3 shows that $p^* < 1$ if and only if the remaining debt defaults with positive probability, which is exactly the second inequality in condition (IA.8). Taking expectations proves the result. □

The recovery rate affects the competitive price p^*, but it does not otherwise enter the state-by-state comparison in equation (IA.9). This is the precise sense in which the shareholder result and its proof are essentially the same for every $\alpha \in [0,1]$. The necessary and sufficient condition in equation (IA.8) says that shareholders receive a strictly positive payoff with positive probability after the repurchase and that the remaining debt defaults with positive probability. The first inequality does not require shareholders to receive a positive payoff in the absence of a leverage adjustment; it also covers the case in which their payoff is zero without the repurchase but strictly positive after the repurchase.

The effect on the original debtholders depends on α. They receive $p^*\Delta F$ for the face value that is repurchased and retain debt worth $p^*(F - \Delta F)$. Their total value after the repurchase is therefore p^*F, compared with $p(F)F$ in the absence of a leverage adjustment.

Proposition IA.6 (Debtholders' payoff)**.** *Fix* (M,F) *and a feasible debt repurchase amount* $\Delta F \in (0,F)$*, and let* $p^* \in [0,1]$ *denote its associated competitive price.*

(i) If $\alpha = 0$*, then*

$$p^* \geq p(F),$$

with strict inequality if and only if

$$\mathbb{P}\big(F - (1 - p^*)\Delta F \leq \widetilde{X} + M < F\big) > 0. \tag{IA.10}$$

Thus, the original debtholders weakly benefit from the repurchase and benefit strictly if and only if condition (IA.10) *holds.*

(ii) If $\alpha = 1$*, then*

$$V^{\text{cash}}(F; \Delta F) - V(F) = F\big(p(F) - p^*\big). \tag{IA.11}$$

Thus, the original debtholders are weakly worse off after the repurchase. They are strictly worse off if and only if shareholders strictly gain, in which case their aggregate loss equals the shareholder gain.

Proof. When $\alpha = 0$, equations (IA.1) and (IA.5) reduce to

$$\begin{aligned} p(F) &= \mathbb{P}(\widetilde{X} + M \geq F), \\ p^* &= \mathbb{P}\big(\widetilde{X} + M \geq F - (1 - p^*)\Delta F\big). \end{aligned}$$

Because $(1 - p^*)\Delta F \geq 0$, the second repayment threshold is weakly lower than the first. The difference between the two probabilities is exactly the probability in equation (IA.10). This proves part (i).

When $\alpha = 1$, debt and equity exhaust the firm's assets in every state. In the absence of a leverage adjustment,

$$V(F) + p(F)F = \mathbb{E}[\widetilde{X} + M].$$

After the repurchase, the original debtholders receive $p^*\Delta F$ for the face value that is repurchased and retain debt worth $p^*(F - \Delta F)$. Therefore,

$$V^{\text{cash}}(F; \Delta F) + p^*F = \mathbb{E}[\widetilde{X} + M].$$

Subtracting the two identities gives equation (IA.11). The result then follows from Proposition IA.5.

$\square$

Proposition IA.7 (Recovery-rate cutoff)**.** *Fix* (M, F) *and* $\Delta F \in (0, F)$*. For this result, make the dependence on the recovery rate explicit and write the pre-transaction and post-transaction debt prices as* $p(F; \alpha)$ *and* $p^*(\alpha)$*, respectively. Suppose the conditions of Lemma IA.4 hold and* $p^*(1)\Delta F \leq M$*. Suppose further that condition* (IA.10) *holds when evaluated at* $\alpha = 0$*, whereas condition* (IA.8) *holds when evaluated at* $\alpha = 1$*. Then there exists a unique* $\alpha^* \in (0, 1)$ *such that*

$$p^*(\alpha) > p(F; \alpha) \quad \textit{for every } \alpha \in [0, \alpha^*),$$

and

$$p^*(\alpha) < p(F; \alpha) \quad \textit{for every } \alpha \in (\alpha^*, 1].$$

At $\alpha = \alpha^*$*, the original debtholders are indifferent:*

$$p^*(\alpha^*) = p(F; \alpha^*).$$

Thus, the original debtholders strictly benefit from the repurchase below the cutoff and are strictly worse off above the cutoff.

Proof. Lemma IA.4 establishes that $p^*(\alpha)$ is continuous and that ΔF is feasible for every $\alpha \in [0, 1]$. By Proposition IA.6, the original debtholders strictly benefit at $\alpha = 0$. At $\alpha = 1$, condition (IA.8) and Propositions IA.5 and IA.6 imply that they are strictly worse off. Continuity therefore implies that there exists $\alpha^* \in (0, 1)$ such that $p^*(\alpha^*) = p(F; \alpha^*)$.

