A Unified Theory of Delegated Capital Management*

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Abstract

We develop a theory of delegated capital management that extends Berk and Green (2004) from mutual funds to alternative assets. In perfectly competitive capital markets, we derive the optimal contract and explain key empirical regularities in private markets, including performance fees, persistent alpha, fee dispersion, and binding limits on fund size. The key distinction between mutual funds and alternatives is the liquidity and opacity of the underlying assets. When assets are liquid and performance is informative, investors optimally rely on realized performance, fund size adjusts through flows, investors earn zero net alpha, and the standard mutual fund contract is optimal. When assets are illiquid and opaque, capital must be committed for extended periods and costly due diligence becomes efficient. This creates a free-rider problem: if all investors receive identical terms, no one has an incentive to become informed. We show that the standard private-market contract is optimal in this environment and implements the first best: informed investors receive preferential terms, uninformed investors pay higher fees, and fundraising is capped. Limited liability rationalizes option-based carried interest with a hurdle rate. Finally, we show that positive ex post alpha for informed investors is consistent with competitive equilibrium, and that delegation is essential: bilateral contracting leaves information rents that distort scale, whereas delegated fundraising with uninformed capital restores the first best.

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The capital markets have seen a seismic shift in the last 70 years. In 1950 individuals and institutions invested their capital directly. In that environment, financial economists derived the now widely accepted paradigm that explains the risk-return tradeoff these individuals and institutions face.[1] By making the dual assumptions of competitive capital markets (an infinite supply of investment capital for positive net present value (NPV) investment opportunities) and rational expectations, financial economists derived the important result that, with the exception of the necessarily small set of investors who have a competitive advantage in collecting, processing and trading on information, for the rest of investors the expected return of any investment in the capital markets is determined solely by its riskiness. Because of the information revealed in prices, for these investors, other factors, like the quality of the underlying business, or the skill of the company's managers, are irrelevant.

Today capital markets are organized differently. Rather than investing directly, most market participants invest indirectly though intermediaries known as money managers. Although the shift was largely completed by the turn of the century, financial economists were slow in extending the paradigm to explain the new market structure. Indeed, for many years financial economists theorized that investors who invested through money managers should face a different risk-return tradeoff to investors who invested directly, and they were perplexed when the empirical evidence suggested otherwise. However, Berk and Green (2004) showed that under the same assumptions of competitive capital markets and rational expectations (hereafter "perfectly competitive capital markets"), investors who invest in one sector of asset management, namely mutual funds, all face the same risk-return tradeoff. Their expected return is only a function of the risk of the fund and does not depend on the skill of the manager. In this case, rather than the price revealing information, information is revealed by the flow of funds.

Our objective in this article is to complete the process of extending the paradigm by deriving the implications of perfectly competitive capital markets for the rest of the money management sector. In broad terms, money managers are divided into two sectors, publicly available investments, mainly in the form of mutual funds, and privately available investments known as "alternatives." Although, superficially, managers in both sectors appear to do the same thing — manage money on behalf of investors — as Kaplan and Schoar (2005) first pointed out, important differences between the sectors remain puzzling given the apparent similarities. In particular, although the logic in Berk and Green (2004) can explain some of the empirical regularities observed in the private sector, such as a strong flow-performance

[1] The seminal contributions that coalesced into the paradigm are Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1961) (who derived the relation between risk and return), and Muth (1961), Fama (1965), LeRoy (1976), Grossman (1976) and Grossman and Stiglitz (1980) (who derived the role of private information).

relation over a long enough horizon, it does not appear to explain other central features of alternatives, including the apparent existence of positive investment alpha and the persistence in that alpha, along with limits on the flow of incoming capital. Why do successful managers appear to leave rents with investors rather than extracting them through higher fees or larger fund size?

Other differences point in the same direction. Likely because performance in the mutual fund sector is unpredictable, there is no evidence that some investors in that sector systematically pick outperforming funds. In contrast, there is evidence that some investors in alternative assets have earned superior returns in certain periods and market segments, particularly where access and due diligence appear especially important. This pattern suggests that some investors possess an informational advantage in selecting managers. The form of the managerial contract also differs sharply across the two sectors. In the alternatives sector, managers are compensated as a function of performance as well as a fraction of assets under management, whereas in the mutual fund sector managers are compensated only as a fraction of assets under management. Many funds in the alternatives sector also limit their own size, rationing capital even from existing investors, and, as Begenau and Siriwardane (2024) document, investors in the same fund often receive different fee terms and therefore earn different returns. Taken together, these facts suggest that the information and contracting problems in alternatives differ fundamentally from those in the mutual fund sector.

We show that a simple parsimonious explanation for all of these puzzles exists. Just as in the mutual fund sector, the behavior observed in private markets can be understood within the paradigm of perfectly competitive capital markets. The crucial difference between the two sectors lies in the nature of the underlying assets under management. Unlike the mutual fund sector, the alternatives sector is characterized by illiquid and opaque investments. Opacity limits investors' ability to infer managerial ability from realized performance, while illiquidity limits their ability to respond to that information by moving capital. As a result, investors in private markets have a much stronger incentive to acquire information about managerial ability before committing capital, rather than learning gradually through performance. In this sense, the industry is endogenously bifurcated by the information problem created by the underlying assets: when assets are liquid and transparent, investors optimally rely on performance and adjust capital over time; when assets are illiquid and opaque, investors optimally acquire information in advance.

In perfectly competitive capital markets, all investment opportunities earn zero alpha. But once investors must spend resources to acquire information, informed investors must earn positive alpha after they invest in order to recover those costs. This immediately creates a free-rider problem. If all investors received identical terms, then once one investor became informed, other investors could simply mimic that investor's capital allocation and earn the same return without bearing the cost of becoming informed. Anticipating this, no investor would choose to acquire information in the first place. The only way to sustain information acquisition in equilibrium is therefore to offer different terms to informed and uninformed investors. That is, different investors will earn different returns, and if some investors have a competitive advantage in acquiring information, these return differences will persist across investors, consistent with the evidence in Begenau and Siriwardane (2024). In this sense, our result mirrors Grossman and Stiglitz (1980): when information is costly, the investors who produce it must earn higher returns ex post.

Perhaps the most surprising result of our model is that the standard contract observed in the alternatives industry is optimal. Conversely, the mutual fund contract, which Berk and Green (2004) show is optimal in the public sector, is suboptimal in private markets. The reason is that once some investors acquire information ex ante, the contract must do more than simply compensate the manager. It must both ensure that informed investors choose to invest efficiently and allow the manager to extract the surplus created by the investment opportunity. In other words, the contract must implement the same allocation that would arise if the manager were fully informed and investing his own capital, even though the relevant information is held by an outside investor.

These dual requirements imply that the optimal private-market contract must have three components and must offer different terms to informed and uninformed investors. The management fee charged to the informed investor aligns investment incentives by ensuring that she commits capital if and only if funding the manager is efficient. A performance fee allows the manager to extract the surplus once information has been produced. And a binding cap on fundraising is required to commit the manager to charge higher fees to uninformed investors. Without such a cap, the manager would have an incentive ex post to lower uninformed fees in order to raise additional capital, thereby diluting informed investors and undermining their incentive to acquire information in the first place. In equilibrium, informed investors therefore receive preferential terms, while uninformed investors pay higher fees and break even. This logic explains why the standard private-market contract differs from the mutual fund contract: in private markets, a fee based only on assets under management cannot simultaneously induce efficient information production, implement efficient investment, and extract the rents.

Although some investors acquire information, managers ultimately bear the cost of inducing them to do so. Informed investors earn positive ex post alpha, but only because the contract must compensate them for the cost of investigating and validating managerial skill. Consistent with perfectly competitive markets, managers still extract all rents once those information costs are taken into account. The cap on fundraising is therefore not incidental. Once information has been produced, the manager has an ex post incentive to attract additional uninformed capital by lowering their fees. Doing so allows the manager to dilute the informed investor for his own benefit: the manager captures the gains from the additional capital, while the return to the informed investor is reduced. Anticipating this, informed investors would have no incentive to acquire information unless the manager can commit in advance to limit fundraising. The cap is thus binding even though no investor earns economic rents ex ante, and it explains why private funds ration capital rather than simply lowering fees and expanding without limit.

It is important to emphasize that, in our framework, the excess returns earned by informed investors do not arise from a premium for holding illiquid assets. If managerial skill were directly observable, all investors would earn zero alpha even though capital would still have to be committed for an extended period. The source of positive alpha in our model is instead the limited ability of capital markets to endogenously communicate information when underlying assets are illiquid and opaque. That is not to say that a liquidity premium does not exist in practice, only that it is not needed to explain the contractual and return patterns we study here.

Finally, the model implies that delegation *itself* plays an essential economic role. At first glance, one might think that once an informed investor has identified a skilled manager, she could simply hire that manager directly and dispense with uninformed capital altogether. But that logic is incomplete. Even when the informed investor has sufficient capital to fund the project herself, bilateral contracting between the manager and the informed investor leaves information rents that distort scale and can prevent socially efficient funds from being launched. Delegated fundraising changes the problem fundamentally. By bringing in uninformed investors on different terms, the manager can break the link between the informed investor's participation and the total scale of the fund, eliminate the bilateral information-rent distortion, and restore the first-best allocation.

The paper is organized as follows. We build the theory in four steps. First, in Section 2, we abstract from delegation and analyze the efficient information acquisition and capital allocation problem when a manager invests his own capital. This benchmark isolates how the liquidity and opacity of the underlying assets determine whether it is optimal to acquire information in advance or instead learn gradually from performance. We then turn, in Section 3, to the delegated setting and derive the core equilibrium in a linear benchmark environment. There we show how differential investor terms and a binding fundraising cap implement the first-best allocation, characterize the roles of fees, carry, and uninformed capital, and explain why delegation with uninformed capital is necessary to do so. In Section 4, we then replace the linear carry with the option-based performance contract observed in practice and show that limited liability preserves the central economics of the benchmark while continuing to implement the first-best allocation. Section 5 discusses the broader implications of the model, including the interpretation of alpha in private markets, the role of uninformed investors, and the open questions the theory leaves unresolved. Section 6 concludes.

1 Background

An early paper to systematically analyze compensation contracts in venture capital is Gompers and Lerner (1999), which also notes that similar contractual terms are common across private equity funds more broadly. In these settings, general partners are typically compensated through both a management fee based on assets under management and carried interest. Gompers and Lerner (1999) develops learning- and signaling-based interpretations of these contracts, taking the basic contractual form as given. In their learning model, investors gradually learn about GP ability over time, so the contract reflects the manager's evolving reputation. That is, information is revealed gradually through realized performance. This approach leaves open the central question we address here: why, in alternative assets, the illiquidity and opacity of the underlying investments make ex ante due diligence more important and thereby alter the optimal contract relative to the mutual fund sector.

One of the most influential studies of performance in the alternatives industry is Kaplan and Schoar (2005), which documents evidence consistent with positive and persistent performance in private equity. Interpreting these findings is complicated by important measurement issues, because much of the early evidence relies on voluntarily reported data rather than the more comprehensive disclosures available in the mutual fund sector. Even so, Kaplan and Sensoy (2015) survey the subsequent literature using alternative data sources and conclude that the broad finding of persistence is robust, though the strength of that evidence appears to vary across segments of the industry.[2] Related reporting concerns arise in hedge funds, where Barth, Joenväärä, Kauppila and Wermers (2023) find that funds that do not report to commercial databases outperform those that do, suggesting that voluntary reporting need not bias measured performance upward.

Sorensen, Wang and Yang (2014) model key contractual features of private equity, including management fees, carried interest, waterfall provisions, and the illiquidity borne by investors. Calibrating their model to the data, they argue that observed outperformance may largely reflect compensation for illiquidity and for the option-like features of the contract, rather

[2]Korteweg (2023) provides a comprehensive review of both the empirical findings and the conceptual difficulties in measuring performance in private markets.

than abnormal risk-adjusted returns. Our mechanism is different. In our model, investors need not demand any premium for holding illiquid investments; the central friction is instead that illiquidity and opacity limit the ability of capital markets to aggregate and reveal information, making ex ante due diligence valuable. As a result, the model predicts zero alpha for uninformed investors, while informed investors earn positive ex post alpha to compensate for information costs.

Even setting aside the direct evidence on fund-level performance, there is indirect evidence that some investors systematically earn higher returns in alternatives. Lerner, Schoar and Wongsunwai (2007) show that certain investors, particularly endowments, earned superior returns in private equity, suggesting that they were better able to identify and access high-quality managers. Subsequent work qualifies this pattern, indicating that such outperformance was concentrated in particular periods and segments of the market, especially venture capital, rather than uniformly present across all private-market investments.[3] One prominent explanation, emphasized by David Swensen, is that some institutions possess a persistent advantage in the human capital devoted to manager selection and due diligence. Recent evidence is consistent with this view: Binfarè, Brown, Harris and Lundblad (2017, 2023) find that investor human capital and governance are associated with better performance, while Mittal (2024) finds that investors with weaker human capital perform worse. Taken together, this evidence is consistent with the idea that, in alternative assets, some investors have a comparative advantage in acquiring and acting on information about manager quality.

As Kaplan and Sensoy (2015) emphasize, persistence in alternatives is puzzling relative to the mutual fund benchmark, where competition implies that investors should not earn persistent abnormal returns. In a competitive delegated-management setting, successful managers should be able to extract those rents through higher fees or larger fund size, so the continued presence of abnormal returns to investors requires explanation. The puzzle is not equally strong across all segments of the industry: more recent evidence suggests that persistence remains more pronounced in venture capital than in buyout funds.[4] Hochberg, Ljungqvist and Vissing-Jørgensen (2014) propose that persistence may arise because incumbent investors acquire an informational advantage, giving rise to a hold-up problem. That mechanism may help explain persistence within existing relationships, but it does not by itself explain how a new fund can initially raise capital on terms that leave rents to investors in a perfectly competitive capital market.

Begenau and Siriwardane (2024) show that investors in alternatives do not all earn the

[3] See Lerner, Schoar and Wang (2008), Brown, Garlappi and Tiu (2010), and Sensoy, Wang and Weisbach (2014).

[4] Subsequent work continues to find evidence of persistence in venture capital funds, but suggests that persistence in buyout funds has weakened; see Harris, Jenkinson, Kaplan and Stucke (2023).

same returns: fees differ substantially across investors, and these differences imply persistent heterogeneity in net performance. This contrasts sharply with the mutual fund sector, where fee schedules are largely standardized, publicly disclosed, and primarily a function of the size of the investment. In alternatives, by contrast, Begenau and Siriwardane (2024) show that some investors consistently obtain better terms, and that this persistence is not explained simply by investment size. This evidence is consistent with our central mechanism: when some investors have a comparative advantage in acquiring and acting on information about manager quality, competitive equilibrium can involve persistent differences in fees and returns across investors. More broadly, it suggests that any positive and persistent net alpha in alternatives may accrue disproportionately to a subset of informed investors rather than uniformly to all capital in the sector, and that differential terms across investors are a central feature to be explained rather than a peripheral detail.

