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Liquidity in the Cross Section of OTC Assets

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Liquidity in the Cross Section of OTC Assets*

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Abstract

We develop a dynamic model of a multi-asset over-the-counter (OTC) market that operates via search and bargaining and empirically test its implications regarding liquidity in the cross section of assets. The key novelty in our model is that investors can hold and manage portfolios of OTC-traded assets. We characterize the stationary equilibrium in closed form and derive natural proxies for asset-specific measures of market liquidity including trade volume, price dispersion, and price impact. Our theoretical results uncover how the general equilibrium (GE) effects shape the patterns of liquidity measures in the cross section of OTC-traded assets. For example, heightened search frictions in one asset trigger *fire sales* in other assets by increasing other assets' trade volume but also making them trade with larger price impact and price dispersion. Based on data from the US corporate bond market and the credit default swap (CDS) market, our empirical tests confirm these key cross-sectional liquidity implications of our general-equilibrium OTC framework.

JEL classification: C73, C78, D53, D61, D83, E44, G11, G12

Keywords: OTC markets, portfolio management, general equilibrium, search and matching, liquidity

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1 Introduction

Many investors hold portfolios of assets that are traded over the counter (OTC). For such an investor, are two OTC-traded assets substitutes or complements market liquidity-wise? Following a negative shock to one asset's liquidity, will other assets become more or less liquid? What happens to the cross section of liquidity after an uncertainty shock to an asset? Do the answers to these questions differ depending on which empirical liquidity measure is used? Through an interplay of empirical analysis and theoretical discourse, this paper embarks on a comprehensive exploration of such questions related to cross-sectional liquidity within and across OTC markets, delving into its nuanced drivers, participants' decision-making processes, and profound implications for measurement methodologies.

On the theoretical side, our paper makes a contribution to the search-based OTC market literature by exploring the understudied area of managing portfolios containing multiple OTC assets. More precisely, we construct a dynamic general equilibrium model, in which investors can invest in portfolios of OTC assets. These assets are traded in a segmented, fully decentralized market that operates via search and bilateral bargaining. The assets differ from one another in their exposure to an aggregate risk factor and in the severity of search frictions in the particular segment of the OTC market where they are traded (i.e., investors are subject to asset-specific contact rates). In the empirical part of the paper, we calculate empirical counterparts from transaction-level data for certain equilibrium objects related to market liquidity. By also utilizing empirical proxies for asset-specific search friction and risk parameters, we test the implications of our theoretical model. This analysis highlights the importance of studying a model with arbitrary joint distribution of payoff risk and contact rates in the cross section of assets.

In our theoretical model, a continuum of risk-averse investors with stochastic hedging needs contact one another pairwise in different segments of the OTC market and bargain bilaterally over the terms of trade including price and quantity of the asset that is traded in that particular market segment. When negotiating over the terms of trade, investors take as given the equilibrium distribution of asset positions in order to evaluate the value of their outside option, i.e., the value of continuing search. In turn, these negotiated terms generate the distribution of asset positions. Thus, the distribution of investors' positions and their strategies must be jointly pinned down as a fixed point in the function space, which complicates the equilibrium analysis. However, employing the characteristic function techniques and focusing on an asymptotic case in which investors are averse to systematic risk only, we show that the model is fully tractable.[1]

The presence of search frictions in our model in the sense of inability to instantly access a competing counterparty makes investors' current state a determinant of their marginal valuation. This means that when bargaining over the terms of trade for an asset, an investor will take into account her current state including her hedging need and her positions in all other assets, unlike all other multi-asset OTC market models which allow investors to hold only one of the many assets at a time. In the characterization of equilibrium, we show that an investor's current state can be summarized by a sufficient statistic which equals her hedging need type plus the weighted sum of her (excess) inventories in all assets with weights being the assets' exposure to the aggregate risk factor. We term this sufficient statistic "excess risk exposure" because it is equal to the difference between the investor's current exposure to systematic risk and the per capita endowment of systematic risk in the economy at large. We derive all the stationary equilibrium objects in closed form including investors' valuations, terms of trade, and the characteristic function of the distribution of investors' excess risk exposures.

When each pair of buyer and seller contact, their negotiated trade quantity is determined such that their excess risk exposures are pairwise equalized. This implies that investors tend to trade safer assets in larger quantities, and vice-versa, riskier assets are traded in smaller quantities. Therefore, controlling for the contact rate (i.e., the inverse of the exposure to search frictions), safer assets have larger trade volume than riskier assets. However, high contact rate in the market for a particular asset allows investors to have more frequent opportunities to exchange that asset, and so tends to increase its trade volume. As a result, we show that upward-sloping iso-trade-volume curves arise on the plane of systematic risk and contact rate because systematic risk and contact rate have an opposite impact on equilibrium trade volume. We argue that, if risk and contact rate are (locally) inversely related in the cross section of assets, this can rationalize some of the puzzling empirical observations such as the apparent flees from quality in the Euro-area government bond market and the non-monotonicity of liquidity in credit rating in the US corporate bond market.[2]

In addition to the intuitive within-asset trade volume results explained above, we also obtain

[1] Praz (2014) and Üslü (2019) also use the same "source-dependent" risk aversion approach combined with characteristic functions (Fourier transform) in their respective single-asset models. Although we have a multi-asset model, our characterization is even more explicit than theirs because we do not assume any *ex ante* heterogeneity in investors' characteristics or asymmetric information. As a result, we are able to obtain an explicit expression for the characteristic function of the distribution of investors' excess risk exposures, while Praz (2014) and Üslü (2019) can only characterize the moments explicitly.