It remains to establish the single-crossing property. Consider any $\bar{\alpha} \in (0, 1)$ such that $p^*(\bar{\alpha}) = p(F; \bar{\alpha}) = \bar{p}$. At this recovery rate, the two pricing equations imply

$$\begin{aligned}
\bar{p} &= \mathbb{P}(\widetilde{X} + M \geq F) + \frac{\bar{\alpha}}{F}\mathbb{E}\Big[(\widetilde{X} + M)\mathbf{1}_{\{\widetilde{X}+M<F\}}\Big] \\
&= \mathbb{P}(\widetilde{X} + M - \bar{p}\Delta F \geq F - \Delta F) \\
&\quad + \frac{\bar{\alpha}}{F - \Delta F}\mathbb{E}\Big[(\widetilde{X} + M - \bar{p}\Delta F)\mathbf{1}_{\{\widetilde{X}+M-\bar{p}\Delta F<F-\Delta F\}}\Big].
\end{aligned}$$

Because $\bar{p} \leq 1$, every state in which the original debt is repaid in full is also a state in which the remaining debt is repaid in full. Since $\bar{\alpha} > 0$, the two equal prices therefore imply

$$\begin{aligned}\frac{1}{F-\Delta F}&\mathbb{E}\Big[(\widetilde{X}+M-\bar{p}\Delta F)\mathbf{1}_{\{\widetilde{X}+M-\bar{p}\Delta F<F-\Delta F\}}\Big]\\&\leq \frac{1}{F}\mathbb{E}\Big[(\widetilde{X}+M)\mathbf{1}_{\{\widetilde{X}+M<F\}}\Big].\end{aligned} \tag{IA.12}$$

Now consider $\alpha > \bar{\alpha}$. At the fixed candidate price $\bar{p}$, the expected payoff to the remaining debt rises with α at the rate given by the left-hand side of inequality (IA.12). By contrast, $p(F;\alpha)$ rises at the rate given by the right-hand side. Moreover, $p(F;\alpha) \geq \bar{p}$, and the expected payoff to the remaining debt is weakly decreasing in the candidate repurchase price. Hence, at the candidate price $p(F;\alpha)$, the expected payoff to the remaining debt is no greater than $p(F;\alpha)$. The strict monotonicity of the fixed-point residual then implies

$$p^*(\alpha) \leq p(F;\alpha).$$

The same argument in reverse shows that, for $\alpha < \bar{\alpha}$,

$$p^*(\alpha) \geq p(F;\alpha).$$

Condition (IA.10) at $\alpha = 0$ implies that the right-hand side of inequality (IA.12) is strictly positive because $F - (1 - p^*(0))\Delta F > F - \Delta F > 0$. If inequality (IA.12) is strict at a crossing, the preceding comparisons are strict away from that crossing. If it holds with equality, the left-hand side is also strictly positive, so the expected payoff to the remaining debt is strictly decreasing in the candidate price whenever the recovery rate is positive. The comparisons are therefore again strict away from the crossing; strictness at $\alpha = 0$ follows directly from condition (IA.10). Applying this single-crossing property at $\bar{\alpha} = \alpha^*$ proves the result and shows that the cutoff is unique. □

Modigliani–Miller benchmark with full recovery. The full-recovery case $\alpha = 1$ is the Modigliani–Miller benchmark. At full recovery, bankruptcy creates no deadweight loss, so equation (IA.11) shows that any gain to shareholders is exactly offset by an equal aggregate loss to the original debtholders.

For an intermediate recovery rate, the corresponding accounting relation is

$$\begin{aligned} V^{\text{cash}}(F;\Delta F) - V(F) = F\big(p(F) - p^*\big) & \\ + (1-\alpha)\Bigg\{ & \mathbb{E}\Big[(\widetilde{X} + M)\mathbf{1}_{\{\widetilde{X}+M<F\}}\Big] \\ & - \mathbb{E}\Big[(\widetilde{X} + M - p^*\Delta F)\mathbf{1}_{\{\widetilde{X}+M-p^*\Delta F<F-\Delta F\}}\Big]\Bigg\}. \end{aligned} \tag{IA.13}$$

The sign of $p^* - p(F)$ can therefore be positive or negative when $\alpha \in (0,1)$. At $\alpha = 0$, the original debtholders do not lose cash in default because their recovery in default is zero, and part (i) of Proposition IA.6 applies. At $\alpha = 1$, the bankruptcy-loss term in equation (IA.13) vanishes, and part (ii) applies.