Lerner and Schoar (2004) study the role of liquidity in private equity, but their focus is on investors' demand for liquidity rather than on the liquidity and opacity of the underlying assets. More recently, Maurin, Robinson and Strömberg (2023) develop a theory in which private equity contracts and returns reflect investors' exposure to illiquidity risk. Our mechanism is different. We assume, for simplicity, that investors do not require any premium for holding illiquid investments, or equivalently, that there is sufficient capital available to fund any investment opportunity with positive net present value regardless of how long capital must be committed. In our model, the key role of illiquidity is not to generate a premium directly, but to limit the ability of capital markets to reveal information and reallocate capital, thereby making ex ante due diligence valuable. If investors also demanded compensation for illiquidity, our results would carry through, with informed investors earning that liquidity premium in addition to the returns associated with their informational advantage.

We abstract from moral hazard in the agency relationship between investors and managers. Moral hazard is clearly important in practice, and a number of papers study its implications for private equity contracting. For example, Axelson, Strömberg and Weisbach (2009) show that agency considerations can help explain observed contractual features of the industry, including the use of debt financing in underlying investments, and can also imply that limited partners retain rents in equilibrium. Our mechanism is different. In our model, rents left to informed investors arise because costly information acquisition must be sustained in competitive equilibrium, which in turn requires differential terms across investors and a binding cap on fundraising. More broadly, our point is not that agency frictions are unimportant, but rather that many of the central features of private-market contracts can be explained even in their absence.

The papers closest to ours are those that extend Berk and Green (2004) to frictions that are especially relevant in alternatives. Like us, Marquez, Nanda and Yavuz (2015) extend Berk and Green (2004) to private markets, but they focus on matching with entrepreneurs rather than on investor information acquisition. In their model, entrepreneurs want to match with high-ability managers, but they cannot perfectly infer ability from observed fund performance because performance also reflects the quality of the firms the manager attracts. Managers may therefore optimally limit fund size in order to be more selective and thereby attract better entrepreneurs.

Glode and Green (2011) also departs from the Berk and Green (2004) benchmark, but in a different direction. In their model, investors and managers learn over time about the profitability of an innovative strategy or sector rather than about manager-specific ability. Because that information can spill over and attract imitation, incumbent managers may allow initial investors to share in the rents so that they are less likely to reallocate capital to competing managers and thereby dissipate those rents. Our mechanism is different from both of these approaches. In our model, the key friction is that illiquidity and opacity make ex ante due diligence valuable, while also limiting the ability of capital markets to aggregate and reveal information through performance. This creates a free-rider problem among investors and, in equilibrium, gives rise to differential fees across investors, positive ex post alpha for informed investors, and binding limits on fund size.

2 Optimal Information Acquisition and Capital Allocation

We begin by abstracting from the delegation aspect of asset management and consider the optimal information acquisition decision and capital allocation to a potentially skilled manager. In this simple setting, the manager's skill (or the quality of the investment strategy) is uncertain (to everyone, including the manager herself), but can be learned upfront via costly due diligence, or over time from the strategy's realized performance. We use this case to derive the first-best efficient allocation and the optimal information acquisition decision, which will serve as the benchmark for the delegated setting analyzed later.

The value of conducting initial due diligence depends critically on the ability to reallocate capital ex post, and therefore on the liquidity and the opacity of the investment strategy. When the manager can rebalance frequently, as in liquid public market securities such as mutual funds, the opportunity to update the portfolio using realized returns substitutes for upfront due diligence. By contrast, when rebalancing opportunities are infrequent due to illiquidity and the value of the underlying assets are not easily observable, as in private equity, the benefits of acquiring information in advance are much greater. Having established this relationship, in the next section we focus on the delegation problem in which the manager and investor are distinct parties, and derive an optimal contract that elicits first-best information acquisition and capital allocation.

We model investment skill as in Berk and Green (2004). Managers, or investment strategies, differ in their ability to generate expected returns in excess of those provided by alternative investment opportunities available to all investors with the same risk. We denote by α the initial level of this excess return for a given manager/strategy.

We also assume there are diminishing returns associated with the scale of investment, as individual investment opportunities may be limited in scale or subject to price impact. As a result, the effective return of the strategy falls by $\beta > 0$ for each additional dollar invested.

Therefore, given total invested capital q, the realized return of the investment in period t in excess of the expected return on a market investment of equivalent risk (i.e., the appropriate passive benchmark) is given by

$$R_t(q) = \alpha - \beta q + \epsilon_t. \tag{1}$$

We assume that the strategy's exposure to any systematic sources of risk is known and can be hedged via benchmarks, so that the residual risk, ϵ, is idiosyncratic and mean zero conditional on α and any prior information.

There is also a fixed cost $\delta \geq 0$ of implementing this strategy, which represents the manager's opportunity cost and any necessary resources (data feeds, trading infrastructure, etc.). We assume both β and δ are known and independent of the manager's ability α.

Incorporating these costs, the total value added of the manager at time t, given committed capital q_t, is given by

$$\pi_t = q_t \ R_t(q_t) - \delta. \tag{2}$$

We assume the manager allocates capital in order to maximize the expected value added each period based on his perception of his ability. Specifically, let $\hat{\alpha}_s$ denote the manager's expectation of his ability given the information he has available at time s:

$$\hat{\alpha}_s \equiv E\left[\alpha | I_s\right].$$

Naturally, the optimal allocation will depend on the perceived profitability, $\hat{\alpha}$, of the strategy given the available information, relative to its expected price impact, β.

Suppose the manager must decide on date s the capital allocation for a future date $t \geq s$. Then, maximizing (2) with respect to q_t, we have the following result regarding the optimal capital allocation and the expected value added:

Proposition 1. *The optimal capital allocation on date t, given the information available on* *date $s \leq t$, is given by*

$$q^*_{t|s} = \frac{\hat{\alpha}_s}{2\beta}.$$

With that allocation, the fund's expected return is $\hat{\alpha}_s/2$, and the (per period) expected value added of the fund is[5]

$$E\left[\pi_t | I_s\right] = \frac{\hat{\alpha}_s^2}{4\beta} - \delta.$$

Proof of Proposition 1. Maximizing the expectation of (2) with respect to q_t, we have the first-order condition $E[\alpha - 2\beta q_{t|s} | I_s] = 0$, which is solved by $q^*_{t|s} = \hat{\alpha}_s/(2\beta)$, and therefore

$$E[\pi_t | I_s] = E[\alpha | I_s] q^*_{t|s} - \beta (q^*_{t|s})^2 - \delta = \hat{\alpha}^2/(4\beta) - \delta.$$

□

The optimal allocation increases with the expected alpha of the strategy. Consequently, to optimally manage money, the manager must adjust the amount of invested capital in response to new information about alpha. An important difference between public- and private-market investments is the frequency with which the manager can respond to new information by adjusting the capital allocation.

Suppose that the capital can only be adjusted every T periods. That is, the capital allocation decision q_t on date t is fixed until period $t + T$. The optimal allocation will then be determined based on the information available on each "reallocation date" according to Proposition 1. To keep the model simple, we will make two further assumptions. First, we assume that the excess return that is earned at any point in this interval only depends on the initial capital investment q_t.[6] Second, we assume that the manager does not have the option to shut down the fund.[7] Under these assumptions the (per period) expected value added is constant between adjustment periods, so the expected value added of the strategy will then be a sequence of T-period annuities, which can be evaluated as follows:

Proposition 2. *Suppose the amount of capital invested can be adjusted every T periods. Given risk-free rate r, the present value of investing is given by*

$$E\left[\int_0^\infty e^{-rt} \pi_t dt\right] = \left(\frac{1 - e^{-rT}}{r}\right) \sum_{n=0}^{\infty} e^{-rnT} \left(\frac{E\left[\hat{\alpha}_{nT}^2\right]}{4\beta} - \delta\right). \tag{3}$$

[5] For simplicity, here we implicitly assume $\hat{\alpha}_s \geq 0$ (or, equivalently, that the manager can short the strategy when $\hat{\alpha}_s < 0$). Later we will introduce constraints on the minimum expected alpha required for investment.

[6] The economic justification for this assumption is that between adjustment periods the scale is fixed, that is, and any intermediate excess returns earn a passive, zero alpha return (or are consumed).

[7] We relax this assumption shortly.

Proof of Proposition 2. Recall that $q_t = q^*_{t|s}$ for $t \in (s, s+T]$. The expected value added is therefore constant over this period, and equal to $\hat{\alpha}_s^2/(4\beta) - \delta$, which we can evaluate as annuity starting on date s:

$$e^{-rs}\left(\frac{1-e^{-rT}}{r}\right)\left(\frac{E\left[\hat{\alpha}_s^2\right]}{4\beta} - \delta\right).$$

The result then follows by summing this result over each adjustment date nT for all n. □

Now suppose the manager can pay a cost $C \geq 0$ to learn his true α. To compute the gain associated with paying this cost, we can compare equation (3) with and without the ex-ante information. When the manager pays the cost and acquires the information, he will learn his true alpha upfront and allocate capital optimally at all times. In that case, the term $E\left[\hat{\alpha}_{nT}^2\right]$ in (3) becomes $E\left[\alpha^2\right]$ when calculating the expected value added of the fund. Thus, the value of information depends on the difference between $E\left[\alpha^2\right]$ and $E\left[\hat{\alpha}_{nT}^2\right]$.

By the law of total variance, for any s, we have

$$E\left[\alpha^2\right] - E\left[\hat{\alpha}_s^2\right] = Var(\alpha) - Var(\hat{\alpha}_s) = E\left[Var(\alpha|I_s)\right].$$

In words, the gain from knowing the true α at any date is proportional to the expected residual uncertainty that would remain without that information. This gain declines with s since the residual uncertainty falls when additional returns are observed.

So, with full information, capital can be efficiently deployed at the optimal level. Without full information, capital is inefficiently deployed, but this inefficiency will be reduced on every adjustment date by using information that is learned in the interim to refine the estimate of α. The longer the interval of illiquidity, the more costly the inefficiency. The cost also increases when returns are noisier, slowing the rate of learning. The inefficiency is maximal when it is impossible to ever adjust capital, so if the costs of acquiring information are strictly less than the size of this inefficiency, there will be a sufficiently long illiquidity period such that it makes sense to pay for information. The following proposition formalizes these insights:

Proposition 3. *Given illiquidity period T, the gain from learning the true alpha of the strategy upfront is given by*

$$G(T) = \left(\frac{1-e^{-rT}}{4\beta r}\right)\left(\sum_{n=0}^{\infty} e^{-rnT} E\left[Var(\alpha|I_{nT})\right]\right). \tag{4}$$

The value of information, G, is higher when the illiquidity period T is longer, the manager's skill is more uncertain, or when returns are noisier. In particular, let $\sigma_\alpha^2 \equiv E\left[Var(\alpha|I_0)\right]$, *then note that*

$$G(T) \to \frac{\sigma_\alpha^2}{4\beta r} \quad as \quad T \to \infty.$$

Proof of Proposition 3. The expression for G follows immediately from (3) and the law of total variance. Because the residual variance of alpha, $Var(\alpha|I_t)$, declines with t (as more information is learned), we can think of (4) as the left Riemann sum approximation of the integral of a decreasing function, which declines with a shorter step size (T). Thus, G is monotone in T. G also increases if the initial uncertainty σ_α^2 is higher, or returns are noiser (in the usual sense), since that will raise the residual uncertainty that remains at any point. Finally, when T approaches infinity, there is no opportunity to update, so the residual uncertainty is constant in perpetuity. □

The limiting value of G provides a necessary condition to pay for information. So long as

$$C < \frac{\sigma_\alpha^2}{4\beta r}, \tag{5}$$

it will be profitable to learn α when the rebalancing horizon T is sufficiently long.

Thus far, we have assumed the manager invests every period and ignored the option to shut down the fund. In Appendix A we generalize the above results to include this option and show that again, the value of information increases with the illiquidity period.

From Proposition 1, when the manager has the option to shut down the fund (earning an outside option of zero), it is profitable to operate the fund given the information available on date s if

$$\frac{\hat{\alpha}_s^2}{4\beta} - \delta \geq 0. \tag{6}$$

We can use (6) to define the threshold belief, α_M, that determines when the manager will operate the fund — that is, when the value added from investing exceeds the manager's opportunity cost δ.[8] Solving (6) gives an explicit expression for the threshold α_M:

$$\alpha_M \equiv 2\sqrt{\beta\delta}. \tag{7}$$

Indeed, consider the case with $T = \infty$, so that there is no opportunity to adjust capital. Then from Proposition 1, without acquiring information, the time 0 expected value added of

[8]Note that this condition is sufficient but not necessary. Because the manager has the option to adust the size of the fund on a future adjustment date, it might be optimal to operate the fund when $\hat{\alpha}_s < \alpha_M$ because of operating affords the option to learn about ability. Obviously when $T = \infty$ the option does not exist and so then the condition is also necessary.

the fund is given by

$$E\left[\int_0^\infty e^{-rt}\pi_0 dt\right] = \frac{1}{r}E\left[\pi_0\right] = \frac{1}{r}\left(\frac{\hat{\alpha}_0^2}{4\beta} - \delta\right). \tag{8}$$

Depending on the prior belief about alpha, this expression need not be positive. Specifically, if $\hat{\alpha}_0 < \alpha_M$, (8) is negative and the fund will fail to launch.

Now consider the decision to acquire information. Again, once alpha is learned, the manager will only invest if $\alpha > \alpha_M$. Therefore, investigating is optimal only if the expected gain from running fund when it is worthwhile to do so exceeds the cost of acquiring information:

$$\frac{1}{r}E\left[\left(\frac{\alpha^2}{4\beta} - \delta\right)\mathbf{1}_{\{\alpha > \alpha_\mathbf{M}\}}\right] - C > 0. \tag{9}$$

Here, the indicator function captures the fact that the fund will only launch when the manager learns that $\alpha > \alpha_M$. That is, once the manager learns his skill, he will choose not to manage money if $\alpha < \alpha_M$.

Going forward, we assume $\hat{\alpha}_0 < \alpha_M$ so that (8) is negative — without the opportunity to acquire information or rebalance, the fund would not open.[9] We also assume that (9) holds — for large enough T, it is optimal to investigate and learn the quality of the investment opportunity.

3 Delegated Capital Management: A Linear Benchmark

We now turn to the delegation problem between managers and investors. We consider a potentially skilled manager with no capital, who must raise capital from investors. As in the previous section, initially neither the manager nor the investors know the skill of the manager. But some investors have the ability to conduct costly due diligence and become informed.

The case in which the underlying assets are liquid and returns are sufficiently informative so that it is suboptimal to acquire information upfront is the focus of Berk and Green (2004). In that setting, investors rationally wait to learn from performance, capital flows adjust over time, and the standard mutual fund contract implements the first-best allocation. Our focus here is the complementary case in which the underlying assets are sufficiently illiquid and opaque that it is optimal to acquire information before capital is committed.

The goal of this section is to derive the core delegated equilibrium in the simplest

[9] If $\hat{\alpha}_0 < \alpha_M$, it might still be socially optimal to run the fund to learn about α, but doing so would require access to outside capital via some type of long-term contract. We do not model that here, but it does not change our fundamental results.

setting that makes the economics transparent. We therefore begin with linear contracts. This benchmark allows us to isolate the free-rider problem created by costly information acquisition, to characterize the roles of differential investor terms and fundraising caps, and to show how delegation can implement the first-best allocation. In Section 4, we then relax the linearity assumption and show how limited liability leads to the standard option-based private-market contract with carried interest and a hurdle rate.