[2] See Beber, Brandt, and Kavajecz (2009) and Geromichalos, Herrenbrueck, and Lee (2023) for further details about these puzzling empirical facts.

a general *substitutability* result regarding trade volume as a cross-asset comparative static. We show that while a decline in the contact rate in a market decreases the equilibrium trade volume in that market, it increases the volume in all other markets. As the contact rate in a particular market declines, investors have less frequent opportunities in that market to equalize their excess risk exposures, which means that there will be more misallocation in their excess risk exposures when they meet in other markets. This raises the volume they trade in other markets. This is an important cross-asset comparative static that could not be deduced from single-asset models or from multi-asset models with independent asset payoffs, whose comparative statics typically imply a positive relationship between contact rate and trade volume.[3]

As is the case with negotiated quantity, when each pair of buyer and seller contact, their negotiated price also depends on their current excess risk exposures. This gives rise to equilibrium price dispersion. We calculate two price-related measures of liquidity: price dispersion and price impact. Price dispersion is defined to be the standard deviation of the equilibrium price distribution, while price impact is defined to be price dispersion divided by the standard deviation of the equilibrium quantity distribution. While the former is a natural definition for price dispersion, the latter is a model-informed measure for price impact. We show that the negotiated prices in equilibrium are equal to the mid-point of the negotiating parties' marginal valuations. Thus, a natural measure of price impact in a certain market segment is the sensitivity of an investor's counterparty's marginal valuation to the traded quantity of the asset traded in that market segment. We show that, in equilibrium, this sensitivity measure coincides with (a normalized version of) the ratio of price dispersion to quantity dispersion, and hence, we define it to be price impact.

We find that, while a general substitutability holds regarding the effect of a change in the contact rate of an asset on the trade volume of other assets, a general *complementarity* holds regarding price dispersion and price impact. This is due to investors' ability to hold multiple OTC assets at the same time, which is a unique feature of our model. As a result of this feature, the sensitivity of investors' marginal valuation to excess risk exposure depends on the total contact rate of all markets, i.e., the sum of asset-specific contact rates. Accordingly, price dispersion and price impact in an individual market also depend on the total contact rate of all markets, instead of the contact rate of that particular market only. In turn, we find that a decline in the contact rate in a market increases the price dispersion and price impact in all markets by the same factor. In other words, the *cross-sectional* differences in price dispersion

[3] See Üslü (2019), Hugonnier, Lester, and Weill (2022), and Li (2023), for example.

and price impact are solely determined by risk differentials across assets, although the cross-sectional trade volume patterns are determined by both risk and contact rate differentials.

To understand the extent to which the within-asset and cross-asset comparative statics results summarized above hold in real-world OTC markets, we test our theoretical formulas for trade volume, price dispersion, and price impact, in the cross section of bonds traded in the US corporate bond market. The corporate bond market is a textbook example of an OTC market where majority of trades are purely bilateral and subject to significant search frictions. Overall, our results from empirical tests of liquidity are mostly consistent with the implications of the theoretical model. We interpret this as pointing to the usefulness of the search-theoretic approach as a unifying framework to study the determinants of endogenous liquidity differentials across OTC assets, especially considering its ability to lead to parsimonious and tractable models as exemplified by our theoretical model.[4]

In the last part of the paper, we broaden the scope of our empirical tests by utilizing data from multiple asset classes, namely, from the bond and the credit default swap (CDS) markets. We start by extending our theoretical model to allow for endogenous contact rate in one market. This allows us to make causal prediction for the effect of liquidity in the CDS market on the liquidity of the bond issued by the same CDS entity. Our results echo the cross-market comparative statics of trade volume explained above: As the CDS market gets more liquid, investors do not have to rely on the bond market as much. In turn, this reduces the liquidity in the bond market. We show that the empirical correlations confirm such a negative relation. In other words, we show that an entity with a deeper CDS market has a bond subject to more severe search frictions. This is also consistent with the arguments of Oehmke and Zawadowski (2017) that bond and CDS markets are alternative trading venues for hedging and speculation.

The remainder of the paper is organized as follows. We next discuss how our paper relates to the pertinent literature. Section 2 describes the model environment. Section 3 studies the stationary equilibrium in this environment, while Section 4 discusses the main results about the various endogenous measures of liquidity. Section 5 analyzes the extent to which our theoretical findings are consistent with liquidity differentials across corporate bonds in practice. Section 6 presents further theoretical and empirical results on liquidity across asset classes. Section 7

[4]To our knowledge, there is no other dynamic model that studies bilateral trade or price impact in large markets when investors are allowed to hold a rich portfolio of assets. State-of-the-art work only studied single-asset dynamic environments (e.g., Sannikov and Skrzypacz, 2016) or multi-asset static environments (e.g., Malamud and Rostek, 2017).

concludes.

Related literature Search-theoretic approach to OTC market structure, spurred by Duffie, Gârleanu, and Pedersen (2005), has proven useful in analyzing the determinants and various measures of market liquidity and become a leading approach in modeling OTC markets.[5] See Weill (2020) for a recent survey of the search-theoretic OTC market literature. Our paper contributes to this literature by considering a multi-asset trading model, where investors are allowed to hold portfolios of OTC assets. In particular, our model follows the approach of having only search frictions like the single-asset models of Gârleanu (2009), Lagos and Rocheteau (2009), Afonso and Lagos (2015), and Üslü (2019) and does not impose any restrictions on portfolio holdings. This sets apart our model from the existing search-theoretic multi-asset OTC models such as Vayanos and Wang (2007), Vayanos and Weill (2008), Weill (2008), Milbradt (2017), An (2020), Li and Song (2021), and Sambalaibat (2022a), whose investors can only hold an indivisible position in some asset. Thus, to our knowledge, our paper is the first to analyze investors' dynamic portfolio management strategies in OTC markets and the resulting effect of asset characteristics in general equilibrium (GE). Li (2023) also studies portfolio choice with multiple OTC assets and unrestricted holdings. Her model assumes independent asset payoffs, and so, does not have any GE effects, while our *one-factor* model makes all assets interdependent liquidity-wise.

Within the search-theoretic OTC market literature, our paper belongs to the group of fully decentralized trading models with all-to-all trading, i.e., without any exogenously assigned trading roles like dealer or customer. For single-asset, fully decentralized trading models, see Afonso and Lagos (2015), Bethune, Sultanum, and Trachter (2016), Chang and Zhang (2019), Farboodi, Jarosch, Menzio, and Wiriadinata (2019), Üslü (2019), Gabrovski and Kospentaris (2021), Bethune, Sultanum, and Trachter (2022), Hugonnier, Lester, and Weill (2022), and Farboodi, Jarosch, and Shimer (2023), among others.[6] Compared to these papers, our contribution

[5]Another leading approach to modeling OTC markets builds on network theory, e.g., Gofman (2011), Babus and Hu (2017), Malamud and Rostek (2017), Aymanns, Georg, and Golub (2018), Babus and Kondor (2018), Manea (2018), and Farboodi (2023). Some work also use a hybrid approach, integrating elements from search and network models, e.g., Atkeson, Eisfeldt, and Weill (2015), Chang and Zhang (2019), Colliard and Demange (2021), Colliard, Foucault, and Hoffmann (2021), Dugast, Üslü, and Weill (2022), and Frei, Capponi, and Brunetti (2022).