IA.A.2 The Motivating Example with an Arbitrary Recovery Rate

In the motivating example, $M = 100$, $F = 250$, $\Delta F = 150$, and

$$\widetilde{X} = \begin{cases} 100, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

The firm defaults in both states in the absence of a leverage adjustment, so

$$p(F) = \frac{3\alpha}{5} \qquad \text{and} \qquad V(F) = 0.$$

After a debt repurchase with cash, the firm repays the remaining debt in full when $\widetilde{X} = 100$ and defaults when $\widetilde{X} = 0$. The competitive pricing equation is

$$p^* = \frac{1}{2} + \frac{\alpha}{2}\left(\frac{100 - 150p^*}{100}\right),$$

which gives

$$p^* = \frac{2(1+\alpha)}{4+3\alpha}. \tag{IA.14}$$

Moreover,

$$p^*\Delta F = \frac{300(1+\alpha)}{4+3\alpha} \leq \frac{600}{7} < 100 = M,$$

so $\Delta F = 150$ is feasible for every $\alpha \in [0, 1]$. Moreover, $p^*(0) = 1/2$ and $p^*(1) = 4/7$, so the repayment thresholds in condition (IA.7) range from 175 to 1300/7. Because $\widetilde{X} + M \in \{100, 200\}$, this interval contains no atom. Lemma IA.4 therefore proves that p^* is continuous in α, and Proposition IA.7 applies even though $\widetilde{X}$ has atoms. The shareholder gain is

$$V^{\text{cash}}(F; \Delta F) - V(F) = \frac{50}{4 + 3\alpha} > 0 \qquad \text{for every } \alpha \in [0, 1]. \tag{IA.15}$$

The two probabilities in condition (IA.8) both equal 1/2: shareholders receive a positive payoff in the high-payoff state, whereas the firm defaults in the low-payoff state. Thus, the exact condition covers the motivating example even though $V(F) = 0$.

By contrast, the change in the debt price is

$$p^* - p(F) = \frac{10 - 2\alpha - 9\alpha^2}{5(4 + 3\alpha)}. \tag{IA.16}$$

Thus, the original debtholders are better off when $\alpha < \frac{\sqrt{91}-1}{9}$, are indifferent when $\alpha = \frac{\sqrt{91}-1}{9}$, and are worse off when $\alpha > \frac{\sqrt{91}-1}{9}$. Thus, the cutoff in Proposition IA.7 is $\alpha^* = (\sqrt{91} - 1)/9$ in this example. In particular:

- If $\alpha = 0$, then $p(F) = 0$ and $p^* = 1/2$. The original debtholders' aggregate value rises from 0 to 125, while shareholders gain 25/2.
- If $\alpha = 1$, then $p(F) = 3/5$ and $p^* = 4/7$. The original debtholders lose 50/7, exactly equal to the shareholders' gain.

The example therefore makes the distinction sharp. The shareholder result holds throughout $\alpha \in [0, 1]$, but the effect on the original debtholders depends on the recovery rate.

IA.A.3 Three Economic Effects

A debt repurchase with cash has three distinct effects.

1. It weakly reduces the probability of default. In the absence of a leverage adjustment, the firm repays its debt in full when $\widetilde{X} + M \geq F$. After the repurchase, it repays the remaining debt in full when

$$\widetilde{X} + M \geq F - (1 - p^*)\Delta F.$$

Because the second threshold is weakly lower, this effect weakly benefits both shareholders and the remaining debtholders.

2. It reduces the firm's debt obligations when the firm repays the remaining debt in full. The firm reduces face value by ΔF but uses only $p^*\Delta F$ of cash. Hence, its cash payment at date $t = 1$ is smaller than the reduction in its terminal debt obligations whenever $p^* < 1$.
3. It reduces the cash available to debtholders if the firm nevertheless defaults. After the repurchase, the firm has used $p^*\Delta F$ of cash that would otherwise have remained available at date $t = 2$. The importance of this effect increases with α and disappears when $\alpha = 0$.

Shareholders benefit from the first two effects. Debtholders benefit from the first effect but are hurt by the third. When $\alpha = 0$, the third effect is absent, so the original debtholders weakly benefit and benefit strictly under condition (IA.10). When $\alpha = 1$, equation (IA.11) shows that the original debtholders lose whenever shareholders gain.

IA.A.4 Relation to the Continuous-Time Model

The continuous-time model assumes zero recovery. Accordingly, the third effect above—the loss of cash that debtholders would otherwise recover in bankruptcy—is absent. Cash depletion is not entirely irrelevant in the dynamic model: using cash to repurchase debt can move the firm closer to bankruptcy and shorten the expected duration of future coupon payments. Nevertheless, the marginal price-impact result shows that the benefit of reducing face value dominates this adverse liquidity effect.