3.1 An Optimal Fully-Revealing Linear Contract

We begin by deriving an optimal fully revealing contract with a linear carry, meaning that the manager receives a constant share of the fund's net return. This linear contract serves as a benchmark that makes the economics transparent. It attains the first-best allocation given the information constraints and delivers the same payoff to the manager as would arise if the manager were fully informed and investing his own capital. It also makes clear why the mutual fund contract, which uses only a fee proportional to assets under management, cannot implement the efficient outcome in private markets. In Section 4, we then show how these same economic forces lead, under limited liability, to the standard option-based private-market contract with carried interest and a hurdle rate.

We begin by specifying the simplest contract that can implement the first-best allocation. As before, we assume that α is the manager's skill, with decreasing return to scale determined by β, and subject to a fixed cost δ, so that the total value added is given by (2). The manager, who does not know α, proposes the following contract to an informed investor:

- A management fee of f_i per dollar invested by the informed investor;
- A fractional share ("carry") s of the net return that the manager will retain;
- A cap $\bar{q}$ on overall fundraising that depends on the amount of capital q_i that the informed investor commits.

Within this framework, we show that

- There exist terms $(f_i, s, \bar{q})$ that implement the first-best allocation and represent an optimal contract for the manager;
- Given the capital commitment q_i of the informed investor, the manager's contract with uninformed investors has the same carry, s, but has a higher management fee f_u, and the capital constraint $\bar{q}$ is binding.

The above contract features a linear sharing rule s, so that the manager must pay the investor if the net return is negative. This linear sharing rule greatly simplifies the analysis, but is unrealistic, and so after establishing the main intuition here we will generalize our results in the next section to the case when carried interest is paid only on returns in excess of a target hurdle rate.

With the linear sharing rule, we will show that first-best allocation can be obtained with a cap on overall fundraising that is linear in the amount of informed capital, q_i:

$$\bar{q}(q_i) = q_0 + k \cdot q_i \tag{10}$$

We will show later that the manager will raise funds up to this cap, so that q_0 represents the minimum fund size, and $k > 0$ limits how much the fund can expand relative to the contribution of the informed investor. Importantly, the choice of k, hereafter called the *participation factor*, limits the amount of uninformed capital the manager can raise along side the informed investor. Specifically, if we subtract the level of informed capital from (10), the cap on the uninformed capital that the manager can raise is:

$$q_u = \bar{q}(q_i) - q_i = q_0 + (k-1)q_i \tag{11}$$

For now, we restrict attention to the case with $k \geq 1$, so that the amount of uninformed capital is increasing in the amount of informed capital. Later we will discuss the equilibrium determinants of k.[10]

The key question is whether this contract gives the informed investor the right incentive both to investigate and, once informed, to choose the efficient capital commitment. We therefore begin with the informed investor's problem.

3.1.1 Informed Investor's Problem

Consider the informed investor's decision regarding whether to become informed and the amount of capital to commit. In analyzing this decision, we assume for now that the informed investor expects that the manager will always choose to raise the maximum amount of capital $\bar{q}$, and show later that it will indeed be optimal for the manager to do so.

Suppose the investor chooses to investigate and learn alpha. Once informed, the investor will then choose an amount q_i to invest to maximize her profits from investing (that is, the

[10] The case $k \geq 1$ is certainly more consistent with practice, and, because we will show later that there is a degree of indeterminacy in k, choosing $k > 1$ may be without loss of generality. We will also show that the linear form of the cap is uniquely optimal in this case — that is, it is the only functional form that supports the efficient outcome.

value added on her invested capital net of fees):

$$q_i(1-s)E\underbrace{\left[R_t - \beta\bar{q}(q_i) - f_i\right]}_{\text{pre-carry net return}} = q_i\,(1-s)\,(\alpha - \beta(q_0 + kq_i) - f_i)\,. \tag{12}$$

The optimal q_i solves the first order condition:

$$(1-s)\,(\alpha - \beta(q_0 + kq_i) - f_i) - q_i\,(1-s)\,\beta k = 0.$$

Solving for q_i provides the amount of capital the informed investor will choose to invest as a function of the manager's skill:

$$q_i(\alpha) = \frac{\alpha - (\beta q_0 + f_i)}{2\beta k} = \frac{\alpha - \alpha_I}{2\beta k}, \tag{13}$$

where α_I is defined as

$$\alpha_I \equiv \beta q_0 + f_i. \tag{14}$$

Notice that the informed investor will only invest $q_i > 0$ if $\alpha > \alpha_I$; that is, α_I is the minimum level of managerial ability required for the informed investor to commit capital, hereafter the *investment threshold*. At this level of managerial ability, the informed investor's expected net return is zero.

Given this investment rule, the net return of the informed investor when she chooses to invest is equal to

$$(1-s)\,(\alpha - \beta(q_0 + kq_i) - f_i) = (1-s)\,\frac{\alpha - \alpha_I}{2}. \tag{15}$$

Multiplying (15) by the optimal investment, (13), the informed investor's expected profit is given by:

$$\pi_i(\alpha) = \begin{cases} \left(\frac{1-s}{k}\right)\frac{(\alpha-\alpha_I)^2}{4\beta} & \text{if } \alpha > \alpha_I \\ 0 & \text{if } \alpha \leq \alpha_I \end{cases} \tag{16}$$

Recall that to become informed, the informed investor incurs a cost C upfront. In order to ensure that she can recoup this cost over the T-period life of the investment, the informed investor's per-period expected profit must be at least c_T, where,[11]

$$c_T \equiv \left(\frac{r}{1 - e^{-rT}}\right) C, \tag{17}$$

[11] Because the informed investor's information will be fully revealed in equilibrium, we assume he must recoup his investigation cost before the subsequent funding round in period T. In principle, the manager could write a longer term contract and the informed investor could recoup the investment over a longer horizon. In this case the net alpha will remain positive and will persist over funding rounds.

that is, the equivalent per-period annuitized cost.

Using our earlier notation $\hat{\alpha}_0$ to represent the expected ability of the manager prior to investigating, the informed investor therefore finds it optimal to investigate as long as

$$E\left[\pi_i(\alpha)\right] - c_T \geq E\left[\pi_i(\hat{\alpha}_0)\right], \tag{18}$$

where $E\left[\pi_i(\hat{\alpha}_0)\right]$ is the expected profit from investing without information. Note that if $\hat{\alpha}_0$ is below the investment threshold α_I, then the right-hand side of (18) is zero, and the optimality condition for investigation becomes,

$$E\left[\pi_i(\alpha)\right] \geq c_T. \tag{19}$$

3.1.2 Manager's Problem

Having characterized how the informed investor responds to the contract, we now turn to the manager's problem of choosing contract terms so that the induced investment rule implements the first-best allocation. We will show that by choosing the contract parameters (s, f_i, q_0, k) appropriately, the manager can achieve the first-best allocation and extract all surplus. The reason the first-best allocation is attainable is that once the manager and uninformed investors observe the informed investor's capital commitment q_i, they can infer α from (13) and information becomes symmetric. Importantly, because the first best is attained, the resulting contract must be optimal for the manager.

To set the terms of the contract to achieve the first best, there are two conditions that must be satisfied. First, the fund should operate whenever the value added from investing exceeds the manager's opportunity cost; that is, if $\alpha > \alpha_M$ as defined by (7). Hence, the manager's and the informed investor's investment thresholds must coincide:

$$\alpha_I = \alpha_M = 2\sqrt{\beta\delta}. \tag{20}$$

Second, from Proposition 1, the total capital $\bar{q}$ that is invested should match the efficient level. Therefore,

$$\bar{q}(q_i) = q_0 + kq_i = \frac{\alpha}{2\beta}. \tag{21}$$

Substituting the optimal investment function $q_i(\alpha)$ of the informed, (13), into the above, we have

$$q_0 + \frac{\alpha - \alpha_I}{2\beta} = \frac{\alpha}{2\beta}. \tag{22}$$

Because (22) must hold for all α, we can solve for q_0 and apply (20) to derive

$$q_0 = \frac{\alpha_I}{2\beta} = \frac{\alpha_M}{2\beta} = \sqrt{\frac{\delta}{\beta}}. \tag{23}$$

Finally, substituting this result for q_0 into the definition of α_I, (14), implies that the optimal management fee for the informed investor is

$$f_i = \frac{\alpha_M}{2} = \sqrt{\beta\delta}. \tag{24}$$

Together, (23) and (24) determine the (f_i, q_0) that ensure the capital allocation is optimal. The remaining contract terms (s, k) and the management fee f_u paid by the uninformed are determined so that all investors break even, thereby ensuring that all rents accrue to the manager.

The uninformed fee is set to ensure that the uninformed investor's earns zero net return:

$$(1 - s)\left(\alpha - \beta\bar{q}(q_i) - f_u\right) = 0, \tag{25}$$

which, using (21), requires setting the fee to be

$$f_u = \alpha - \beta\left(\frac{\alpha}{2\beta}\right) = \frac{\alpha}{2}. \tag{26}$$

That is, the uninformed management fee is set equal to the gross expected excess return of the fund, so that the uninformed earn a zero net alpha.[12] While α is not known to the uninformed investors (or the manager) initially, it is revealed by the size of the informed investor's commitment q_i (or equivalently by the size of the cap, $\bar{q}$, on the total capital raised). Thus, the fee f_u can be implemented conditional upon the informed investor's committing to invest.[13] Note that because the informed investor only invests when $\alpha > \alpha_M$, if the fund is launched, $f_u > f_i$ implying "most favored nation" treatment of the informed investor.

Lastly, the manager sets the contract so that the informed investor earns zero rents net of her investigation cost. That is, under the assumption that $\hat{\alpha}_0 \leq \alpha_M$ the participation constraint, (19), holds with equality, implying that the final parameters (s, k) satisfy:

$$E\left[\pi_i(\alpha)\right] = \left(\frac{1-s}{k}\right) E\left[\frac{(\alpha - \alpha_M)^2}{4\beta} \cdot \mathbf{1}_{\{\alpha > \alpha_M\}}\right] = c_T. \tag{27}$$

[12] Note that because the carry is charged symmetrically, uninformed investors do not pay carry in expectation.

[13] It is common in practice for funds to use the commitment of a high reputation investor to raise additional capital.

Notice that (s, k) are not uniquely determined. There is a one-dimensional family of solutions that satisfy (27). That is, the manager can either set the ratio of informed to uninformed capital by specifying k and then let the carry be determined by the distribution of skill in the market, or he can set the carry and let the ratio be a function of the distribution of skill in the market. This indeterminacy mirrors a similar result in the mutual fund space. As Berk and Green (2004) show, a mutual fund manager can either choose the management fee and let the market determine the size of the fund he manages or he can choose the size of the fund and let the market decide the management fee. Because the cross-sectional variation in fund size is much larger than management fees, in that space it appears that managers opt to set the fee. It appears that a similar result holds in the alternative space. The cross-sectional variation of the carry charge is low, likely implying that the participation factor k, which controls the ratio of informed to uninformed capital, adjusts based on variation in market perception of managerial ability.

3.2 Why the Capital Constraint Binds

Thus far, we have taken as given that the manager raises capital up to the cap and does not exceed it. We now show that both conclusions are endogenous: the manager will indeed raise capital up to the cap, and the cap itself is necessary because without it he would continue raising capital by lowering uninformed fees.

Note first that because the manager sets q_u and f_u after α is revealed, f_u can be set so that the uninformed earn zero expected net return, and the uninformed investors will willingly invest the level of capital allowed. As a result, we can assume the manager captures all rents attributed to uninformed capital. Therefore, the manager's payoff for a given choice of q_u is given by

$$\begin{aligned} \pi_m(q_u) &= E[\text{ informed fees} + \text{uninformed fees} - \delta\ ] \\ &= q_i(f_i + s\,[\alpha - \beta(q_i+q_u) - f_i]) + q_u(f_u + s\,\underbrace{[\alpha - \beta(q_i+q_u) - f_u]}_{=0}) - \delta \\ &= (sq_i + q_u)\,[\alpha - \beta(q_i+q_u)] + (1-s)q_i f_i - \delta. \end{aligned} \tag{28}$$

The first-order condition for the manager's choice of q_u, if total capital is below the cap, is then

$$\begin{aligned} \pi_m'(q_u) &= [\alpha - \beta(q_i+q_u)] - \beta\,(sq_i + q_u) \\ &= \alpha - 2\beta(q_i+q_u) + \beta(1-s)q_i. \end{aligned} \tag{29}$$

The final term in (29), $\beta(1-s)q_i$, is always strictly positive given $s < 1$. This term equals the reduction in the expected profit of the informed agent, whose net return falls by $\beta(1-s)$ for each additional dollar of capital raised. Then, given a total quantity below the efficient level — that is, $q_i + q_u \leq \alpha/(2\beta)$ — the first term in (29) is also non-negative:

$$\begin{aligned} \pi'_m(q_u) &> \alpha - 2\beta(q_i + q_u) \\ &\geq \alpha - 2\beta(\alpha/2\beta) = 0, \end{aligned} \tag{30}$$

Hence, if the cap is set to the efficient level of capital, taking additional uninformed capital both increases the total value added of the fund and lowers the net return to the informed investor, both of which benefit the manager. Therefore, the investment cap (21) strictly binds.

The result that the capital constraint binds explains one of the most perplexing features of private market investing. Unlike open-ended mutual funds, private investment funds often have binding limits on the capital raised. It might therefore appear that the fund manager could increase his rents by raising fees or taking additional capital.

In our model, the cap is binding even while the manager extracts all rents. The reason is that without the cap, since $\pi'_m(\bar{q}) > 0$, the manager has an incentive to take more capital by lowering the fee to uninformed investors. Increasing the amount of capital beyond the first best is a second-order loss, but the manager captures a first-order gain by reducing the return to the informed investor. Knowing this, the informed investor will not invest without an ex ante commitment from the manager to limit the total amount of capital to the first-best level.

The following proposition summarizes the equilibrium:

Proposition 4. *A contract (s, f_i, q_0, k) satisfying (23), (24), and (27) is optimal for the manager and achieves the first-best capital allocation. Under this contract, the manager starts the fund and raises capital $q_u = \bar{q}(q_i) - q_i$ from uninformed investors, charging $f_u > f_i$ given by (26), if and only if the informed investor commits $q_i > 0$. All investors break even, and the manager earns an expected per period profit of*[14]

$$E\left[\left(\frac{\alpha^2}{4\beta} - \delta\right) \mathbf{1}_{\{\alpha > \alpha_{\mathrm{M}}\}}\right] - c_T, \tag{31}$$

consistent with (9).

Proof of Proposition 4. Follows from the analysis in the text. □

[14]As a reminder, the fact that the fund is only launched when $\alpha > \alpha_M$ is equivalent to $\alpha^2/(4\beta) > \delta$.