[6]Similar to these papers, we do not assume *ex ante* who is a customer and who is a dealer. Instead, any customer-like or dealer-like trading behavior emerges endogenously. Another approach in the literature is to assume exogenously designated dealers operating either in a frictionless interdealer platform (e.g., Lagos and Rocheteau, 2009, Pintér and Üslü, 2022, Kargar, Passadore, and Silva, 2023, and Li, 2023.) or in a frictional interdealer platform (e.g., Hugonnier, Lester, and Weill, 2020, Sambalaibat, 2022b, and Yang and Zeng, 2023). Because we focus on the aggregate implications of our model, introducing exogenously designated dealers would

is to obtain cross-market comparative statics regarding market liquidity, which are not possible to obtain in single-asset models. Similar to Farboodi, Jarosch, Menzio, and Wiriadinata (2019), An (2020), Hendershott, Li, Livdan, and Schürhoff (2020), Hugonnier, Lester, and Weill (2020), Shen, Wei, and Yan (2020), Bethune, Sultanum, and Trachter (2022), Brancaccio and Kang (2022), Li (2023), Lu, Puzzelo, and Zhu (2023), and Pintér and Üslü (2023), not only do we construct an OTC trading model, but we also empirically test our model's key implications. While these papers only test market-wide implications or test cross-sectional implications via comparative statics of model parameters, our theoretical portfolio choice model with GE effects allows us to formulate precise cross-sectional hypotheses and directly test them.

Malamud and Rostek (2017) and Aymanns, Georg, and Golub (2018) study static network-based models of multi-asset OTC markets, where investors engage in one-shot trading game in multiple segmented markets at the same time. Our dynamic model, instead, analyzes how investors optimally manage their portfolios over time by fully internalizing the option value of waiting and continuing search. Our paper is also related to the literature that studies price impact in dynamic environments such as Chapter III of Praz (2014, co-authored with Julien Cujean), Rostek and Weretka (2015), Sannikov and Skrzypacz (2016), Du and Zhu (2017), and Antill and Duffie (2021), among others. Because this literature considers only single-asset environments, our multi-asset model contributes to this literature by analyzing how dynamic portfolio considerations affect price impact. There are also multi-asset models in the search-theoretic literature on monetary economics. See, among others, Rocheteau (2011), Li, Rocheteau, and Weill (2012), Hu (2013), Lagos (2013), Hu, In, Lebeau, and Rocheteau (2021), Geromichalos and Herrenbrueck (2023), and Geromichalos, Herrenbrueck, and Lee (2023). These papers also focus on analyzing liquidity differentials across assets. However, the concept of liquidity they analyze is mainly the assets' ability to serve as medium of exchange, while we focus on market liquidity, i.e., the ease of sale and purchase.

Finally, our paper is related to the recent literature on demand-system asset pricing. In particular, focusing on an asymptotic case in which investors are averse to only systematic risk makes our model a *one-factor* model like Koijen and Yogo (2019a).[7] While this literature mainly focuses on the implications of *ex ante* heterogeneity among investors and estimating the different demands of different class of investors from the holdings data, our paper is concerned with the *ex post* heterogeneity that the market frictions create among *ex ante* homogeneous investors.

not change our qualitative results.

[7]See also Koijen and Yogo (2019b), Allen, Kastl, and Wittwer (2023), and Koijen, Richmond, and Yogo (2023), among others.

Because our main focus is on market liquidity in the cross section of OTC-traded assets, we utilize transactions data, which is high-frequency, instead of the low-frequency holdings data.

2 Environment

Time is continuous and has an infinite horizon. We fix a probability space $(\Omega, \mathcal{F}, \Pr)$ and a filtration $\{\mathcal{F}_t, t \geq 0\}$ of sub-σ-algebras satisfying the usual conditions (see Protter, 2004). An economy is populated by a continuum of banks with a normalized mass of 1.[8] Each bank comprises of $J \in \mathbb{Z}_+$ traders, who are von Neumann-Morgenstern expected utility maximizers with a constant absolute risk aversion (CARA) coefficient of $\gamma > 0$. Within a bank, all traders share risk perfectly. Namely, a trader's net consumption is equal to $1/J$ of her bank's traders' total consumption at any point in time. Traders discount the future at rate $r > 0$ and are also able to borrow and lend a risk-free asset, which we designate as the *numéraire*, frictionlessly at the same exogenous rate r.

There are also J risky assets, which are indexed by $j \in \mathcal{J} \equiv \{1, 2, ..., J\}$ and each of which are in zero net supply. Banks can trade these assets over the counter. The assets' cumulative dividend flows, D_j, evolve according to

$$dD_{jt} = m_j dt + \sigma \psi_j dB_t + \nu_j dB_{jt} \tag{1}$$

for $j \in \mathcal{J}$, where B_t is a standard Brownian motion. The first term of (1) captures the expected dividend flow. The second term captures the systematic risk and depends on the aggregate volatility parameter σ. The last term captures the asset-specific risk; i.e., B_{jt}s are i.i.d. standard Brownian motion processes, which are also independent of B_t.

From its operations outside the explicitly modeled OTC markets, bank $i \in [0, 1]$ has a cumulative background income process Z^i:

$$dZ_t^i = m_Z dt + \eta_t^i \sigma dB_t,$$

where

$$d\eta_t^i = \sigma_\eta dB_t^i. \tag{2}$$

For $j \in \mathcal{J}$, the exogenous object $\eta_t^i \sigma^2 \psi_j$ captures the instantaneous covariance between the payoff of OTC asset j and the bank i's background income. This covariance is time-varying and

[8] In practice, many non-bank institutions including hedge funds, pension funds, and insurance companies engage in trading in the OTC markets. We follow Atkeson, Eisfeldt, and Weill (2015), Dugast, Üslü, and Weill (2022), and Frei, Capponi, and Brunetti (2022) in labeling all of them as banks for brevity.

heterogeneous across banks. Thus, this heterogeneity creates the fundamental gains from trade. We interpret this heterogeneity as stemming from instantaneous hedging need differentials across banks.