For any infinitesimal debt repurchase with cash, $dM = p\,dF$, so

$$dp = p_M\,dM + p_F\,dF = (pp_M + p_F)dF.$$

By Lemma 2, $pp_M + p_F < 0$. Hence, a debt repurchase with $dF < 0$ satisfies $dp > 0$: the price of the remaining debt rises during a smooth repurchase. In this sense, the continuous-time model resembles the $\alpha = 0$ static benchmark rather than the $\alpha = 1$ Modigliani–Miller benchmark. Shareholders voluntarily use cash reserves to repurchase debt even though the remaining debtholders also benefit. Because the operating cash-flow process is unchanged by the repurchase, this comparison does not invoke asset substitution. Thus, under full recovery, any strict shareholder gain is a wealth transfer from the original debtholders. However, neither wealth extraction nor asset substitution is essential to the broader mechanism or to the continuous-time result.

IA.B Discrete Optimization

In the spirit of DeMarzo and He (2021), we ask what shareholders would choose if they could commit to a debt adjustment over a short interval $[t, t+dt)$. At time $t+dt$, they revert to the equilibrium financial policy. Fix $(M, F) \in \mathcal{R}$ and a constant $\overline{G} > 0$ such that $|G^{\star}(M, F)| < \overline{G}$. Shareholders choose an adjustment $\Delta \in [-\overline{G}\, dt, \overline{G}\, dt]$. Thus, the rate Δ/dt may take any value in the fixed interval $[-\overline{G}, \overline{G}]$. The constant $\overline{G}$ is arbitrary and may be chosen as large as desired. We show that, for sufficiently small dt, shareholders have a unique global optimum over the entire interval and that the chosen rate converges to $G^{\star}(M, F)$. We then show that $G^{\star}(M, F)$ is the unique global solution of the limiting problem over all finite adjustment rates. Specifically, suppose that the timing unfolds as follows.

(1) At time t, shareholders observe the financial position $(M, F) \in \mathcal{R}$.

(2) Shareholders adjust the debt level by $\Delta \in [-\overline{G}\, dt, \overline{G}\, dt]$. Let $\tilde{p}(M, F; \Delta)$ denote the competitive price per unit of face value at which the entire adjustment is implemented. As in our continuous-time model, issuing debt raises cash, whereas repurchasing debt uses cash. Hence, the cash reserves $\widetilde{M}_{-}$ after leverage adjustment satisfy $\widetilde{M}_{-} = M + \tilde{p}(M, F; \Delta)\Delta$. [1]

(3) The operating cash flow increment $\widetilde{X} - X$ realizes over $[t, t+dt)$. Then, the coupon is paid at time $t+dt$, which determines the cash holdings $\widetilde{M}$ at time $t+dt$:

$$\widetilde{M} = \widetilde{M}_{-} + (1-\tau_c)\Big[\widetilde{X} - X - c(F+\Delta)dt\Big].$$

In this environment, shareholders solve

$$\max_{\Delta \in [-\overline{G}\, dt, \overline{G}\, dt]} \mathbb{E}\left[e^{-r\, dt} V\Big(\widetilde{M}, F+\Delta\Big) \,\Big|\, M_{t^-} = M,\ F_{t^-} = F \right]. \tag{IA.17}$$

Let $J_{dt}(\Delta)$ denote the objective in (IA.17). The value and price functions are smooth at the interior

[1] The transaction price $\tilde{p}(M, F; \Delta)$ and the post-adjustment cash reserves are determined simultaneously, and every unit in the adjustment trades at the same price. Shareholders therefore account for how the size of the adjustment affects its price. The restriction $|\Delta| \le \overline{G}\, dt$ makes the choice set compact. The constant $\overline{G}$ may be any fixed finite number greater than $|G^{\star}(M, F)|$, so it does not restrict the limiting comparison among finite rates. The solution derived below satisfies $\Delta^{\star}_{dt} = O(dt)$ and $\tilde{p}(M, F; \Delta^{\star}_{dt}) = p(M, F) + O(dt)$. Consequently, $\tilde{p}(M, F; \Delta^{\star}_{dt})\Delta^{\star}_{dt} = p(M, F)\Delta^{\star}_{dt} + o(dt)$, so the continuous-time limit is consistent with the predictable cash dynamics in our model.

position (M, F), so the Taylor expansions below hold uniformly over the feasible interval.