3.3 The Roles of Fees, Carry, and the Capital Constraint

Having established the linear benchmark and shown why the fundraising cap binds, we can now interpret the economic role of each component of the contract. A distinguishing feature of private equity contracts compared to mutual fund contracts is the existence of a cap on AUM and a performance-based carry fee in addition to a management fee. As we have seen, unlike the mutual fund setting, to achieve the first best two conditions need to be satisfied. First, the set of states in which the fund invests must be efficient. Second, all extracted rents must accrue to the manager. Because the informed investor determines the set of states when the fund invests, his contract must satisfy this dual purpose. As shown in (24), the management fee is set to ensure that the set of states in which investment occurs is optimal by determining the critical alpha threshold. The role of carry and the cap on total capital, on the other hand, is to ensure that the informed investor breaks even.

As we have already noted, there is an indeterminacy between k and s, that is the participation factor and the carry fee. The reason for this indeterminacy is that both variables affect the rents. The manager can directly reduce the rents to informed capital by raising the carry s. But he can also reduce rents by increasing the ratio of uninformed capital to informed capital because uninformed capital does not earn rents. Intuitively, because free riding by uninformed investors comes at the expense of informed investors, the manager can extract rents from informed investors either by increasing the amount of free riding (by increasing k), or limiting the free riding (by keeping k low) and instead increasing the carry, s.

Notice that in this equilibrium, the management fee paid by the informed is not set to extract rents. Instead, it is set to ensure that the manager makes at least his opportunity cost even in the worst-case scenario when the manager turns out to have ability α_M. The form of the optimal contract also makes it clear why the manager has to take uninformed capital at all. The informed investor is indifferent about investing in a manager with ability α_M, which necessarily means that the optimal investment at that level is zero. But when a α_M skilled manager invests his own capital, he has strictly positive investment (he breaks even at this level). Therefore, to achieve first best investment, the manager must take uninformed capital. At this minimal capital level, the management fee exactly covers the fixed costs of managing capital.

This role of the management fee accords well with the common justification given in the industry for charging a management fee — to ensure that the fixed costs of managing money are covered. If the fee was not set in this way, there would be states in which either the manager would choose to quit when his ability is revealed, or the informed investor would choose not to invest even though starting the fund is optimal.

In contrast, in a public-market mutual fund the assets are liquid and investors have insufficient incentive to investigate ex ante. Because managers and investors are therefore symmetrically informed, they always choose to invest in the same states. As a result, the manager can use the management fee to extract the rents. In private markets, when the manager and investor are asymmetrically informed, an additional instrument is required for rent extraction.

3.4 Feasibility and Implementation of the Linear Contract

The economic roles of the contract terms are now clear. What remains is to characterize when this linear implementation can be supported with economically natural parameter values.

A technical issue we have not yet addressed is when (27) admits a solution with both $s \geq 0$ and $k \geq 1$. As long as the investigation cost satisfies

$$c_T \leq E\left[\frac{(\alpha - \alpha_M)^2}{4\beta} \cdot \mathbf{1}_{\{\alpha > \alpha_M\}}\right], \tag{32}$$

then (27) implies $(1 - s)/k \leq 1$, and a solution with $s \geq 0$ and $k \geq 1$ is possible. However, for an investigation cost in the range

$$E\left[\frac{(\alpha - \alpha_M)^2}{4\beta} \cdot \mathbf{1}_{\{\alpha > \alpha_M\}}\right] < c_T < E\left[\frac{(\alpha^2 - \alpha_M^2)}{4\beta} \cdot \mathbf{1}_{\{\alpha > \alpha_M\}}\right], \tag{33}$$

then, while it is still efficient to investigate, the informed investor's participation constraint may require $k < 1$ or $s < 0$.

To see why this wedge exists, consider the extreme case when the costs of investigation equals the total available surplus. In that case, all rents must accrue to informed capital to compensate the informed investor for investigating. But recall that the manager captures all the rents associated with uninformed capital via the uninformed management fee. Hence, in states in which management talent is high, these fees must be rebated to informed capital (so that, in expectation, the manager earns nothing after paying his fixed costs), which necessarily requires a negative s. Indeed, in this scenario, we might expect delegation to fail, with the informed investor hiring the manager directly instead. In summary, the costs of investigation need to be low enough to ensure that any rents the manager earns from uninformed fees do not exceed the total expected rents from managing money.

When these conditions hold, the contract can be implemented in a simple sequence. First the manager calculates the minimum skill level at which he would still choose to manage money, α_M, from his opportunity cost, δ, and the technology parameter β using (7). Next, he uses this parameter to calculate f_i using (24) and q_0 using (23), and picks any reasonable s. He then calculates k using his choice of s based on the investigation cost and (27). Using these parameters, he presents potentially informed investors with the contract (f_i, s, q_0, k). On seeing this contract, an investor chooses to incur the cost C to acquire information, and if the resulting α exceeds α_M she chooses to invest q_i as given by (13). On seeing this investment level, uninformed investors (and the manager) infer α by inverting (13) as follows:

$$\alpha = \alpha_M + 2\beta k q_i.$$

The manager then offers the remaining (uninformed) investors the contract $\{f_u, s\}$, where f_u is given by (26). Note that since $\bar{q}(q_i) = \alpha/2\beta$, we can equivalently set the uninformed management fee proportional to the maximum fund size:

$$f_u = \beta \bar{q}. \tag{34}$$

The manager then solicits the maximum allowable capital from uninformed investors, which will equal $\bar{q}(q_i) - q_i$ as specified in (11).

This equilibrium implementation closely models reality. New managers almost always seek a cornerstone investor with a successful reputation. In return for investing, this investor is offered a larger capital allocation and a break in fees. The manager then uses the information that this investor chose to invest in his fund to solicit capital from other investors. Moreover, managers routinely cap the size of their funds, that is, they agree up front to limit the amount of capital they will accept.

The main difference between this benchmark contract and those observed in practice is the linear carry charge. Because the manager would have to pay investors when realized returns are sufficiently negative, the contract requires unlimited liability. In practice, of course, managerial liability is limited by both personal wealth and bankruptcy law, so the linear contract is not directly implementable.

The next section shows that this limitation does not overturn the economics of the benchmark. We replace the linear carry with the option-based carry used in practice, under which carry is earned only when returns exceed a hurdle. We show that limited liability then rationalizes the standard private-market contract, while preserving the central economic logic of the model and, most importantly, the first-best allocation.

4 Option-Based Performance Contracts

Although the linear benchmark in Section 3.1 attains the first-best allocation, it requires unlimited liability for the manager. When the fund's return is sufficiently negative, the manager must reimburse investors for a fraction s of the loss. Such a contract is therefore not directly implementable in practice, both because the manager's ability to make such payments is uncertain and because bankruptcy law limits investors' ability to enforce them.

In practice, private-market contracts restore limited liability by making carry asymmetric: the manager receives a share s only of returns in excess of a target hurdle rate. As long as the hurdle rate is non-negative, the manager's minimum carry payment is also non-negative, and liability is limited. Typical hurdle rates in private equity range from zero to ten percent, depending on the asset class, with zero common in U.S. venture funds and 8% the most common in private equity.[15] We now show that replacing the linear carry with this option-based performance fee preserves the central economics of the benchmark and continues to implement the first-best allocation.

We now formalize that contract and show how it modifies the linear benchmark. Let $h \geq 0$ be the hurdle rate. Investors are charged the carry s on the portion of their return net of management fees that exceeds the hurdle h. When the investor's net return is below h, carry is zero (and the manager does not rebate the investors). That is, the carry owed to the manager by an investor with management fee f, given (gross) return R, can be calculated as[16]

$$\text{Carry}(R) = s(R - f - h)^+$$

We refer to this carry formula as "option-based" carry, as opposed to the "linear" carry of the prior section.

This modification changes the shape of the informed investor's incentives, but not the basic structure of the problem. As we show below, all of the main insights of the prior section continue to hold. The important contract features, such as the cap on uninformed capital, and a lower management fee for the informed investor, continue to be optimal. The main qualitative change is that the amount of capital invested by the informed, as well the cap on uninformed capital, are no longer linear in the manager's perceived alpha.

[15] See Goodwin Insight, November 2023. Note that hurdles may be soft or hard; hard hurdles compensate the manager only on the portion of the return that exceeds the hurdle, as modeled here, whereas soft hurdles compensate the manager based on the entire return once the hurdle is met.

[16] As a simplification, we compute carry each period. In private equity, carry is computed based on the current cumulative return, and may include clawback provisions if the fund subsequently underperforms. Hedge funds pay carry periodically but often contain a high water mark, effectively paying carry on cumulative returns only.

4.1 Gross versus Net Returns

The key new object in the option-based setting is the mapping from expected pre-carry returns to expected net returns. Recall that a manager whose trading strategy has quality α, and who invests total capital q, generates a gross return equal to

$$R(q) = \alpha - \beta q + \epsilon. \tag{35}$$

For an investor with management fee f, the expected *pre*-carry return is therefore

$$\bar{R}^{pre} = E\left[R(q) - f\right] = \alpha - \beta q - f. \tag{36}$$

Given carry s with hurdle h, this investor has an expected *net* return, post carry, given by

$$\text{Expected Net Return} = \bar{R}^{net} = F(\bar{R}^{pre}) \equiv \bar{R}^{pre} - sE\left[\left(\bar{R}^{pre} + \epsilon - h\right)^{+}\right]. \tag{37}$$

For example, with this definition of F, the expected net return for the informed investor given a manager of quality α managing a fund at the efficient investment level, $\frac{\alpha}{2\beta}$, is equal to $F(\alpha/2 - f_i)$. In the linear carry model, $F(\alpha/2 - f_i) = (1-s)(\alpha/2 - f_i)$, and therefore the expected net return is linear in alpha. With option-based carry, the net return is increasing and concave in alpha. The concavity arises because better performance raises the probability that carry will be charged.

The concavity of F has two important implications. First, because expected net returns are no longer linear in alpha, the informed investor's optimal capital commitment will no longer increase linearly with alpha. Second, the concavity of F raises the break-even level of the pre-carry return. To see why, note that $F(0) < 0$, since even with a zero expected pre-carry return, the expected carry is positive — the manager will earn carry when the idiosyncratic shock is above the hurdle (that is, the option component of managerial compensation is always positive regardless of managerial ability).

The second implication is central for the contract design problem. In the linear model, investors invest whenever the pre-carry return is positive. With option-based carry, a positive pre-carry return is no longer sufficient. Instead it must exceed the break-even pre-carry expected return, r_0, which is strictly positive, and is defined as the unique solution to

$$F(r_0) \equiv 0. \tag{38}$$

Note that, solving this equation using (37) implies that r_0 is equal to the expected carry charge at break even,

$$r_0 = sE\left[(r_0 + \epsilon - h)^+\right] > 0.$$

Intuitively, $r_0 > 0$ is the pre-carry return that just offsets the value of the manager's carry option. The informed investor will invest whenever his expected pre-carry return exceeds r_0 (rather than zero, as in the linear case). Managers set fees so that uninformed investors earn zero expected net return, which requires their pre-carry expected return to equal r_0 (rather than zero, as in the linear case). Notice that although r_0 is an increasing function of the carry charge, s, it does not depend on the manager's ability, α. Figure 1 illustrates the difference the option contract makes.

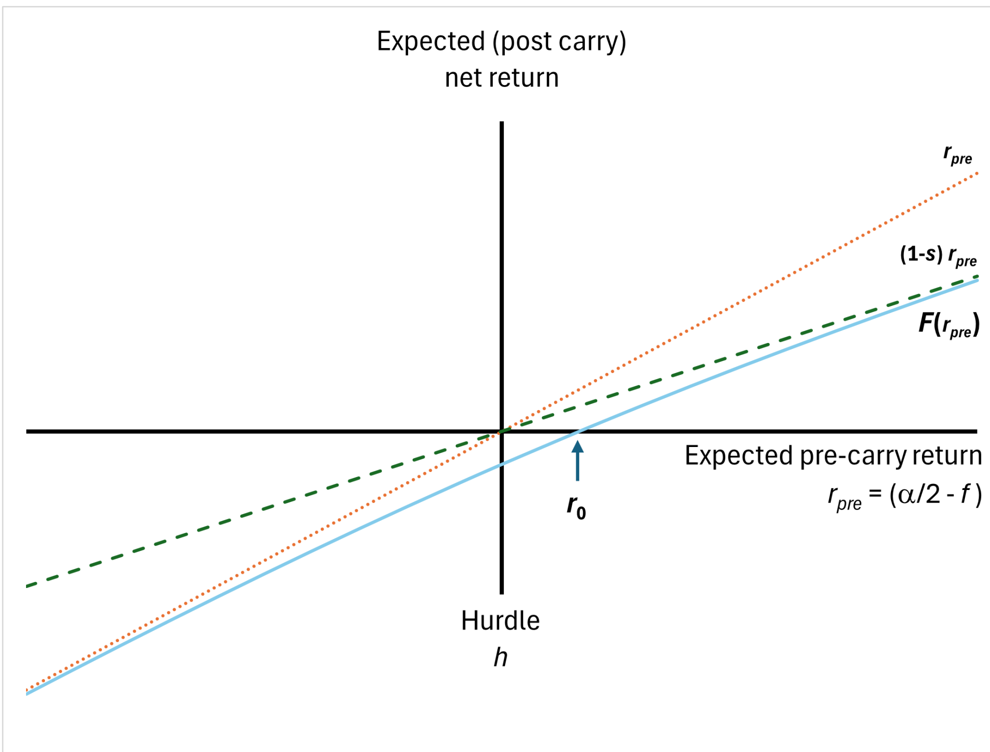


Figure 1: Break-even Expected Pre-Carry Return r_0. The green dashed line is the net return earned in the linear case. The solid blue line is the net return earned in the option based carry case. The optimal investment policy in the linear case is to invest when ever the net return exceeds the hurdle, while in the option case, it is optimal to invest only when the net return exceeds r_0.

4.2 Informed Investor's Problem

With linear carry, the informed investor's expected net return, and therefore her optimal capital commitment, varied linearly with α. With option-based carry, expected net returns are concave in α, so the informed investor's capital commitment is no longer linear. As a result, we can no longer solve directly for the manager's inference and the associated fundraising cap as functions of the informed investor's quantity choice.