Importantly, the heterogeneity-driving coefficient η_t^i is stochastic itself. Banks continuously receive idiosyncratic shocks to the covariance between the asset payoffs and their background risk, which creates the motive to trade even in steady state.[9] Arrival of these shocks is governed as a diffusion by the standard Brownian motion processes B_t^i, which are i.i.d. in the cross section of banks and independent of B_t and B_{jt}s, as well. Since the assets are in zero net supply and (2) does not have a drift term, all traded and non-traded risks net out to zero once aggregated across all banks.

Trades are fully bilateral and take place in segmented markets for assets 1, 2, and so on. There are J different types of traders, indexed by j. A trader indexed by j has specialization in trading in market j. All banks are *ex ante* identical. Namely, each bank has a complete set of traders, and so, has access to all OTC markets. In each market, pairwise meetings among traders follow standard random search and matching dynamics.[10] A given trader in market j meets another trader at Poisson arrival times with intensity $\lambda_j > 0$, where $1/\lambda_j$ reflects the expected trading delay in market j for $j \in \mathcal{J}$. Conditional on a meeting in market j, the counterparty is drawn randomly and uniformly from the pool of all traders operating in market j.

Let a_{-j} refer to a $J-1$-dimensional vector that represents a bank's asset positions in all markets except for j. A meeting in market j between the trader who works for bank (η, a_j, a_{-j}) and another trader who works for bank (η', a_j', a_{-j}') is followed by a bargaining process over quantity q_j and unit price P_j. The resulting number of assets that the trader from the former bank purchases is denoted by $q_j\left[(\eta, a_j, a_{-j}), (\eta', a_j', a_{-j}')\right]$. Thus, her bank's position in asset j will become $a_j + q_j\left[(\eta, a_j, a_{-j}), (\eta', a_j', a_{-j}')\right]$ after this trade, while her counterparty's bank's position in asset j will become $a_j' - q_j\left[(\eta, a_j, a_{-j}), (\eta', a_j', a_{-j}')\right]$. The per unit price, the bank (η, a_j, a_{-j}) will pay, is denoted by $P_j\left[(\eta, a_j, a_{-j}), (\eta', a_j', a_{-j}')\right]$. The specific bargaining protocol we employ is the axiomatic bargaining *à la* Nash (1950) in which traders are symmetric in their bargaining powers.

[9] To generate trade volume, Lo, Mamaysky, and Wang (2004), Chapter III of Praz (2014, co-authored with Julien Cujean), and Sannikov and Skrzypacz (2016) also utilize hedging needs that follow diffusion processes. Antill and Duffie (2021) allow for Lévy processes which include pure jump, pure diffusion, and jump-diffusion processes.

[10] See Duffie, Qiao, and Sun (2017) for a formal treatment of the existence of continuous-time independent random matching in a continuum population. See also Sun (2006), Duffie and Sun (2007), and Duffie and Sun (2012) for discrete-time analogues.

3 Equilibrium

We solve for the equilibrium of the economy described in the previous section in two steps. First, we study a "partial equilibrium" determination of banks' stationary trading rules by taking as given the dynamics of the joint distribution of banks' types and asset positions, denoted by Φ. In the second part, we endogenize the dynamics of the equilibrium joint distribution generated by banks' stationary optimal trading rules.

3.1 Trader's problem

Let $\mathbf{a} \in \mathbb{R}^J$ denote the vector of a bank's asset positions. Let $U^j(W, \eta, \mathbf{a})$ and $U^j(W, \eta, a_k, a_{-k})$, that we use interchangeably, both refer to the maximum attainable continuation utility of a type-j trader who works for a bank of type $(\eta, \mathbf{a})$ with current numéraire holding of W. They satisfy

$$U^j(W, \eta, \mathbf{a}) = \sup_c \mathbb{E}_t \left[-\int_t^\infty e^{-r(s-t)} e^{-\gamma c_s} ds \,\middle|\, W_t = W,\ \eta_t = \eta,\ \mathbf{a}_t = \mathbf{a} \right],$$

subject to

$$dW_t = (rW_t - c_t - \tilde{c}_t + m_Z)\, dt + \eta_t \sigma dB_t + \sum_{k=1}^{J} \left\{ a_{kt-} dD_{kt} - P_k \left[(\eta_{t-}, \mathbf{a}_{t-}), (\eta'_t, \mathbf{a}'_t) \right] da_{kt} \right\},$$

$$da_{kt} = \begin{cases} q_k \left[(\eta_{t-}, \mathbf{a}_{t-}), (\eta'_t, \mathbf{a}'_t) \right] & \text{if } (\eta'_t, \mathbf{a}'_t) \text{ is contacted in market } k \\ 0 & \text{if no contact in market } k, \end{cases}$$

$$c_t = \frac{\tilde{c}_t}{J-1},$$

where

$$\begin{aligned} &\left\{ q_k \left[(\eta, \mathbf{a}), (\eta', \mathbf{a}') \right], P_k \left[(\eta, \mathbf{a}), (\eta', \mathbf{a}') \right] \right\} \\ &= \arg\max_{q,P} \left[U^k(W - Pq, \eta, a_k + q, a_{-k}) - U^k(W, \eta, a_k, a_{-k}) \right]^{\frac{1}{2}} \\ &\qquad\qquad \left[U^k\left(W' + Pq, \eta', a'_k - q, a'_{-k}\right) - U^k\left(W', \eta', a'_k, a'_{-k}\right) \right]^{\frac{1}{2}}, \end{aligned} \tag{3}$$

subject to

$$\begin{aligned} &U^k(W - Pq, \eta, a_k + q, a_{-k}) \geq U^k(W, \eta, a_k, a_{-k}), \\ &U^k\left(W' + Pq, \eta', a'_k - q, a'_{-k}\right) \geq U^k\left(W', \eta', a'_k, a'_{-k}\right). \end{aligned}$$