Recall that the marginal price impact from an infinitesimal debt issuance is

$$\mathcal{I}(M, F) := p(M, F) p_M(M, F) + p_F(M, F).$$

Competitive pricing implies the one-period present-value relationship

$$\tilde{p}(M, F; \Delta) = \mathbb{E}\left[e^{-r\,dt} \left\{ c\,dt + p\Big(\widetilde{M}, F + \Delta\Big) \right\} \,\Big|\, M_{t^-} = M,\ F_{t^-} = F \right] + o(dt). \tag{IA.18}$$

For $\Delta = O(dt)$, a first-order expansion yields

$$\begin{aligned}
\tilde{p}(M, F; \Delta) = e^{-r\,dt} \Bigg[& c\,dt + p(M, F) + p_F(M, F)\Delta \\
& + \Big((1 - \tau_c)(\mu - cF)dt + p(M, F)\Delta\Big) p_M(M, F) \\
& + \frac{1}{2}(1 - \tau_c)^2 \sigma^2 dt\, p_{MM}(M, F) \Bigg] + o(dt) \\
= p(M, F) + \Bigg[& c + \left(\frac{e^{-r\,dt} - 1}{dt} \right) p(M, F) + (1 - \tau_c)(\mu - cF) p_M(M, F) \\
& + \frac{1}{2}(1 - \tau_c)^2 \sigma^2 p_{MM}(M, F) \Bigg] dt \\
& + \mathcal{I}(M, F)\Delta + o(dt).
\end{aligned}$$

Here, the additional coupon paid on Δ changes the level of cash reserves by only $O(\Delta dt) = o(dt)$. Its derivative with respect to Δ, however, is of order dt and must therefore be retained when differentiating the shareholders' objective. Uniformly over the feasible interval, the expansion also implies

$$\frac{\partial \tilde{p}}{\partial \Delta}(M, F; \Delta) = \mathcal{I}(M, F) + o(1). \tag{IA.19}$$

For any adjustment in the interior of the feasible interval, differentiating the objective in (IA.17) and using the equilibrium indifference condition gives

$$\begin{aligned}
J'_{dt}(\Delta) = \mathbb{E}\Bigg[& e^{-r\,dt} V_M\left(\widetilde{M}, F + \Delta\right) \Bigg\{ \tilde{p}(M, F; \Delta) + \frac{\partial \tilde{p}}{\partial \Delta}(M, F; \Delta)\Delta \\
& - (1 - \tau_c)c\,dt - p\Big(\widetilde{M}, F + \Delta\Big) \Bigg\} \,\Bigg|\, M_{t^-} = M,\ F_{t^-} = F \Bigg] + o(dt).
\end{aligned}$$

Define

$$N(M,F) := rp(M,F) - c - (1-\tau_c)(\mu - cF)p_M(M,F) - \frac{1}{2}(1-\tau_c)^2\sigma^2 p_{MM}(M,F).$$

A direct expansion around (M,F), using the equity HJB equation and the shareholder-indifference condition, gives, uniformly for $\Delta \in [-\overline{G}\,dt, \overline{G}\,dt]$,

$$J'_{dt}(\Delta) = V_M(M,F)\big[\mathcal{I}(M,F)\Delta - N(M,F)dt\big] + o(dt).$$

Equation (21) implies

$$N(M,F) = \mathcal{I}(M,F)G^\star(M,F).$$

To compare adjustment rates directly, write $\Delta = g\,dt$. Integrating the preceding expansion for J'_{dt} from 0 to $g\,dt$ gives, uniformly for $g \in [-\overline{G}, \overline{G}]$,

$$\frac{J_{dt}(g\,dt) - J_{dt}(0)}{dt^2} = V_M(M,F)\left[\frac{1}{2}\mathcal{I}(M,F)g^2 - N(M,F)g\right] + o(1).$$

Because $\overline{G}$ is arbitrary, this limit applies on every bounded interval of rates. Its right-hand side is a strictly concave quadratic function of g because $V_M(M,F) > 0$ and $\mathcal{I}(M,F) < 0$. Moreover, its unique maximizer is

$$g = \frac{N(M,F)}{\mathcal{I}(M,F)} = G^\star(M,F).$$

It follows that $G^\star(M,F)$ is the unique global maximizer of the limiting problem over all $g \in \mathbb{R}$.

A second-order expansion yields

$$J''_{dt}(\Delta) = V_M(M,F)\mathcal{I}(M,F) + o(1)$$

uniformly for $\Delta \in [-\overline{G}\,dt, \overline{G}\,dt]$. Under Assumption 1, $V_M(M,F) > 0$ and $\mathcal{I}(M,F) < 0$ throughout $\mathcal{R}$. Hence, for sufficiently small dt, J_{dt} is strictly concave over the entire feasible interval.