We therefore proceed differently from the linear case. Suppose the informed investor follows an investment schedule $q_i(\alpha)$, with $q_i(0) = 0$ and $q_i'(\alpha) > 0$ whenever $q_i(\alpha) > 0$. Given such a schedule, the manager can infer α by inverting q_i and can then set the fundraising cap $\bar{q}(q_i)$ equal to the efficient fund size. We begin by characterizing the schedule $q_i(\cdot)$ that is incentive compatible for the informed investor; once that schedule is determined, the remaining contract terms can be chosen to implement the efficient allocation and extract the rents.[17]

Consider such an equilibrium investment schedule $q_i(\alpha)$. Suppose that after learning the true α, the informed investor deviates and instead communicates $\hat{\alpha}$ by choosing to invest $q_i(\hat{\alpha}) > 0$. In that case, from Proposition 1, the manager will infer that the efficient size of the fund is $\hat{\alpha}/(2\beta)$. Assuming the manager then runs the fund at that inferred efficient size, the informed investor's expected pre-carry return is

$$\alpha - \beta\left(\frac{\hat{\alpha}}{2\beta}\right) - f_i = \alpha - \frac{\hat{\alpha}}{2} - f_i.$$

The informed investor will choose $\hat{\alpha}$ to maximize her net investment payoff:

$$\max_{\hat{\alpha}} \quad q_i(\hat{\alpha})F(\alpha - \hat{\alpha}/2 - f_i). \tag{39}$$

The first order condition is therefore

$$q_i'F - \tfrac{1}{2}q_iF' = 0. \tag{40}$$

Incentive compatibility requires that (39) be maximized when $\hat{\alpha} = \alpha$.[18] When $q_i > 0$, this requirement implies that (40) can be written as follows:

$$\frac{q_i'(\alpha)}{q_i} = \frac{\frac{1}{2}F'(\alpha/2 - f_i)}{F}$$

[17]This part of the model can be viewed either as a signaling problem or as a screening mechanism in which the manager commits ex ante to the mapping from the informed investor's capital commitment to total fundraising. Methodologically, however, it is closer to signaling, as the key object is the incentive-compatible capital-commitment schedule $q_i(\alpha)$ and the relevant deviations are deviations in that signal.

[18]Given $s < 1$, the objective in (39) satisfies the Spence-Mirrlees single-crossing property $\frac{\partial^2}{\partial q_i \partial \alpha} q_i F = F' > 0$. Thus, we can use the approach of Mailath (1987 CITATION) to characterize the separating equilibrium in our setting with continuous types, which we follow here.

Integrating both sides in the range when $q_i > 0$, we get $\ln q_i(\alpha) = \ln F(\alpha/2 - f_i) + \ln\theta$, where $\ln\theta$ is an arbitrary constant of integration. Exponentiating both sides gives

$$q_i(\alpha) = \theta\, F(\alpha/2 - f_i). \tag{41}$$

For ease of comparison with the linear case, we introduce a new constant of integration k,

$$k \equiv \frac{1}{\beta\theta(1-s)}. \tag{42}$$

Substituting this expression leads to the following general form for any incentive compatible investment schedule:

$$q_i(\alpha) = \begin{cases} \frac{F(\alpha/2-f_i)}{\beta k(1-s)} & \text{if } F(\alpha/2 - f_i) > 0 \\ 0 & \text{if } F(\alpha/2 - f_i) \leq 0 \end{cases} \tag{43}$$

where the scale factor $k > 0$, the arbitrary constant of integration is still to be determined.[19] To see that this result matches that of the linear model, note that in the linear model, $F(\alpha/2 - f_i) = (1-s)(\alpha/2 - f_i)$. Hence in that case, when $\alpha > \alpha_I = 2f_i$, (43) becomes

$$q_i(\alpha) = \frac{(1-s)(\alpha/2 - f_i)}{\beta k(1-s)} = \frac{\alpha - 2f_i}{2\beta k} = \frac{\alpha - \alpha_I}{2\beta k}. \tag{44}$$

matching (13).

Equation (43) states that for any contract in which the informed investor reveals α and the manager implements the efficient investment level $\alpha/(2\beta)$, the informed investor commits capital proportional to her expected net return. Note also that the investment threshold α_I is defined by

$$F(\alpha_I/2 - f_i) = 0 = F(r_0),$$

and therefore,

$$\frac{\alpha_I}{2} - f_i = r_0. \tag{45}$$

The left side of (45) is the expected return of the fund, $\alpha_I/2$, less the management fee, that is, the informed investor's pre-carry expected return. So the investment threshold is the level of managerial ability at which the investor's pre-carry expected return is equal to r_0, the

[19] The boundary for positive investment, $\alpha_0 = \inf\{\alpha | q_i(\alpha) > 0\}$, can be derived as follows. Since the informed investor earns a non-negative profit, $F(\alpha/2 - f_i) \geq 0$ for $\alpha > \alpha_0$, and so by continuity, $F(\alpha_0/2 - f_i) \geq 0$. But if $F(\alpha_0/2 - f_i) > 0$, then type $\alpha_0 - \epsilon$ could profit by deviating from choosing zero investment, to choosing $q_i(\alpha_0 + 2\epsilon) > 0$, thereby earning net return $F(\alpha_0 - \epsilon - (\alpha_0 + 2\epsilon)/2 - f_i) = F(\alpha_0/2 - 2\epsilon - f_i) > 0$ for small enough ϵ. Hence $F(\alpha_0/2 - f_i) = 0$.

break-even pre-carry expected return. (Again, this matches the linear case, for which $r_0 = 0$.)

4.3 Manager's Problem

Having characterized the implementable investment schedules of the informed investor, we now turn to the manager's problem of choosing contract terms that extract the first-best surplus. To do so, the manager again commits to the following upfront:

- A management fee of f_i per dollar invested by the informed investor,
- A fractional share ("carry") $s \in [0, 1)$ of the net return in excess of a hurdle rate $h \geq 0$ that the manager will retain;
- A cap $\bar{q}(\cdot)$ on overall fundraising that depends on the amount of capital q_i that the informed investor commits.

First, we must ensure the informed investor's investment threshold α_I matches the efficient level α_M defined in (7). From (45), this requires

$$f_i = \frac{\alpha_M}{2} - r_0. \tag{46}$$

Given any schedule $q_i(\alpha)$ based on (43), the manager and other investors can invert the schedule to assess the informed investor's determination of alpha based on his capital commitment. The cap on total capital is then set to the efficient level:

$$\bar{q}(q_i) = \frac{\alpha}{2\beta} = \frac{q_i^{-1}(q_i)}{2\beta}. \tag{47}$$

Note that because q_i increases with alpha, $\bar{q}$ will increase with q_i. To compare with the linear model, we can compute the slope of $\bar{q}$ as follows:

$$\begin{aligned} \bar{q}'(q_i) &= \frac{1}{2\beta q_i'(\alpha)} = \frac{k(1-s)}{F'(\alpha/2 - f_i)} = \frac{k(1-s)}{1 - s\,\Pr(\alpha/2 - f_i + \epsilon > h)} \\ &= \frac{k(1-s)}{1 - s\,\Pr(\text{carry}|\alpha)} \end{aligned} \tag{48}$$

Therefore, $\bar{q}$ is convex in q_i, with a slope that increases and approaches k as the likelihood that carry will be charged approaches one (matching the linear model).

Uninformed investors, upon observing the capital cap $\bar{q}$, can also infer $\alpha = 2\beta\bar{q}$ and will therefore correctly anticipate a gross return of the fund equal to $\alpha - \beta(\alpha/(2\beta)) = \alpha/2$. Given that, the manager will set the fee f_u to make the uninformed investors indifferent. That is, the uninformed expected net return will be set to zero, implying that the uninformed investors' pre-carry expected return must be set equal to r_0, the break-even pre-carry expected return:

$$\frac{\alpha}{2} - f_u = r_0,$$

implying

$$f_u = \frac{\alpha}{2} - r_0. \tag{49}$$

Note that, like f_i, the uninformed fee is similarly reduced by r_0, the break-even expected carry. Consequently, the difference in management fees,

$$f_u - f_i = \frac{\alpha - \alpha_M}{2},$$

is the same as in the linear case, and because $\alpha > \alpha_M$ if the fund launches, we have $f_u > f_i$.

Thus, given any parameters (s, h, k), by choosing $(f_i, \bar{q}, f_u)$ according to (46), (47), and (49), the manager can be assured that the schedule q_i given by (43) is incentive compatible for the informed investor. The final condition for efficiency is that the informed investor finds it worthwhile to investigate. The informed investor's profit upon investigating is given by

$$\pi_i(\alpha) = \begin{cases} q_i(\alpha)F(\alpha/2 - f_i) = \frac{F(\alpha/2-f_i)^2}{\beta k(1-s)} & \text{if } \alpha > \alpha_M \\ 0 & \text{if } \alpha \leq \alpha_M \end{cases} \tag{50}$$

Therefore, for a given option-based carry (s, h), we can choose k so that the expected profit of the informed investor from investigating equals the investigation cost c_T:[20]

$$E\left[\pi_i(\alpha)\right] = \frac{1}{\beta k(1-s)} E\left[F(\alpha/2 - f_i)^2 \, \mathbf{1}_{\{\alpha > \alpha_M\}}\right] = c_T \tag{51}$$

The final step is to show that, in this equilibrium, the manager will indeed choose to accept capital until the cap, $\bar{q} = \alpha/2\beta$, is reached. As in the linear model, the manager will find it optimal to do so. Because the manager earns all rents attributed to the uninformed investors, the manager's expected profit is equal to the total value added of the fund less the amount captured by the informed investor. If the total capital is below the efficient level — i.e. the cap — then each dollar raised both improves efficiency, and so increases the total value added of the fund, while it reduces the net return earned by the informed investor, both of which improve the manager's payoff. We prove this formally as part of the following

[20]Note that we continue to assume $\hat{\alpha}_0 < \alpha_M$, so that the investor would never invest without investigating first. (Indeed, with option-based carry, an investor who does not investigate earns an even lower expected payoff than $\pi_i(\hat{\alpha}_0)$, since their higher residual uncertainty raises the expected cost of the carry option.)

proposition:

Proposition 5. *For a given (s, h), if the manager chooses $(k, f_i, \bar{q}, f_u)$ according to (46)-(51), then the informed investor will choose to investigate and invest according to (43), and the manager will raise uninformed capital until the cap $\bar{q}$ binds at the first-best allocation. This contract is optimal for the manager, who will earn the first-best surplus (31).*

Proof of Proposition 5. The essence of the proof is contained within the main text. Here we formalize two details. First, we verify the global optimality of the informed agent's optimal investment choice. Recall that the first-order condition (40) holds by construction of $q_i(\alpha)$. To see that the first-order condition is sufficient, note that F is increasing and concave, and therefore q_i, which is proportional to F, is also increasing and concave. The second derivative with respect to $\hat{\alpha}$ of the objective in (39) is then:

$$q_i'' F - q_i' F' + \tfrac{1}{4} q_i F'' < 0.$$

Second, we verify that the manager will choose raise capital until $\bar{q}$ binds. Here we use the fact that, because the uninformed earn zero rents, the manager's expected payoff is equal to the total surplus net of the informed agent's profit:

$$\pi_m = (q_i + q_u)(\alpha - \beta(q_i + q_u)) - \delta - \pi_i.$$

Therefore,

$$\tfrac{\partial}{\partial q_u} \pi_m = (\alpha - 2\beta(q_i + q_u)) - \tfrac{\partial}{\partial q_u} \pi_i.$$

Because higher q_u reduces the expected return for the informed investor, the second term, $-\frac{\partial}{\partial q_u}\pi_i$, is strictly positive. The first term is weakly positive as long as $q_i + q_u \leq \alpha/(2\beta)$. Therefore, the cap $\bar{q} = \alpha/(2\beta)$ strictly binds. □

4.4 Quantitative Implications

Having characterized the option-based contract analytically, we now illustrate its quantitative implications with a simple calibration. We identify the main parameters as follows. First, we consider a minimum fund size $\bar{q}(0) = \$50$ million, with manager opportunity cost $\delta = \$1$ million per year. Because the minimum efficient alpha is α_M, the minimum fund size must equal $\frac{\alpha_M}{2\beta}$, with total value added $\bar{q}(0)\alpha_M/2 = \delta$. Therefore, we can calibrate

$$\alpha_M = \frac{2\delta}{\bar{q}(0)} = 4\% \quad \text{and} \quad \beta = \frac{\alpha_M}{2\bar{q}(0)} = 0.04\%.$$

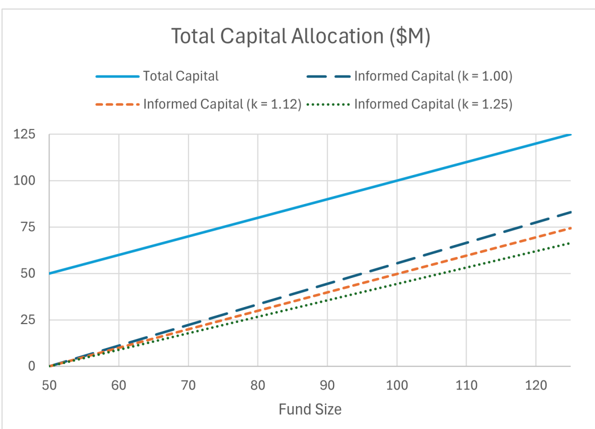
We consider a standard carry rates of 10–20% and use a hurdle rate of 0%. Assume idiosyncratic risk $\epsilon \sim N(0, \sigma_\epsilon^2)$. Then we can calculate r_0 from (38) as follows:

	σ_ϵ		
s	10%	15%	20%
10%	0.420	0.630	0.841
15%	0.648	0.972	1.297
20%	0.890	1.335	1.780

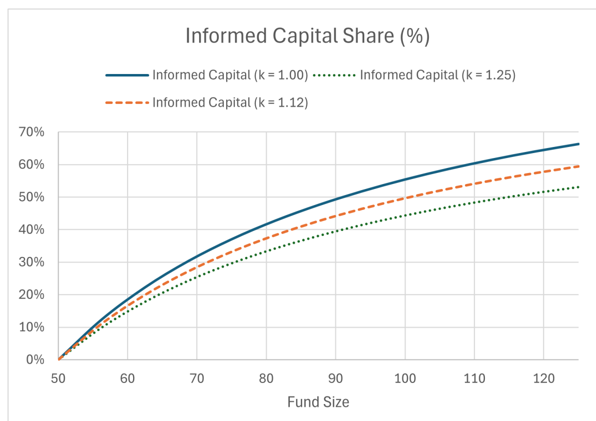
Table 1: Implied r_0 (in percent) given carry s, zero hurdle, and idiosyncratic risk σ_ϵ.

Hence with a 15% volatility and 20% carry, $r_0 = 1.34\%$. Recall that we can interpret r_0 as the minimum expected carry payment. Therefore, the optimal informed management fee with these parameters is $f_i = \alpha_M/2 - r_0 = 0.66\%$.

The figures below show the allocation of capital and surplus for different realizations of alpha, which correspond to different fund sizes. We show the capital allocation to the informed investor for different choices of the co-investment factor k.