Thanks to the translation invariance property of CARA preferences, terms of trade are independent of numéraire holdings as will be clear shortly. Therefore, in writing down the dynamic budget constraint, the law of motion for asset positions, and the Nash product above, we dropped W and W' from the arguments of terms of trade functions. To prevent Ponzi schemes from arising in the optimal solution, we impose the transversality condition

$$\lim_{T\to\infty} e^{-r(T-t)}\mathbb{E}_t\left[e^{-r\gamma\frac{W_T}{J}}\right]=0.$$

We use the technique of stochastic dynamic programming to derive the optimal rules. Assuming sufficient differentiability and applying the Ito's lemma for Lévy processes, the trader's value function $U^j(W,\eta,\mathbf{a})$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
0=\sup_c &\Bigg\{-e^{-\gamma c}+U_W^j(W,\eta,\mathbf{a})\left(rW-c-\tilde{c}+m_Z+\sum_{k=1}^J a_k m_k\right)\\
&+\frac{1}{2}U_{WW}^j(W,\eta,\mathbf{a})\left(\eta^2\sigma^2+2\eta\sigma^2\sum_{k=1}^J\psi_k a_k+2\sigma^2\sum_{l=1}^J\sum_{k>l}^J\psi_l\psi_k a_l a_k+\sigma^2\sum_{k=1}^J\psi_k^2 a_k^2+\sum_{k=1}^J\nu_k^2 a_k^2\right)\\
&+\frac{1}{2}U_{\eta\eta}^j(W,\eta,\mathbf{a})\sigma_\eta^2-rU^j(W,\eta,\mathbf{a})+\dot{U}^j(W,\eta,\mathbf{a})\\
&+\sum_{k=1}^J\Bigg(2\lambda_k\int_{\mathbb{R}}\int_{\mathbb{R}^J}\left[U^j\left(W-q_k(\mu,\mu')P_k(\mu,\mu'),\eta,a_k+q_k(\mu,\mu'),a_{-k}\right)\right.\\
&\qquad\left.-U^j(W,\eta,a_k,a_{-k})\right]\Phi(d\mathbf{a}',d\eta')\Bigg)\Bigg\},
\end{aligned}\tag{4}$$

subject to

$$c=\frac{\tilde{c}}{J-1},$$

where $\mu\equiv(\eta,\mathbf{a})$ and $\mu'\equiv(\eta',\mathbf{a}')$.

Note that the HJB equation (4) is written under the assumption that optimal consumption is a coalitional choice, although a trader's optimal trading behavior depends only on her own continuation utility. In other words, not only does (4) pin down the optimal consumption of trader j but also every other trader's optimal consumption level with perfect equality constraint.[11] Thus, any optimizer of the constrained HJB equation (4) must satisfy

$$c^j(W,\eta,\mathbf{a})=c^{j'}(W,\eta,\mathbf{a})\tag{5}$$

[11] Alternatively, one can think of traders' optimal consumption being determined by a multilateral proportional bargaining *à la* Kalai (1977) in which traders get to consume the same amount as one another.

for all $(j, j') \in \mathcal{J}^2$.

Noting that (4) is symmetric across traders, we look for a symmetric solution in which $U^j(W, \eta, \mathbf{a}) = U(W, \eta, \mathbf{a})$ for all $j \in \mathcal{J}$. We solve (4) by making the standard *Ansatz*

$$U(W, \eta, \mathbf{a}) = -e^{-\frac{r\gamma}{J}\left(W + V(\eta, \mathbf{a}) + \overline{V}\right)},$$

where

$$\overline{V} = \frac{1}{r}\left(m_Z + \frac{J \log r}{\gamma}\right)$$

is a constant and $V(\eta, \mathbf{a})$ is the *bank's* wealth-equivalent continuation value that will determine the terms of trade. Indeed, a natural interpretation following from the Ansatz is that the total flow value of a bank, $r\left(W + V(\eta, \mathbf{a}) + \overline{V}\right)$, is distributed equally to its J traders each of whom has the CARA coefficient of γ.

Using the Ansatz, we find that a trader's optimal consumption is

$$c = -\frac{\log r}{\gamma} + \frac{r}{J}\left(W + V(\eta, \mathbf{a}) + \overline{V}\right).$$

Uniqueness of the optimal consumption implies that the coalitional consistency condition (5) is satisfied. Substituting c into (4) and dividing by $\frac{r\gamma}{J}U(W, \eta, \mathbf{a})$, we find at steady state that (4) is satisfied if and only if

$$\begin{aligned} rV(\eta, \mathbf{a}) &= \sum_{j=1}^{J} m_j a_j - \frac{1}{2}\frac{r\gamma}{J}\sigma^2\left(\eta^2 + 2\eta\sum_{j=1}^{J}\psi_j a_j + 2\sum_{j=1}^{J}\sum_{k>j}^{J}\psi_j\psi_k a_j a_k + \sum_{j=1}^{J}\psi_j^2 a_j^2\right) \\ &- \frac{1}{2}\frac{r\gamma}{J}\sum_{j=1}^{J}\nu_j^2 a_j^2 - \frac{1}{2}\sigma_\eta^2\left[\frac{r\gamma}{J}\left(V_\eta(\eta, \mathbf{a})\right)^2 - V_{\eta\eta}(\eta, \mathbf{a})\right] \\ &\quad + \sum_{j=1}^{J}\left(2\lambda_j \int_{\mathbb{R}}\int_{\mathbb{R}^J} \frac{1 - e^{-\frac{r\gamma}{J}\left[V(\eta, a_j + q_j(\mu,\mu'), a_{-j}) - V(\eta, a_j, a_{-j}) - q_j(\mu,\mu')P_j(\mu,\mu')\right]}}{\frac{r\gamma}{J}}\Phi(d\mathbf{a}', d\eta')\right). \end{aligned} \tag{6}$$