Moreover,

$$\frac{J'_{dt}(-\overline{G}\,dt)}{dt} = V_M(M,F)\mathcal{I}(M,F)\big[-\overline{G} - G^\star(M,F)\big] + o(1) > 0,$$
$$\frac{J'_{dt}(\overline{G}\,dt)}{dt} = V_M(M,F)\mathcal{I}(M,F)\big[\overline{G} - G^\star(M,F)\big] + o(1) < 0.$$

The inequalities follow from $|G^\star(M,F)| < \overline{G}$. Thus, the objective has a unique stationary point in the interior of the feasible interval. Strict concavity implies that this point is the unique global maximizer over the entire interval. Denoting it by $\Delta^\star_{dt}$, its first-order condition gives

$$\Delta^\star_{dt} = \frac{N(M,F)}{\mathcal{I}(M,F)}dt + o(dt) = G^\star(M,F)dt + o(dt).$$

Therefore, for any fixed bound $\overline{G} > |G^\star(M,F)|$, the uniquely optimal rate $\Delta^\star_{dt}/dt$ converges to $G^\star(M,F)$ as $dt \downarrow 0$. The argument encompasses every finite rate because $\overline{G}$ may be chosen arbitrarily large. It does not consider rates that diverge as $dt \downarrow 0$, or equivalently, block adjustments that do not shrink in proportion to dt.

IA.C Proofs of the Asset-Pricing Implications

This section first establishes the concavity of the debt price in cash reserves and then uses this property to prove the volatility and credit-spread comparative statics reported in the main text.

Lemma IA.8. *Under Assumption 1, the equilibrium debt price satisfies*

$$p_{MM}(M,F) < 0 \tag{IA.20}$$

for every $(M,F) \in \mathcal{R}$.

Proof. Assumption 1 and $F \in (0, \mu/c)$ imply

$$0 < \frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} < \frac{1}{3}. \tag{IA.21}$$

Recall that

$$M^\star(F) = \frac{2(1-\tau_c)\sigma^2}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}} \operatorname{arctanh}\left(\frac{\mu - cF}{\sqrt{(\mu - cF)^2 + 2r\sigma^2}}\right). \tag{IA.22}$$

Differentiating the closed-form debt price with respect to M gives

$$
\begin{aligned}
p_M(M,F) = \frac{c}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \Bigg\{ & \left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}\,(M^\star(F)-M)}{(1-\tau_c)\sigma^2} + \frac{(\mu-cF)\sqrt{(\mu-cF)^2+2r\sigma^2}}{2r\sigma^2} \right] \\
& \times \operatorname{sech}^2\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \\
& - \tanh\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \Bigg\}.
\end{aligned} \tag{IA.23}
$$

A second differentiation yields

$$
\begin{aligned}
p_{MM}(M,F) = & -\frac{2c}{(1-\tau_c)\sigma^2} \operatorname{sech}^2\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \\
& \times \Bigg\{ 1 + \left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}\,(M^\star(F)-M)}{(1-\tau_c)\sigma^2} + \frac{(\mu-cF)\sqrt{(\mu-cF)^2+2r\sigma^2}}{2r\sigma^2} \right] \\
& \times \tanh\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \Bigg\}.
\end{aligned} \tag{IA.24}
$$

Because $0 < M < M^\star(F)$,

$$
\left| \tanh\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \right| < \frac{\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}}. \tag{IA.25}
$$

Using $\operatorname{arctanh}(x) < \frac{x}{1-x^2}, \quad x \in (0,1)$, together with (IA.21) and (IA.22), the expression in braces in (IA.24) satisfies

$$
\begin{aligned}
& 1 + \left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}\,(M^\star(F)-M)}{(1-\tau_c)\sigma^2} + \frac{(\mu-cF)\sqrt{(\mu-cF)^2+2r\sigma^2}}{2r\sigma^2} \right] \\
& \quad \times \tanh\left[\frac{\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2} \left(M - \frac{M^\star(F)}{2} \right) \right] \\
& > 1 - \frac{\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \left[2\operatorname{arctanh}\left(\frac{\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}} \right) + \frac{(\mu-cF)\sqrt{(\mu-cF)^2+2r\sigma^2}}{2r\sigma^2} \right] \\
& > 1 - \frac{3(\mu-cF)^2}{2r\sigma^2} > \frac{5}{8}.
\end{aligned} \tag{IA.26}
$$