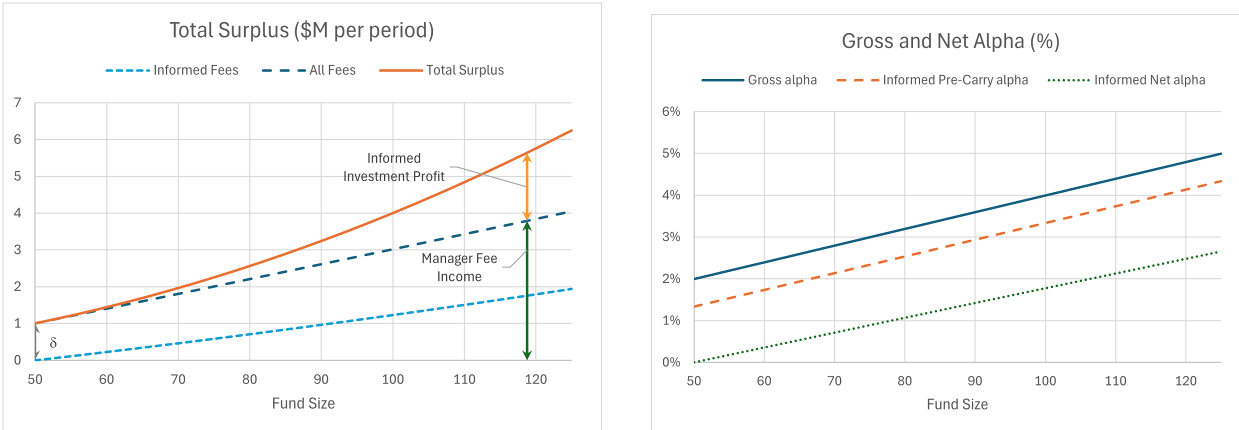
(a) Total Capital Allocation



(b) Informed Share

Figure 2: Total Capital Allocation and Informed Capital Share versus Fund Size

The next set of figures show the total surplus, total fees, and gross and net alpha as function of realized fund size.



(a) Total Surplus and Fees

(b) Gross and Net Alpha

Figure 3: Total Surplus, Fee Income, and Gross and Net Alpha versus Fund Size

5 Discussion

5.1 Industry Bifurcation and Optimal Contracts

The central implication of the model is that the delegated money management industry is optimally bifurcated. In public markets, where underlying assets are liquid and performance is informative, investors can learn about managerial ability over time and reallocate capital in response. In that environment, it is inefficient to incur the cost of ex ante due diligence, and the mutual fund contract, in which investors face common terms and managers are compensated through a fee based on assets under management, implements the efficient allocation. In private markets, by contrast, the illiquidity and opacity of the underlying investments make realized performance a less effective mechanism for revealing skill and adjusting capital. As a result, some investors optimally acquire information in advance, and once they do, efficiency requires a different contractual structure: investors cannot all face the same terms, and the optimal contract includes both management fees and carry, together with limits on the scale of fundraising. The contractual differences between public and private delegated management are therefore not incidental institutional details, but the equilibrium response to different information environments.

5.2 Alpha, Persistence, and Investor Heterogeneity

As in the competitive benchmark for mutual funds, our model implies that managers extract the rents once the costs of information have been accounted for. The implication, however, is not that all investors in private markets earn zero alpha. Uninformed investors break even, while informed investors earn positive ex post alpha because they incur the cost of acquiring and validating information about managerial skill. In the baseline model, where informed investors are competitively supplied and face the same investigation cost, this positive ex post alpha exactly compensates the marginal informed investor for those costs, so informed investors do not outperform on a net basis.

A simple extension changes that conclusion in a natural way. If investors differ in their costs of acquiring information, then those with the lowest costs earn true excess returns, while only the marginal informed investor breaks even. More generally, if the ability to acquire and act on information is itself scarce, then informed investors may capture part of the surplus in equilibrium. In our framework, this is equivalent to a higher effective cost of inducing informed participation. Importantly, this does not alter the efficiency result. The same contractual structure can still implement the efficient allocation; the additional rents required by informed investors are accommodated through different values of the carry, participation factor, or hurdle rate, rather than through a distortion in investment. The model can therefore account not only for positive ex post alpha, but also for persistent outperformance by a subset of investors with a durable advantage in due diligence and manager selection.

More broadly, the model suggests a different interpretation of alpha in private markets. Positive alpha need not reflect a breakdown of competition or an unexplained anomaly. Instead, it can arise because illiquidity and opacity limit the ability of capital markets to aggregate information and reallocate capital efficiently through prices and flows alone. In that environment, informed investors can earn positive ex post alpha even though the overall equilibrium remains competitive.

5.3 Why Uninformed Investors Are Necessary

At first glance, uninformed investors should seem unnecessary in our setting. The manager has the skill, and the informed investor has the information needed to determine whether the fund should be launched and, if so, at what scale. Moreover, uninformed investors are not needed to provide scarce capital: even when the informed investor has sufficient capital to fund the project herself, bilateral contracting between the manager and the informed investor still generally fails to implement the first best.

This is a central and perhaps surprising implication of the model. As we show formally in Appendix C, bilateral contracting necessarily leaves information rents with the informed investor. The reason is that, in a bilateral relationship, the informed investor's capital commitment must simultaneously provide incentives for information acquisition and truth-telling and determine the overall scale of the fund. Those two objectives are in conflict. To limit the rents that must be left to the informed investor, the manager distorts her stake downward, but because her stake is also the entire fund, that distortion reduces total investment and, for some realizations of managerial ability, prevents socially efficient funds from being launched. Introducing uninformed investors changes the problem fundamentally. By breaking the link between the informed investor's participation and the total scale of the fund, delegation allows the manager to reduce the informed investor's stake without sacrificing efficient fund size. In this sense, uninformed investors are essential not because they provide information or scarce capital, but because they make efficiency implementable by eliminating bilateral information rents.

5.4 Open Questions on Contract Design and Implementation

Our analysis identifies the contractual structure needed to implement the efficient allocation in private markets, but it does not fully pin down the exact combination of contract terms. In particular, as shown in Section 3.1 and in the option-contract extension that follows, there is an indeterminacy among the participation factor, the carry, and the hurdle rate. This indeterminacy is not about the division of rents. Rather, it reflects the fact that different combinations of these instruments can deliver the same equilibrium payoffs while implementing the same efficient allocation. Unmodeled considerations may therefore determine which contractual form is selected in practice. For example, if investors face liquidity risk or portfolio-concentration concerns, managers may prefer a higher participation factor and lower carry, since this allows the informed investor to be compensated while holding a smaller share of the fund. Conversely, stronger managerial incentive concerns, which we abstract from, may favor a higher carry and a lower participation factor. Tax considerations may also matter, since carry and management fees are taxed differently.

A second open question concerns the credibility of informed participation. In the baseline model, the informed investor takes only a very small position near the investment threshold. In practice, however, such a small investment may be neither economical nor credible as a signal of manager quality. As we show in Appendix B, once fixed costs of allocating capital are introduced, the informed investor optimally makes a meaningful minimum investment if she invests at all. The logic is straightforward: when becoming informed is costly, a small position does not credibly signal that the investor has incurred those costs. A larger investment makes the signal more believable because it shows that the investor has committed meaningful capital on the basis of her information. This extension suggests that the substantial initial positions often taken by cornerstone investors may reflect not only scale, but also the need to make their information production and endorsement credible.

6 Conclusion

This paper extends the competitive paradigm of delegated money management from mutual funds to private markets. Under the assumptions of competitive capital markets and rational expectations, the same underlying logic continues to apply: after the costs of information have been accounted for, the equilibrium allocation is efficient and managers extract the rents. What differs across sectors is not the presence or absence of competition, but the way information is produced and transmitted.

The crucial distinction is that public-market investments are liquid and transparent, whereas private-market investments are illiquid and opaque. In public markets, investors can learn about managerial ability over time from realized performance and can reallocate capital in response, so it is inefficient to incur the cost of ex ante due diligence. In private markets, by contrast, illiquidity and opacity make realized performance a less effective mechanism for revealing skill and adjusting capital. As a result, some investors optimally acquire information in advance, and the efficient contractual response is different. The model therefore explains why private markets feature management fees together with carry, why different investors receive different terms, and why fund-size limits are binding in equilibrium.

More broadly, the model provides a different interpretation of alpha in private markets. Positive alpha need not reflect a breakdown of competition or an unexplained anomaly. Instead, it can arise because illiquidity and opacity limit the ability of capital markets to aggregate and transmit information through prices, performance, and capital flows alone. In that environment, informed investors can earn positive ex post alpha even though the overall equilibrium remains competitive.

Finally, the model highlights an unexpected role for delegation itself. The manager and the informed investor together possess all the information needed to identify the efficient outcome, and uninformed investors are not needed to provide scarce capital. Yet bilateral contracting between the manager and the informed investor is generally inefficient because it leaves information rents that distort investment. Delegation changes the problem fundamentally: by bringing in uninformed investors on different terms, the manager can break the link between the informed investor's stake and the total scale of the fund, eliminate the bilateral information-rent distortion, and implement the first best. In this sense, uninformed investors are essential not because they provide information or scarce capital, but because they make efficiency implementable. More broadly, our results show that when illiquidity and opacity prevent markets from aggregating information through prices and flows alone, both the optimal contract and the optimal structure of delegation must adjust to restore efficiency.

Appendix

A Option to Shut Down

This appendix characterizes the efficient operating and shutdown policy. At reset dates $0, T, 2T, \ldots$ the fund's capital $q \geq 0$ can be reset and the fund may instead be shut down permanently (yielding payoff 0 thereafter). Between reset dates, capital is held fixed. If the fund continues, the operating cost $\delta > 0$ is incurred each period and beliefs about α evolve as data are observed.[21]

Full information. If α is learned at $t = 0$, the efficient policy operates the fund only if the optimized per-period surplus is nonnegative. Under the quadratic technology, this is equivalent to $\alpha \geq \alpha_M$, where

$$\alpha_M \equiv 2\sqrt{\beta\delta}.$$

The full-information value is

$$V_0^{FI} = \frac{1}{r}\, E\left[\left(\frac{\alpha^2}{4\beta} - \delta\right) \mathbf{1}_{\{\alpha > \alpha_M\}}\right].$$

Comparative statics in T

Let $V_0(T)$ denote the maximal expected discounted surplus when capital and the shutdown decision can be changed only at reset dates $\{0, T, 2T, \ldots\}$ (so shutdown is not allowed within a block). Next we consider lengthening the illiquidity period by an integer multiple.

Proposition 6. ***(Value declines with a longer illiquidity interval.)*** *For any integer $m \geq 1$,*

$$V_0(T) \;\geq\; V_0(mT).$$

Proof. Fix any admissible policy under interval mT. Construct a feasible policy under interval T that makes the same shutdown/continue decisions at dates $0, mT, 2mT, \ldots$ and, conditional on continuing, holds the same capital level fixed over each of the m finer sub-intervals within every coarse block. This policy is feasible because it simply ignores intermediate information and does not change decisions at intermediate reset dates. It generates identical cash flows, hence the same expected discounted payoff. Taking suprema over admissible policies yields $V_0(T) \geq V_0(mT)$. □

[21] We allow learning even if $q = 0$ (e.g., the fund can record trades without executing them). This simplifies notation; alternatively one could impose a minimum investment level.

Remark 1. Proposition 6 gives monotonicity along divisibility chains because $\{0, mT, 2mT, \dots\} \subseteq \{0, T, 2T, \dots\}$. A standard way to obtain monotonicity for all real T is to model T as a *capital lockup*: capital may be reset at stopping times $\tau_0 = 0 < \tau_1 < \dots$ satisfying $\tau_{k+1} - \tau_k \geq T$ (with shutdown allowed only at reset times). Larger T then imposes a tighter restriction on admissible policies, implying $V_0(T)$ is weakly decreasing in T without any distributional assumptions.

Value of information as T becomes large. Full information at $t = 0$ is a refinement of the information available under learning, so it weakly increases the maximal payoff. Proposition 6 implies that $V_0(T)$ weakly decreases as the reset interval is coarsened along multiples, so the value of acquiring information upfront is (weakly) increasing in T.

In the limiting case in which capital is never reset after $t = 0$ (formally, along $mT \to \infty$), the efficient policy either shuts down immediately or operates forever based on the initial posterior mean $\hat{\alpha}_0 \equiv E[\alpha \mid I_0]$. The corresponding *per-period* no-information surplus is

$$\left(\frac{\hat{\alpha}_0^2}{4\beta} - \delta\right) \mathbf{1}_{\{\hat{\alpha}_0 > \alpha_M\}}.$$

Thus acquiring information at cost C is efficient if and only if

$$rC \;<\; E\left[\left(\frac{\alpha^2}{4\beta} - \delta\right) \mathbf{1}_{\{\alpha > \alpha_M\}}\right] - \left(\frac{\hat{\alpha}_0^2}{4\beta} - \delta\right) \mathbf{1}_{\{\hat{\alpha}_0 > \alpha_M\}}. \tag{52}$$

When $\delta = 0$, condition (52) coincides with (5).

Computing $V_0(T)$ in the normal case

To determine the value of information when $T < \infty$, we must solve numerically a dynamic program for the optimal investment and continuation decision. Assume that at reset dates $t = nT$ the posterior satisfies $\alpha \mid I_{nT} \sim \mathcal{N}(\mu_n, s_n^2)$, where

$$\mu_n \equiv E[\alpha \mid I_{nT}], \qquad s_n^2 \equiv \mathrm{Var}(\alpha \mid I_{nT}).$$

Under Gaussian updating with per-period signal noise σ_ε,

$$s_{n+1}^2 = \left(\frac{1}{s_n^2} + \frac{T}{\sigma_\varepsilon^2}\right)^{-1}, \qquad \mu_{n+1} = \mu_n + \eta_n, \quad \eta_n \sim \mathcal{N}\big(0,\; s_n^2 - s_{n+1}^2\big).$$

Define the optimized discounted payoff from operating for one reset interval as a function of the posterior mean,

$$g_T(\mu) \equiv \left(\frac{1-e^{-rT}}{r}\right)\left(\frac{\mu_+^2}{4\beta}-\delta\right), \qquad \mu_+ \equiv \max\{\mu, 0\},$$

and let $\gamma \equiv e^{-rT}$. The one-step recursion is

$$V(\mu_n, s_n^2) = \max\left\{0,\ g_T(\mu_n) + \gamma\, E\big[V(\mu_{n+1}, s_{n+1}^2) \mid \mu_n, s_n^2\big]\right\}.$$

Because s_n evolves deterministically in time, we can compute $V_0(T) = V(\mu_0, \sigma_0^2)$ by backward induction on a grid for μ, evaluating the conditional expectation with Gauss–Hermite quadrature.

B Extension: Minimum Informed Investment

In our optimal contract, the informed investor's stake, given by (13), approaches zero if α is close to $\alpha_I = \alpha_M$. In reality, a di minimus informed investment may be unrealistic — as well as unconvincing to uninformed investors. Indeed, it is likely that there are additional costs to the informed investor from deploying capital, and potentially from monitoring the manager to be sure the investment strategy is correctly implemented. There is also a potential opportunity cost of not being able to invest profitably elsewhere.

To incorporate this idea into the model, suppose that there is a fixed cost δ_i borne by the informed investor upon investing in the fund, and let δ_m be the fixed cost of the manager, so that the total cost of running the fund is $\delta = \delta_m + \delta_i$ and the total available surplus (31) is unchanged.

Let us continue to define $\alpha_M \equiv 2\sqrt{\beta\delta}$ to be the efficient investment threshold as in (7). Then, in the linear model, the optimal contract in this case will adjust as follows:

$$f_i = \frac{\alpha_M}{2} - \sqrt{\frac{\beta\delta_i k}{1-s}} \quad \text{and} \quad q_0 = \frac{f_i}{\beta}, \tag{53}$$

The informed investor will then invest according to

$$q_i(\alpha) = \left(\frac{\alpha/2 - f_i}{\beta k}\right) \mathbf{1}_{\{\alpha > \alpha_M\}}. \tag{54}$$

Then, at the investment threshold α_M, the minimum informed investment is

$$q_i(\alpha_M) = \sqrt{\frac{\delta_i}{\beta k(1-s)}}. \tag{55}$$

Note that we can use equation (55) to calibrate δ_i based on the minimum investment level for an "anchor" investor.