Terms of bilateral trades, $q_j(\mu, \mu')$ and $P_j(\mu, \mu')$, maximize the Nash product (3). By dividing by $U(W, \eta, \mathbf{a})^{\frac{1}{2}}U(W', \eta', \mathbf{a}')^{\frac{1}{2}}$, we simplify (3) as

$$\begin{aligned} &\{q_j[(\eta, \mathbf{a}), (\eta', \mathbf{a}')], P_j[(\eta, \mathbf{a}), (\eta', \mathbf{a}')]\} \\ &= \arg\max_{q,P}\left[1 - e^{-\frac{r\gamma}{J}[V(\eta, a_j + q, a_{-j}) - V(\eta, a_j, a_{-j}) - qP]}\right]^{\frac{1}{2}}\left[1 - e^{-\frac{r\gamma}{J}\left[V\left(\eta', a_j' - q, a_{-j}'\right) - V\left(\eta', a_j', a_{-j}'\right) + qP\right]}\right]^{\frac{1}{2}}, \end{aligned}$$

subject to

$$1 - e^{-\frac{r\gamma}{J}[V(\eta, a_j + q, a_{-j}) - V(\eta, a_j, a_{-j}) - qP]} \geq 0,$$

$$1 - e^{-\frac{r\gamma}{J}\left[V\left(\eta', a_j' - q, a_{-j}'\right) - V\left(\eta', a_j', a_{-j}'\right) + qP\right]} \geq 0,$$

which verifies that there are no wealth effects. Solving this problem is relatively straightforward: We set up the Lagrangian of this problem. Then using the first-order and Kuhn-Tucker conditions, the trade quantity $q_j\left[(\eta,\mathbf{a}),(\eta',\mathbf{a}')\right]$ solves

$$V^{(j)}\left(\eta,a_j+q,a_{-j}\right)=V^{(j)}\left(\eta',a'_j-q,a'_{-j}\right), \tag{7}$$

where $V^{(j)}$ stands for the partial derivative with respect to the argument representing the position in asset j. And, the negotiated price $P_j\left[(\eta,\mathbf{a}),(\eta',\mathbf{a}')\right]$ is determined such that the joint trade surplus is split equally between the negotiating parties:

$$P=\frac{V\left(\eta,a_j+q,a_{-j}\right)-V\left(\eta,a_j,a_{-j}\right)-\left(V\left(\eta',a'_j-q,a'_{-j}\right)-V\left(\eta',a'_j,a'_{-j}\right)\right)}{2q} \tag{8}$$

if $V^{(j)}\left(\eta,a_j,a_{-j}\right)\neq V^{(j)}\left(\eta',a'_j,a'_{-j}\right)$; and $P=V^{(j)}\left(\eta,a_j,a_{-j}\right)$ if $V^{(j)}\left(\eta,a_j,a_{-j}\right)=V^{(j)}\left(\eta',a'_j,a'_{-j}\right)$. From (6), (7), and (8), one can see that the bargaining between two traders is equivalent to a bargaining between the traders' respective banks where V is the banks' value function and $\frac{\gamma}{J}$ is the banks' effective risk aversion.

Letting $\gamma_B=\frac{\gamma}{J}$ and substituting the pricing function (8) into (6), we get

$$\begin{aligned}
rV\left(\eta,\mathbf{a}\right)=&\sum_{j=1}^{J}m_ja_j-\frac{1}{2}r\gamma_B\sigma^2\left(\eta^2+2\eta\sum_{j=1}^{J}\psi_ja_j+2\sum_{j=1}^{J}\sum_{k>j}^{J}\psi_j\psi_ka_ja_k+\sum_{j=1}^{J}\psi_j^2a_j^2\right)\\
&-\frac{1}{2}r\gamma_B\sum_{j=1}^{J}\nu_j^2a_j^2-\frac{1}{2}\sigma_\eta^2\left[r\gamma_B\left(V_\eta\left(\eta,\mathbf{a}\right)\right)^2-V_{\eta\eta}\left(\eta,\mathbf{a}\right)\right]\\
&+\sum_{j=1}^{J}\left(2\lambda_j\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^J}\frac{1-e^{-\frac{r\gamma_B}{2}\left[V(\eta,a_j+q_j(\mu,\mu'),a_{-j})-V(\eta,a_j,a_{-j})+V\left(\eta',a'_j-q_j(\mu,\mu'),a'_{-j}\right)-V\left(\eta',a'_j,a'_j\right)\right]}}{r\gamma_B}\right.\\
&\left.\Phi\left(d\mathbf{a}',d\eta'\right)\right),
\end{aligned} \tag{9}$$

subject to (7).

In order to obtain an asymptotic solution of Equation (9) in closed form, we follow Üslü (2019) and calculate the limit as the CARA coefficient vanishes and the aggregate volatility goes to infinity at the same speed. Mathematically, this leads to the first-order linear approximation $\frac{1-e^{-r\gamma_Bx}}{r\gamma_B}\approx x$ that ignores terms of order higher than 1 in $\left[V\left(\eta,a_j+q,a_{-j}\right)-V\left(\eta,a_j,a_{-j}\right)\right]$.[12] Economically, this approximation can be understood in terms of source-dependent risk aversion; i.e., we assume that banks are averse towards systematic risk (risk generated by B_t in the model),

[12]The same approximation is also used by Biais (1993), Duffie, Gârleanu, and Pedersen (2007), Vayanos and Weill (2008), Gârleanu (2009), and Praz (2014).

while they are neutral towards other types of risk. The assumption does not suppress the impact of risk aversion because the instantaneous mean-variance benefit function (12) associated with asset positions possesses a negative definite quadratic part. Therefore, as is formally stated in the lemma below, this assumption focuses the bank's risk aversion on systematic diffusion risk and eliminates aversion to idiosyncratic diffusion risks and jump risks caused by the Poisson arrival of trade opportunities.