Consequently, (IA.24) implies $p_{MM}(M,F) < 0$. □

Proof of Corollary 2. Because leverage adjustments have finite variation, only shocks to cash reserves contribute to instantaneous asset-return volatility. Itô's formula and the equity and debt-pricing equations therefore give

$$\sigma^S(M,F) = (1-\tau_c)\sigma\frac{V_M(M,F)}{V(M,F)}, \qquad \sigma^B(M,F) = (1-\tau_c)\sigma\frac{p_M(M,F)}{p(M,F)}.$$

We first consider stock return volatility. Since $V > 0$, $V_M > 0$, and $V_{MM} < 0$ throughout $\mathcal{R}$,

$$\frac{\partial\sigma^S(M,F)}{\partial M} = (1-\tau_c)\sigma\frac{V(M,F)V_{MM}(M,F) - V_M(M,F)^2}{V(M,F)^2} < 0. \tag{IA.27}$$

The closed-form equity value also gives

$$\sigma^S(M,F) = \frac{1}{\sigma}\left[\sqrt{(\mu-cF)^2+2r\sigma^2}\coth\left(\frac{M\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2}\right) - (\mu-cF)\right]. \tag{IA.28}$$

Differentiating with respect to F yields

$$\begin{aligned}\frac{\partial\sigma^S(M,F)}{\partial F} = \frac{c}{\sigma}\Bigg[1 - \frac{\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}}\Bigg\{&\coth\left(\frac{M\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2}\right)\\ &- \frac{M\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2}\operatorname{csch}^2\left(\frac{M\sqrt{(\mu-cF)^2+2r\sigma^2}}{(1-\tau_c)\sigma^2}\right)\Bigg\}\Bigg].\end{aligned} \tag{IA.29}$$

For every $z > 0$, $0 < \coth(z) - z\operatorname{csch}^2(z) < 1$. Together with $0 < \frac{\mu-cF}{\sqrt{(\mu-cF)^2+2r\sigma^2}} < 1$, this proves $\frac{\partial\sigma^S(M,F)}{\partial F} > 0$. We next consider bond return volatility. By Lemma IA.8, $p(M,F) > 0, \quad p_M(M,F) > 0, \quad p_{MM}(M,F) < 0$. Consequently,

$$\frac{\partial\sigma^B(M,F)}{\partial M} = (1-\tau_c)\sigma\frac{p(M,F)p_{MM}(M,F) - p_M(M,F)^2}{p(M,F)^2} < 0. \tag{IA.30}$$

Because $R^*(M,F) = c/p(M,F)$ and the established debt-price properties give $p_M > 0$ and $p_F < 0$, $\frac{\partial R^*(M,F)}{\partial M} = -\frac{c\,p_M(M,F)}{p(M,F)^2} < 0$ and $\frac{\partial R^*(M,F)}{\partial F} = -\frac{c\,p_F(M,F)}{p(M,F)^2} > 0$. The credit spread $R^*(M,F)-r$ has the same comparative statics.

□

IA.D Personal Taxation of Interest Income

This section first proves Proposition 6. It then uses the upper bound on the marginal price impact from an infinitesimal debt issuance to show that, when the tax rate on interest income is sufficiently small, cash reserves accumulate at every repurchase position and firms issue debt when M is sufficiently close to zero.

Proof of Proposition 6. Personal taxation of coupon income does not affect the firm's cash dynamics or the shareholders' no-trade problem. Hence, Lemma A.7 gives the same equity value $V = V^0$ as in the baseline model. Shareholder indifference then implies that the equilibrium debt price is also unchanged:

$$p_i(M,F) = -\frac{V_F^0(M,F)}{V_M^0(M,F)} = p(M,F).$$

The debt-pricing equation in the extension is

$$rp = (1-\tau_i)c + [(1-\tau_c)(\mu - cF) + pG_i^\star]\,p_M + \frac{1}{2}(1-\tau_c)^2\sigma^2 p_{MM} + G_i^\star p_F.$$

Subtracting the baseline debt-pricing equation (20) gives

$$(G_i^\star - G^\star)\,(pp_M + p_F) = \tau_i c.$$

Therefore,

$$G_i^\star(M,F) = G^\star(M,F) + \frac{\tau_i c}{p(M,F)p_M(M,F) + p_F(M,F)}. \tag{IA.31}$$

Lemma 2 gives $pp_M + p_F < 0$. Thus, $G_i^\star < G^\star$ whenever $\tau_i > 0$, so every baseline repurchase position remains a repurchase position. □