With these changes, the investment cap continues to provide the efficient level of capital,

$$\bar{q}(q_i(\alpha)) = q_0 + kq_i(\alpha) = \frac{f_i}{\beta} + \frac{\alpha/2 - f_i}{\beta} = \frac{\alpha}{2}, \tag{56}$$

and the minimum rent for the informed investor is equal to

$$\pi_i(\alpha_M) = q_i(\alpha_M)(1-s)(\alpha_M/2 - f_i) = \delta_i, \tag{57}$$

which also implies the manager earns $\alpha_M^2/(4\beta) - \delta_i = \delta_m$, as required.

Finally, the co-investment factor k must be set so that informed investor breaks even:

$$E\left[(\pi_i(\alpha) - \delta_i)^+\right] = c_T. \tag{58}$$

Compared to our earlier setting, compensating the informed investor for their opportunity cost will lead to a somewhat lower co-investment factor.

C Direct Investment and Managerial Rents

C.1 Direct Investment Model Formulation

This appendix rewrites the manager–informed-investor interaction in the notation used to analyze the direct investment setting.

An informed investor investigates and learns the manager's true skill α. After observing α, the investor chooses an investment quantity $q \geq 0$. If $q = 0$, all payoffs are zero. Otherwise, with positive investment, the investment earns the return

$$R = \alpha - \beta q + \epsilon, \qquad \epsilon \sim N(0, \sigma_\epsilon^2), \tag{59}$$

and the investment payoff net of operating costs (δ) is

$$Y = q(1+R) - \delta = q\big(1 + \alpha - \beta q + \epsilon\big) - \delta. \tag{60}$$

Given the realized outcome (Y, q), the investor pays the manager a transfer $\tau(Y, q)$. The transfer payment must satisfy:

- Manager Limited Liability: $\tau(Y, q) \geq 0$.
- Monotone Contracts: $\tau_Y(Y, q) \in [0, 1]$.

First Best Investment. Given the manager's skill α, the *first-best* investment level maximizes the expected net payoff $Y - q$:

$$\max_{q \geq 0} \mathbb{E}[Y - q \mid \alpha] = \max_{q \geq 0} q\alpha - C(q), \tag{61}$$

where $C(q) \equiv \beta q^2 + \delta$ if $q > 0$ and zero otherwise. The first-best investment is therefore:

$$q^{FB}(\alpha) = \frac{\alpha}{2\beta} \mathbf{1}\{\alpha \geq \alpha_0\}, \tag{62}$$

where α_0 is the threshold where the surplus becomes non-negative. Because the expected surplus is equal to

$$\frac{\alpha^2}{4\beta} - \delta, \tag{63}$$

this first-best investment threshold is given by

$$\alpha_0 = 2\sqrt{\beta\delta}. \tag{64}$$

Transfers in Return Space. Given a candidate investment function $q(\alpha)$ which is strictly increasing on the range $q(\alpha) > 0$ — a property we will show holds in equilibrium — we can define the inverse function $\hat{\alpha}(q)$ to be the manager's "inferred skill" given the investor's quantity choice.

Additionally, for $q > 0$, we define the realized return *innovation* as

$$e(Y, q) \equiv \underbrace{\left(\frac{Y + \delta}{q} - 1\right)}_{\text{realized gross return}} - \underbrace{[\hat{\alpha}(q) - \beta q]}_{\text{predicted gross return}} = (\alpha - \hat{\alpha}) + \epsilon. \tag{65}$$

Thus, we can rewrite the transfer function $\tau(Y, q)$ in return space using the relation

$$\tau(Y, q) \equiv q\, t(e, \hat{\alpha}), \tag{66}$$

with the constraints on τ replaced by similar constraints on t; that is, $t \geq 0$ and $t_e \in [0, 1]$. (For the latter, note that since e is linear in Y/q, the chain rule implies $\tau_Y = t_e$.)

C.2 Optimal Direct Mechanism

Revelation Principle. Because the investor is the informed party and the manager designs the contract, the standard revelation principle applies. Hence, without loss of generality, we can consider an optimal mechanism that specifies the investment and transfer rule $(q(\hat{\alpha}), t(e, \hat{\alpha}))$ as a function of the investor's report $\hat{\alpha}$, and for which the investor has the incentive to report truthfully ($\hat{\alpha} = \alpha$). We can restrict attention to mechanisms with $q = 0$ when $\alpha < \alpha_0$ since investment is then socially inefficient even absent information rents. In the optimal mechanism, the investment threshold may be strictly higher, $\alpha^* > \alpha_0$.

Investor's Payoff. Given the mechanism, the investor's problem is

$$\pi_I(\alpha) = \max_{\hat{\alpha}} \; q(\hat{\alpha})\alpha - C(q(\hat{\alpha})) - q(\hat{\alpha})\mathbb{E}[t(\alpha - \hat{\alpha} + \epsilon, \hat{\alpha})] \tag{67}$$

Then, using the envelope theorem and truth-telling,

$$\pi_I'(\alpha) = q(\alpha) \left(1 - \mathbb{E}[t_e(\epsilon, \alpha)]\right). \tag{68}$$

Next define,

$$m(\alpha) = \mathbb{E}[t(\epsilon, \alpha)] \qquad \text{and} \qquad \Delta(\alpha) = \mathbb{E}[t_e(\epsilon, \alpha)]. \tag{69}$$

Because the total surplus is zero at α_0, and both parties have outside options of zero, we must have $\pi_I(\alpha_0) = 0$. Hence, using (68),

$$\pi_I(\alpha) = \int_{\alpha_0}^{\alpha} q(\tilde{\alpha})(1 - \Delta(\tilde{\alpha})) \, d\tilde{\alpha}. \tag{70}$$

In addition, from (67),

$$\pi_I(\alpha) = q(\alpha)\alpha - C(q(\alpha)) - q(\alpha)m(\alpha). \tag{71}$$

Manager's Payoff. Given α, the manager earns the expected transfer

$$\pi_M(\alpha) = q(\alpha)m(\alpha) = q(\alpha)\alpha - C(q(\alpha)) - \pi_I(\alpha). \tag{72}$$

The manager's objective is equivalent to maximizing surplus net of the investor's rents.

Now, note that

$$\mathbb{E}[\pi_M(\alpha)] = \int_{\alpha_0}^{\infty} q(\alpha)m(\alpha)\, dF(\alpha) \tag{73}$$

$$= \int_{\alpha_0}^{\infty} q(\alpha)\alpha - C(q(\alpha)) - \pi_I(\alpha)\, dF(\alpha)$$

$$= \int_{\alpha_0}^{\infty} q(\alpha)\alpha - C(q(\alpha)) - \pi_I'(\alpha)\frac{1-F(\alpha)}{f(\alpha)}\, dF(\alpha) \tag{74}$$

where (74) follows by integrating $\mathbb{E}[\pi_I]$ by parts (the standard Myerson identity, using the boundary conditions $\pi_I(\alpha_0) = 0$ and $\pi_I(\alpha)(1 - F(\alpha)) \to 0$ as $\alpha \to \infty$). Finally, substituting (68) into (74) we can write the expected manager rents as

$$\mathbb{E}[\pi_M(\alpha)] = \int_{\alpha_0}^{\infty} q(\alpha)\left[\alpha - (1-\Delta(\alpha))\frac{1-F(\alpha)}{f(\alpha)}\right] - C(q(\alpha))\, dF(\alpha) \tag{75}$$

Therefore, as in the standard mechanism design setting, we can think of type α having a "virtual" valuation that is adjusted by the required information rent:

$$\alpha_v = \alpha - \underbrace{(1-\Delta(\alpha))\frac{1-F(\alpha)}{f(\alpha)}}_{\text{information rent}} \tag{76}$$

Note that the information rent has two components. The standard term, $(1-F)/f$, is the inverse hazard rate, which captures the cost of preventing higher types from mimicking lower types. This rent is mitigated by the term $1-\Delta$, where $\Delta = \mathbb{E}[t_e(\epsilon, \alpha)] \in [0, 1]$ is the expected sensitivity of the transfer to the return innovation. Intuitively, a more performance-sensitive contract (higher Δ) makes mimicry more costly for a high-α investor who underreports: by committing less capital than is optimal, the investor's realized return innovation $e = (\alpha-\hat{\alpha})+\epsilon$ is on average positive, and a higher Δ means this positive innovation translates into higher expected transfers, thereby deterring the deviation and reducing the information rent the manager must concede. This Δ term thus captures the incentive benefit of the security design.

Security Design. To characterize the optimal mechanism, we use the following security design result:

Proposition 7 (Option Contract)**.** *Assume the idiosyncratic risk ϵ has an absolutely continuous distribution with nondecreasing hazard rate and finite first moment. Then for a given expected transfer m, a call option with strike price k yields the highest possible $\Delta(m)$,* *determined implicitly by*

$$m = \mathbb{E}\left[(\epsilon - k)_+\right] \quad \textit{and} \quad \Delta = \Pr\left(\epsilon > k\right).$$

The strike k is decreasing in m, and Δ is increasing and concave in m, with $\Delta = 0$ when $m = 0$.

Proof. We want to maximize $\Delta(\alpha) = \mathbb{E}[t_e(\epsilon, \alpha)]$ given $m(\alpha) = \mathbb{E}[t(\epsilon, \alpha)]$. Letting λ be the lagrange multiplier on m, and using integration by parts given distribution $G(\epsilon)$ (which is justified since t has at most linear growth), we must maximize

$$\Delta - \lambda m = \mathbb{E}\left[t_e - \lambda t\right] = \mathbb{E}\left[t_e\left(1 - \lambda\frac{1 - G(\epsilon)}{g(\epsilon)}\right)\right].$$

Recall that $t_e \in [0, 1]$. With a nondecreasing hazard rate, the bracketed term in the expectation above is nondecreasing, and so crosses zero at most once. The optimal t_e is therefore bang-bang, with a slope of zero to the left and one to the right of some point k. That is, for each α, the transfer function $t(\cdot, \alpha)$ that maximizes $\Delta(\alpha)$ for a given cost $m(\alpha)$ is a call option with strike price $k(\alpha)$ such that

$$m(\alpha) = \mathbb{E}\left[(\epsilon - k(\alpha))_+\right].$$

Therefore, k is decreasing in m. This option has corresponding delta $\Delta(\alpha) = \Pr\left(\epsilon > k(\alpha)\right)$ which is decreasing in k and hence increasing with m. As $m \to 0$, then $k \to \infty$ and so $\Delta = \Pr(\epsilon > k) \to 0$. Note also that

$$\frac{d\Delta}{dm} = \frac{d\Delta/dk}{dm/dk} = \frac{g(k)}{1 - G(k)},$$

which shows that $d\Delta/dm$ equals the hazard rate, which increases with k and so decreases with m. Hence Δ is concave in m. □

The next result shows that higher idiosyncratic volatility reduces the Δ achievable for any given expected transfer m: since the same m requires a higher strike k when volatility is larger, less probability mass falls in the tail and Δ is smaller.

Lemma 1 (Scaling)**.** *Assume the idiosyncratic risk $\epsilon = \sigma z$. Let $\Delta_\sigma(m)$ be the delta of the call option with value m. Then $\Delta_\sigma(m) = \Delta_1(m/\sigma)$.*

Proof. For the case in which $\epsilon = \sigma z$ is scaled by σ, there is a natural scaling law, directly from the definitions of m and Δ:

$$m(k,\sigma) = \mathbb{E}\left[(\sigma z - k)_+\right] = \sigma \mathbb{E}\left[(z - k/\sigma)_+\right] = \sigma m(k/\sigma, 1).$$

Therefore,

$$\Delta_\sigma(m) = \Pr(\sigma z > k(m)) = \Pr(z > k(m)/\sigma) = \Delta_1(m/\sigma).$$

□

So, doubling the volatility effectively "stretches" the curve of Δ as a function of m horizontally by a factor of two.

Optimal q and Boundary Condition. Given m and Δ, from (75), we can choose q optimally to maximize the manager's rents by maximizing the virtual surplus

$$q(\alpha)\alpha_v - C(q(\alpha))$$

and therefore, $q^*(\alpha) = q^{FB}(\alpha_v) = \alpha_v/(2\beta)$. The manager will choose to invest when the virtual surplus is positive, and therefore when $\alpha_v > \alpha_0$. Therefore, the investment boundary solves

$$\alpha_0 = \alpha_v(\alpha^*_+) = \alpha^* - (1 - \Delta_\sigma(m(\alpha^*_+)))\frac{1 - F(\alpha^*)}{f(\alpha^*)} \tag{77}$$

Note that Δ and m are discontinuous and increasing at the boundary ($m = \Delta = 0$ to the left), so we need to evaluate this condition using the limit as $\alpha \downarrow \alpha^*$.

To evaluate that limit, note that the true surplus at this boundary point is

$$\begin{aligned} q^*(\alpha^*)\alpha^* - C(q^*(\alpha^*)) &= q^{FB}(\alpha_0)\alpha^* - C(q^{FB}(\alpha_0)) \\ &= \frac{\alpha_0\alpha^*}{2\beta} - \frac{\alpha_0^2}{4\beta} - \delta \\ &= \frac{\alpha_0(\alpha^* - \alpha_0)}{2\beta} \\ &= q^*(\alpha^*)(\alpha^* - \alpha_0) \end{aligned} \tag{78}$$

Because the investor receives $\pi_I(\alpha^*) \geq 0$ at the boundary, to maximize Δ we maximize m by setting $q^*(\alpha^*)m(\alpha^*)$ equal to the surplus at the boundary (where we define m and Δ to be right-continuous):

$$m(\alpha^*) = \alpha^* - \alpha_0 \qquad \text{and} \qquad \Delta(\alpha^*) = \Delta_\sigma(\alpha^* - \alpha_0) \tag{79}$$

Hence, we can write the final boundary condition as

$$\alpha_0 = \alpha^* - (1 - \Delta_\sigma(\alpha^* - \alpha_0))\frac{1 - F(\alpha^*)}{f(\alpha^*)}. \tag{80}$$

Note that the RHS of (80) is strictly increasing, and hence α^* is uniquely defined.

Optimal Mechanism The prior results suggest the following iterative procedure to solve for the optimal mechanism (q, m, Δ) as functions of α.

First, we initialize $m_0(\alpha) = (\alpha^* - \alpha_0)\mathbf{1}\{\alpha \geq \alpha^*\}$, which satisfies the boundary condition (79). Under this initialization, $\Delta_0 = \Delta_\sigma(\alpha^* - \alpha_0)$ for all $\alpha \geq \alpha^*$.