Lemma 1. *Fix parameters $\overline{\gamma}_B$ and $\overline{\sigma}$ and let $\sigma = \overline{\sigma}\sqrt{\overline{\gamma}_B/\gamma_B}$. Banks' stationary value function solves the following HJB equation in the limit as $\gamma_B \to 0$:*

$$\begin{aligned} rV(\eta, \mathbf{a}) = &\sum_{j=1}^{J} m_j a_j - \frac{1}{2} r\overline{\gamma}_B\overline{\sigma}^2 \left(\eta^2 + 2\eta\sum_{j=1}^{J}\psi_j a_j + 2\sum_{j=1}^{J}\sum_{k>j}^{J}\psi_j\psi_k a_j a_k + \sum_{j=1}^{J}\psi_j^2 a_j^2\right) \\ &+ \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta, \mathbf{a}) + \sum_{j=1}^{J}\Bigg(\lambda_j \int_{\mathbb{R}}\int_{\mathbb{R}^J} \left[V(\eta, a_j + q_j(\mu, \mu'), a_{-j}) - V(\eta, a_j, a_{-j})\right. \\ &\left. + V\left(\eta', a_j' - q_j(\mu, \mu'), a_{-j}'\right) - V\left(\eta', a_j', a_{-j}'\right)\right]\Phi(d\mathbf{a}', d\eta')\Bigg), \end{aligned} \tag{10}$$

subject to (7).

Notice that the quantity which solves (7) is also the maximizer of the joint trade surplus; i.e.,

$$\begin{aligned} &q_j\left[(\eta, \mathbf{a}), (\eta', \mathbf{a}')\right] \\ &\quad = \arg\max_q V(\eta, a_j + q, a_{-j}) - V(\eta, a_j, a_{-j}) + V\left(\eta', a_j' - q, a_{-j}'\right) - V\left(\eta', a_j', a_{-j}'\right). \end{aligned}$$

Using this and ignoring bars, (10) can be written as

$$\begin{aligned} rV(\eta, \mathbf{a}) = &\, u(\eta, \mathbf{a}) + \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta, \mathbf{a}) \\ &+ \sum_{j=1}^{J}\Bigg(\lambda_j \int_{\mathbb{R}}\int_{\mathbb{R}^J} \max_q \left[V(\eta, a_j + q, a_{-j}) - V(\eta, a_j, a_{-j}) + V\left(\eta', a_j' - q, a_{-j}'\right)\right. \\ &\left. - V\left(\eta', a_j', a_{-j}'\right)\right]\Phi(d\mathbf{a}', d\eta')\Bigg), \end{aligned} \tag{11}$$

where

$$u(\eta, \mathbf{a}) \equiv \mathbf{m}^T\mathbf{a} - \frac{1}{2} r\gamma_B\sigma^2\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) \tag{12}$$

is the instantaneous mean-variance benefit to the bank from holding the portfolio $\mathbf{a}$ when of type η,

$$\mathbf{m} \equiv [m_1 \;\; m_2 \;\; \ldots \;\; m_J]^T ,$$

$$\boldsymbol{\psi} \equiv [\psi_1 \;\; \psi_2 \;\; \ldots \;\; \psi_J]^T ,$$

and

$$\Psi \equiv \boldsymbol{\psi}^T \boldsymbol{\psi}.$$

In order to solve for $V(\eta, \mathbf{a})$, we follow the method of undetermined coefficients. The complete solution is given in Theorem 1. Since (11) is a flow Bellman equation with a negative definite linear-quadratic return function, the solution $V(\eta, \mathbf{a})$ itself inherits the negative definite linear-quadratic functional form as well. As a result, to find the stationary equilibrium value of $V(\eta, \mathbf{a})$, we are required to use the cross-sectional mean of the linear part and of the quadratic part of the return function, i.e., $\mathbb{E}[\mathbf{a}']$ and $\mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}'\right]$, respectively, instead of the entire joint distribution $\Phi(\mathbf{a}', \eta')$.

What is more striking is that determining the banks' equilibrium trading behavior does not require calculating any moment of the endogenous distribution of asset positions. To see this, one can take the derivative of (11) with respect to $\mathbf{a}$ by applying the envelope theorem and arrive at the following vector of partial derivatives:

$$\begin{aligned} r \frac{\partial V}{\partial \mathbf{a}}(\eta, \mathbf{a}) &= \mathbf{m} - r\gamma_B \sigma^2 (\eta \boldsymbol{\psi} + \Psi \mathbf{a}) \\ &+ \sum_{k=1}^{J} \left(\lambda_k \int_{\mathbb{R}} \int_{\mathbb{R}^J} \left[\frac{\partial V}{\partial \mathbf{a}} (\eta, a_k + q_k [(\eta, \mathbf{a}), (\eta', \mathbf{a}')], a_{-k}) - \frac{\partial V}{\partial \mathbf{a}} (\eta, \mathbf{a}) \right] \Phi(d\mathbf{a}', d\eta') \right). \end{aligned} \tag{13}$$

Equation (13) provides us with a flow Bellman equation for the vector of *marginal* valuations, where the j^{th} element of $\frac{\partial V}{\partial \mathbf{a}}(\eta, \mathbf{a})$ is the marginal valuation for asset j, $V^{(j)}(\eta, \mathbf{a})$. As can be seen, the flow Bellman equations for the marginal valuation have a return function that is linear and separable in η and all a_js for $j \in \mathcal{J}$. In turn, the FOC (7) for Nash bargaining and (13) imply that $V^{(j)}(\eta, \mathbf{a})$ is itself linear and separable in all of its arguments. Thus, calculating the equilibrium value of the second line of (13) requires only the first moment of the asset holding distribution for assets $j \in \mathcal{J}$, which equals the exogenous supply of those assets by market clearing: $\mathbb{E}[\mathbf{a}'] = \mathbf{0}$. The following theorem establishes the optimal trading behavior of banks at steady state.