Proposition IA.9 (Cash accumulation and low-cash issuance under interest-income taxation)**.** *Fix $F \in (0,\bar{F})$. There exists $\tau_i^\star(F) > 0$ such that, if $\tau_i < \tau_i^\star(F)$, then*

$$G_i^\star(M,F) < 0 \quad\Longrightarrow\quad (1-\tau_c)(\mu - cF) + p(M,F)G_i^\star(M,F) > 0 \tag{IA.32}$$

for every $M \in (0, M^\star(F))$. Moreover, there exists $\tau_i^{\star\star}(F) > 0$ such that, if $\tau_i < \tau_i^{\star\star}(F)$, then there exists $\varepsilon(F,\tau_i) > 0$ satisfying

$$G_i^\star(M,F) > 0 \tag{IA.33}$$

for every $M \in (0, \varepsilon(F,\tau_i))$. The latter result continues to hold when $\tau_c = 0$.

Proof. We next prove the cash-accumulation claim. Fix $F \in (0,\bar{F})$, let N be defined by (A.45), and write

$$D(M,F) := p(M,F)p_M(M,F) + p_F(M,F) < 0.$$

Equations (A.45) and (31) imply

$$G_i^\star(M,F) = \frac{N(M,F)+\tau_i c}{D(M,F)}. \tag{IA.34}$$

For $M \in (0, M^\star(F)]$, define

$$\widehat{\tau}_i(M,F) := \frac{1}{c}\left[-\frac{(1-\tau_c)(\mu - cF)D(M,F)}{p(M,F)} - N(M,F)\right].$$

The one-sided extension of Lemma A.9 to the payout boundary gives

$$-\frac{(1-\tau_c)(\mu - cF)D(M,F)}{p(M,F)} > rp(M,F).$$

Using (A.45), we obtain

$$\widehat{\tau}_i(M,F) > \tau_c - \frac{(1-\tau_c)^2\sigma^2 p_M(M,F)V_{MM}(M,F)}{cV_M(M,F)} \geq 0.$$

Thus, $\widehat{\tau}_i(M,F) > 0$, including at the payout boundary by one-sided continuity.

Equation (A.46) gives $N(0,F) < 0$. Choose $\delta \in (0, M^\star(F))$ such that $N(\delta,F) < 0$, and let

$$\eta(F) := -\frac{N(\delta,F)}{2c} > 0.$$

Since $N(\cdot,F)$ is strictly increasing, if $\tau_i < \eta(F)$ and $M \leq \delta$, then

$$N(M,F) + \tau_i c \leq N(\delta,F) + \tau_i c < \frac{1}{2}N(\delta,F) < 0.$$

Equation (IA.34) then implies $G_i^\star(M,F) > 0$. Hence, for such tax rates, every repurchase position satisfies $M > \delta$.

The function $\widehat{\tau}_i(\cdot,F)$ is continuous and strictly positive on the compact interval $[\delta, M^\star(F)]$. Therefore,

$$\kappa(F) := \min_{M\in[\delta,M^\star(F)]} \widehat{\tau}_i(M,F) > 0.$$

Define

$$\tau_i^\star(F) := \min\{\eta(F), \kappa(F)\} > 0.$$

Suppose $\tau_i < \tau_i^\star(F)$ and $G_i^\star(M,F) < 0$. Then $M > \delta$ and $\tau_i < \widehat{\tau}_i(M,F)$, so

$$N(M,F) + \tau_i c < -\frac{(1-\tau_c)(\mu - cF)D(M,F)}{p(M,F)}.$$

Using (IA.34) and $D < 0$ now gives

$$(1-\tau_c)(\mu - cF) + p(M,F)G_i^\star(M,F) > 0.$$

Thus, cash reserves have positive instantaneous drift whenever the firm repurchases debt.

Finally, define

$$\tau_i^{**}(F) := -\frac{N(0,F)}{c} = \tau_c + 2(1-\tau_c)\mathcal{G}(x(F)) > 0. \tag{IA.35}$$

If $\tau_i < \tau_i^{**}(F)$, then $N(0,F) + \tau_i c < 0$. Continuity therefore implies that $N(M,F) + \tau_i c < 0$ for all sufficiently small $M > 0$. Since $D < 0$, equation (IA.34) gives $G_i^\star(M,F) > 0$ for all sufficiently small $M > 0$. When $\tau_c = 0$, the sufficient condition reduces to $\tau_i < 2\mathcal{G}(x(F))$, which remains strictly positive for every $F \in (0, \bar{F})$.

The additional term in (31) has the same $O(1/M)$ behavior near zero cash as the baseline leverage-adjustment rate, and its product with p is bounded. The state-process, shareholder-verification, and debtholder-break-even arguments in Section D therefore continue to apply with $(1-\tau_i)c$ in the debt-pricing equation. This proves both claims. $\square$