Next, given Δ_t, solve for q_t from (75) state by state:

$$\begin{aligned} q_t(\alpha) &= \arg\max_q \; q \underbrace{\left[\alpha - (1 - \Delta_t(\alpha_+))\frac{1 - F(\alpha)}{f(\alpha)}\right]}_{\alpha_v} - C(q) && (81) \\ &= \frac{\alpha_v}{2\beta}\mathbf{1}\{\alpha \geq \alpha^*\}. && (82) \end{aligned}$$

Then, given q_t, we solve for m_{t+1} using (70) and (71), by subtracting the implied investor payoff $\pi_I(\alpha)$ from the implied available surplus

$$q_t(\alpha)m_{t+1}(\alpha) = q_t(\alpha)\alpha - C(q_t(\alpha)) - \int_{\alpha_0}^{\alpha} q_t(\tilde{\alpha})(1 - \Delta_t(\tilde{\alpha}))\, d\tilde{\alpha}. \tag{83}$$

Finally, having solved for m_{t+1}, we calculate $\Delta_{t+1}(\alpha) = \Delta_\sigma(m_{t+1}(\alpha))$ for the corresponding call option. This suggests a natural fixed-point algorithm for computing the optimal mechanism. In the numerical implementation, the iteration converges to a stable solution.

C.3 Uniqueness

While the iteration above works well numerically, we provide an alternative ODE characterization below, which also establishes uniqueness.

Proposition 8 (Characterization and Uniqueness of the Direct-Investment Mechanism)**.** *The direct-investment mechanism is uniquely characterized by a first-order ODE together with the boundary condition $q(\alpha^*) = \alpha_0/(2\beta)$.*

Proof. For any $\alpha \geq \alpha^*$, if an investment level q is implemented, the virtual-surplus condition implies

$$q = \frac{\alpha - (1 - \Delta)\frac{1 - F(\alpha)}{f(\alpha)}}{2\beta}.$$

Thus Δ is pinned down pointwise by (α, q):

$$\Delta = 1 - \frac{\alpha - 2\beta q}{(1 - F(\alpha))/f(\alpha)}.$$

Since $\Delta_\sigma(\cdot)$ is strictly increasing, this determines the expected transfer uniquely as

$$m(\alpha, q) \equiv \Delta_\sigma^{-1}\left(1 - \frac{\alpha - 2\beta q}{(1 - F(\alpha))/f(\alpha)}\right). \tag{84}$$

Hence define

$$P(\alpha, q) \equiv q\alpha - \beta q^2 - \delta - q\, m(\alpha, q). \tag{85}$$

Along any equilibrium path,

$$\pi_I(\alpha) = P(\alpha, q(\alpha)).$$

Differentiating with respect to α,

$$\pi_I'(\alpha) = P_\alpha(\alpha, q(\alpha)) + P_q(\alpha, q(\alpha))\, q'(\alpha).$$

By the envelope condition,

$$\pi_I'(\alpha) = q(\alpha)(1 - \Delta(\alpha)).$$

Therefore the equilibrium path satisfies the first-order ODE

$$q'(\alpha) = \frac{q(\alpha)(1 - \Delta(\alpha)) - P_\alpha(\alpha, q(\alpha))}{P_q(\alpha, q(\alpha))}. \tag{86}$$

At the cutoff, $\alpha_v(\alpha^*) = \alpha_0$, so

$$q(\alpha^*) = \frac{\alpha_0}{2\beta}.$$

Thus the equilibrium is characterized by the ODE (86) together with the initial condition at α^*.

It remains to sign P_q. From (84), the increasing hazard rate and monotonicity of Δ_σ^{-1} imply that $m_\alpha < 0$. Therefore,

$$P_\alpha(\alpha, q) = q - q\, m_\alpha(\alpha, q) > q \geq q(1 - \Delta).$$

Substituting into the identity

$$P_\alpha(\alpha, q(\alpha)) + P_q(\alpha, q(\alpha))\, q'(\alpha) = q(\alpha)(1 - \Delta(\alpha)),$$

and using $q'(\alpha) > 0$, we obtain

$$P_q(\alpha, q(\alpha)) < 0 \qquad \text{for all } \alpha \geq \alpha^*.$$

The denominator in (86) is therefore everywhere nonzero. Since the right-hand side is well defined, continuous, and locally Lipschitz in q, standard uniqueness for first-order initial value problems implies that $q(\alpha)$ is unique. Since $\Delta(\alpha)$ is uniquely recovered from $q(\alpha)$, and $m(\alpha)$ is uniquely recovered from

$$m(\alpha) = \Delta_\sigma^{-1}(\Delta(\alpha)),$$

the full direct-investment mechanism is uniquely determined. □

Calibration for Numerical Illustrations. For the figures and table below, we assume

$$\alpha \sim N(\mu_\alpha, \sigma_\alpha^2), \qquad \mu_\alpha = 4\%, \qquad \sigma_\alpha = 6\%.$$

We set

$$\beta = 0.0004, \qquad \delta = 1,$$

so that the first-best threshold is

$$\alpha_0 = 2\sqrt{\beta\delta} = 4\%.$$

For idiosyncratic risk, we consider the cases

$$\sigma_\epsilon \in \{5\%, 20\%\},$$

as well as the benchmark case $\Delta = 0$ (fixed payments only). Unless otherwise noted, all reported quantities are computed using the direct-investment mechanism characterized above.

Table 2: Direct Investment: Expected Payoffs and Loss Decomposition

Scenario	α^* (%)	Init q	Prob (%)	Mgr Rent	Inv Rent	Total Surplus	Entry Loss	Scale Loss
First Best	4.0000	50.0	50.0000	0.5239	0.0000	0.5239	0.0000	0.0000
Manager Rent ($\sigma_\epsilon = 5\%$)	5.1186	50.0	28.7936	0.3056	0.1441	0.4497	0.0631	0.0112
Manager Rent ($\sigma_\epsilon = 20\%$)	5.3581	50.0	24.8553	0.2474	0.1683	0.4158	0.0912	0.0170
Manager Rent ($\Delta = 0$)	5.5036	50.0	22.6085	0.2215	0.1727	0.3942	0.1101	0.0196

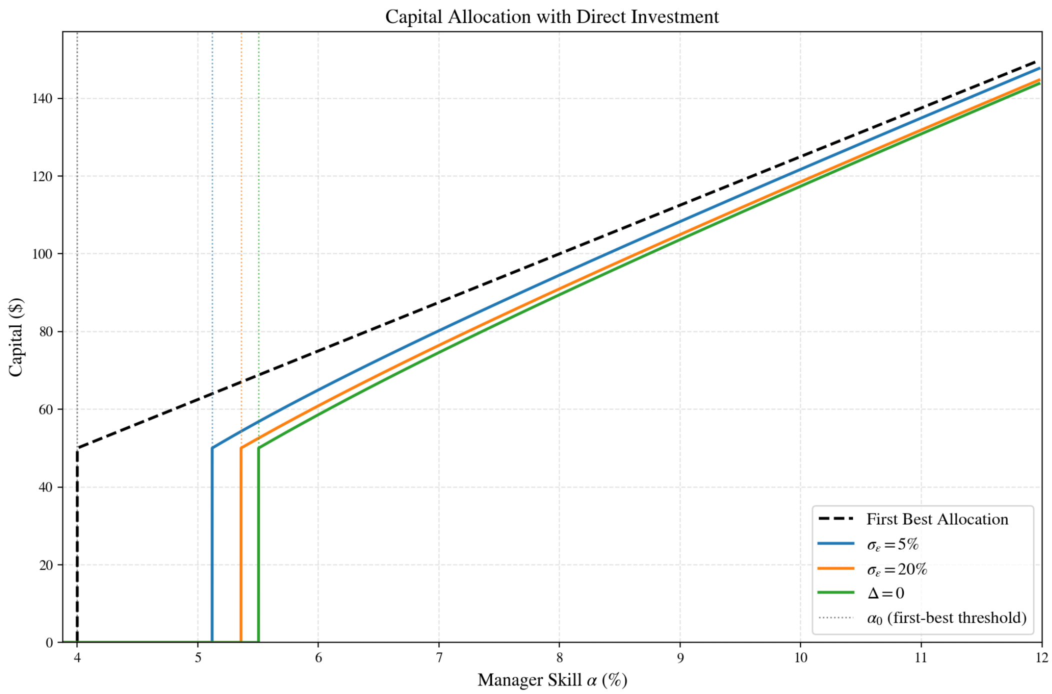


Figure 4: Optimal Capital Allocation across Information Scenarios. The dashed line represents the first-best allocation $q^{\text{FB}}(\alpha) = \alpha/(2\beta)$. The colored lines show the second-best allocations $q(\alpha)$ under different levels of idiosyncratic noise σ_ε, and under the degenerate case $\Delta = 0$ (fixed payment only). Dotted vertical lines mark the exclusion threshold α^* for each scenario; the grey dotted line marks the first-best threshold α_0. Higher noise forces the manager to distort capital further downward to reduce the rents that must be ceded to the investor, resulting in a higher exclusion threshold and greater under-investment relative to the first best.

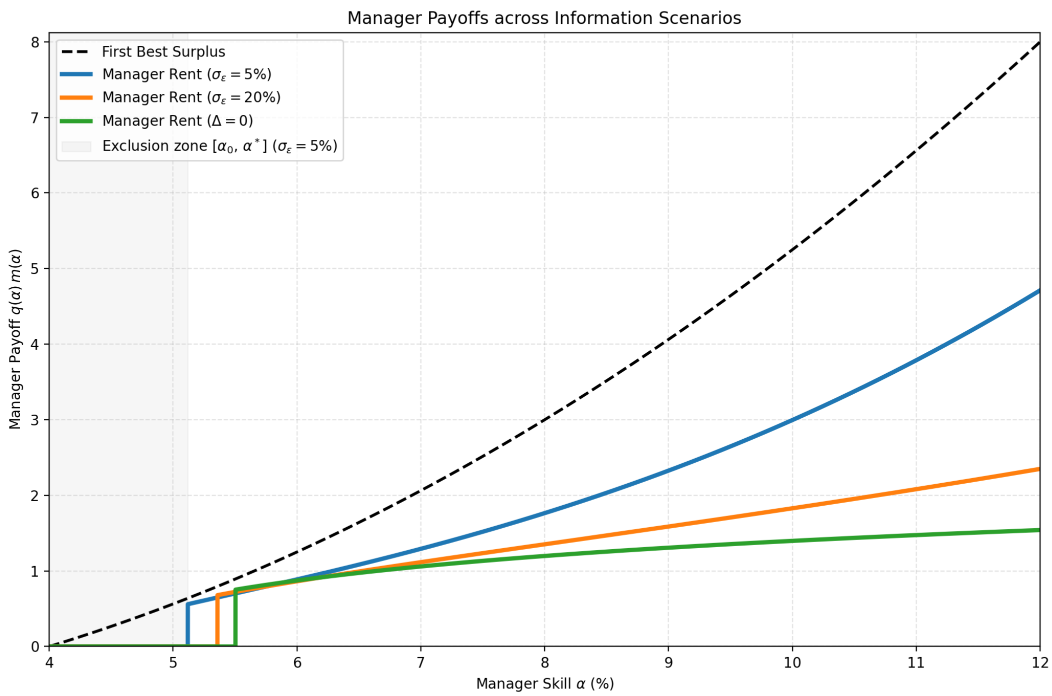


Figure 5: Manager Payoffs across Information Scenarios. The dashed line represents the First Best (FB) social surplus. The colored lines represent the manager's ex-post payoffs under different idiosyncratic noise levels (σ_ϵ), as well as with a fixed payment only ($\Delta = 0$). Note the "Exclusion Zone" between the first-best threshold of 4.0% and α^* where investment is socially efficient but suppressed by the manager to limit information rents the manager must concede to the investor.

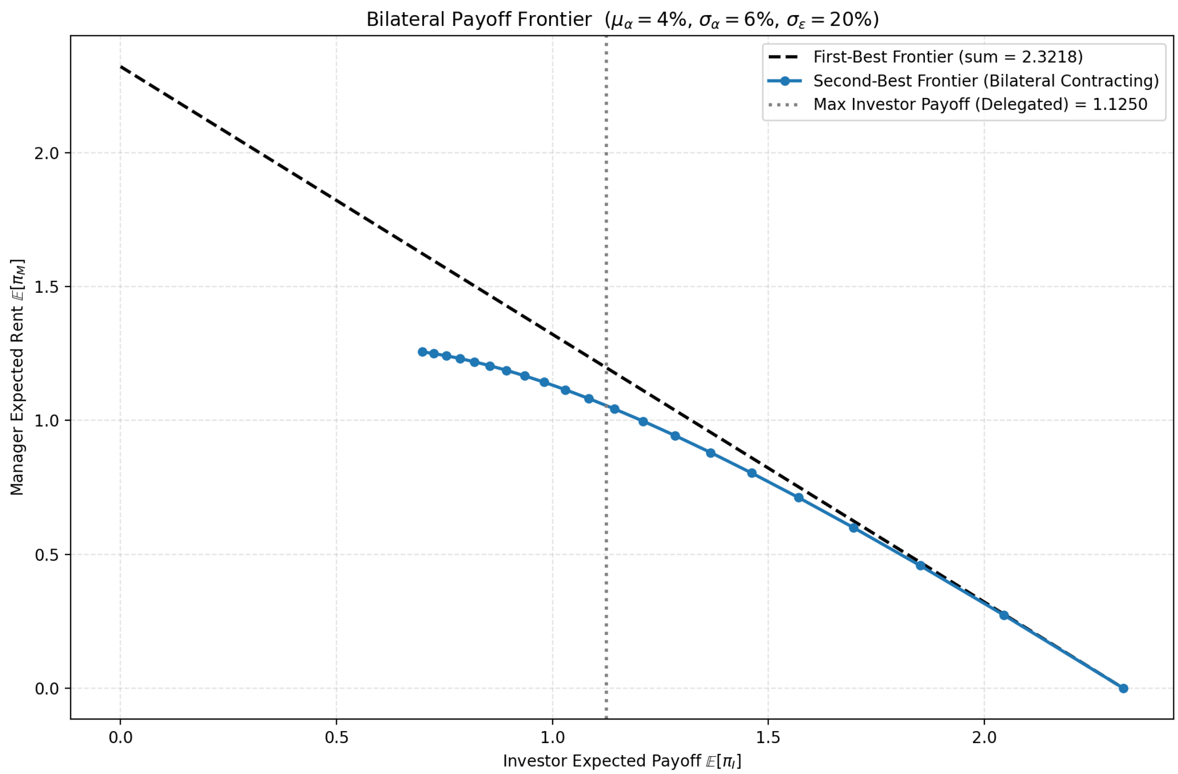


Figure 6: Bilateral Payoff Frontier and Management Benchmarks ($\sigma_\alpha = 6\%$). The figure illustrates the feasible set of expected payoffs for the manager ($\mathbb{E}[\pi_M]$) and the investor ($\mathbb{E}[\pi_I]$) under bilateral contracting vs. delegated management. The *second-best frontier* represents the outcomes achieved via the optimal bilateral mechanism as the level of investor rents is varied. The dashed black line represents the first-best frontier where $\mathbb{E}[\pi_M + \pi_I] = 2.3218$. Under delegated management, the first best is obtained, up to the *max investor payoff*, shown as a vertical line and calculated as $\mathbb{E}\left[\frac{(\alpha-\alpha_0)_+^2}{4\beta}\right] = 1.1250$.

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