Theorem 1. *Let* $\lambda = \sum_{k=1}^{J} \lambda_k$ *and*

$$\theta(\eta, \mathbf{a}) = \eta + \boldsymbol{\psi}^T (\mathbf{a} - \mathbb{E}[\mathbf{a}']) .$$

The unique quadratic stationary value function that solves (11) is

$$V(\eta, \mathbf{a}) = \frac{\gamma_B \sigma^2}{2r + \lambda} \left(-\sigma_\eta^2 + \frac{\lambda}{2} \mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}' \right] \right) - \lambda \frac{\gamma_B \sigma^2}{2r + \lambda} \boldsymbol{\psi}^T \mathbb{E}[\mathbf{a}'] \eta$$
$$+ \left(\frac{1}{r} \mathbf{m} - \lambda \frac{\gamma_B \sigma^2}{2r + \lambda} \boldsymbol{\psi}^T \mathbb{E}[\mathbf{a}'] \boldsymbol{\psi} \right)^T \mathbf{a} - \frac{r \gamma_B \sigma^2}{2r + \lambda} \left(\eta^2 + 2\eta \boldsymbol{\psi}^T \mathbf{a} + \mathbf{a}^T \Psi \mathbf{a} \right), \quad (14)$$

Thus, at steady state, banks' marginal valuations, individual trade sizes, and transaction prices are given by:

$$\frac{\partial V}{\partial \mathbf{a}} (\eta, \mathbf{a}) = \frac{1}{r} \frac{\partial u}{\partial \mathbf{a}} (0, \mathbb{E}[\mathbf{a}']) - \frac{r \gamma_B \sigma^2}{r + \lambda/2} \theta(\eta, \mathbf{a}) \boldsymbol{\psi}, \quad (15)$$

$$q_j [(\eta, \mathbf{a}), (\eta', \mathbf{a}')] = \frac{\theta(\eta', \mathbf{a}') - \theta(\eta, \mathbf{a})}{2\psi_j}, \quad (16)$$

and

$$P_j [(\eta, \mathbf{a}), (\eta', \mathbf{a}')] = V^{(j)} \left(\frac{\eta + \eta'}{2}, \frac{\mathbf{a} + \mathbf{a}'}{2} \right) = \frac{V^{(j)}(\eta, \mathbf{a}) + V^{(j)}(\eta', \mathbf{a}')}{2}, \quad (17)$$

respectively.

Equation (15) reveals important information about the effect of OTC frictions. In a frictionless market, the equilibrium marginal valuation would not depend on the current state as banks would equalize their marginal valuation instantly. The frictionless case is achieved in the limit as $\lambda \to \infty$. When all λ_ks are finite, banks' marginal valuation is dependent on their current state as well. This essentially reflects the time cost of search. When negotiating a trade, traders rationally expect that their banks will spend some time with their post-trade portfolio as a result of limited trading opportunities, even if their preferred portfolio becomes very different. Therefore, this situation creates deviation of the marginal valuation from what would obtain in a frictionless benchmark case.

Combining (15) with the FOC (7) for Nash bargaining, one sees that traders' bilateral trade quantities are determined such that their banks' θs are pairwise equalized. Thus, the composite type θ serves as a sufficient statistic for banks' optimal trading behavior. We name θ a bank's *excess risk exposure* because it is equal to the difference between the bank's exposure to systematic risk, $\eta + \boldsymbol{\psi}^T \mathbf{a}$, and the per-capita systematic risk in the economy at large,

$\boldsymbol{\psi}^T \mathbb{E}[\mathbf{a}']$, which is assumed to be zero for simplicity. Equations (16) and (17) provide us with explicit expression for the bilateral trade sizes and prices. One sees from (16) that the larger the difference between the bargaining parties' excess risk exposures, the larger the trade size implied by the equalization of their post-trade excess risk exposures. In addition, the larger the systematic risk exposure of the traded asset, the smaller the trade size. As expected, banks must exchange smaller quantity of an asset to equalize their excess risk exposures if per-unit systematic risk content of the asset is larger. Finally, (17) tells us that the bilateral trade price is equal to the trading banks' post-trade marginal valuation, which equals the midpoint of their initial marginal valuations due to symmetric bargaining powers and linear marginal valuations.

3.2 Dynamics of the distribution of banks' states

Theorem 1 shows that the excess risk exposure θ is a sufficient statistic for banks' equilibrium trading behavior. Furthermore, as mentioned above, the equilibrium value function $V(\eta, \mathbf{a})$ depends only on two particular moments calculated from the equilibrium distribution: $\mathbb{E}[\mathbf{a}']$ and $\mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}'\right]$. The former is totally pinned down by the market-clearing conditions $\mathbb{E}[\mathbf{a}'] = \mathbf{0}$, and so, the latter is equal to $\mathbb{E}\left[(\theta')^2\right]$. Therefore, determining the equilibrium dynamics of θ is sufficient to analyze the banks' optimal trading and their equilibrium value functions. Accordingly, what we do next is calculate the distribution of θ across banks instead of the joint distribution of banks' hedging need types η and their portfolios $\mathbf{a}$.

Lemma 2. *Let* $\lambda = \sum_{k=1}^{J} \lambda_k$. *If traders trade according to the trade size function* (16), *the pdf* $g(\cdot)$ *of banks' excess risk exposures satisfies the following Kolmogorov Forward Equation:*

$$\dot{g}(\theta) = \frac{1}{2} g''(\theta)\,\sigma_\eta^2 - 2\lambda g(\theta) + 4\lambda \int_{\mathbb{R}} g(\theta')\, g(2\theta - \theta')\, d\theta' \tag{18}$$

for all $\theta \in \mathbb{R}$,

$$\int_{\mathbb{R}} g(\theta)\, d\theta = 1, \tag{19}$$

and

$$\int_{\mathbb{R}} \theta g(\theta)\, d\theta = 0. \tag{20}$$

Equation (19) holds because $g(\cdot)$ is a pdf. Equation (20) is implied by the market-clearing conditions and the fact that η does not have a drift. Equation (18) has the usual inflow-outflow interpretation. The first term represents the net inflow due to the diffusion process that η follows. The second and third terms represent the (gross) outflow and the (gross) inflow due to trading, respectively. Banks with the current excess risk exposure of θ receive trading opportunities at the Poisson rate of 2λ and this gives rise to the outflow term $-2\lambda g(\theta)$. The third term is a convolution integral because any bank of type θ' can become of type θ following a trade with the "right" counterparty. It is easy to see from Theorem 1 that $\theta' + \psi_j\, q_j(\theta', 2\theta - \theta') = \theta$, and hence, the right counterparty in this context is a counterparty of type $2\theta - \theta'$. Because both the bank with the initial type of θ' and the one with the initial type of $2\theta - \theta'$ become of type θ after trading with each other, the coefficient in front of the convolution integral is 4λ. Since the convolution integral complicates the computation of the pdf, we will make use of the characteristic function (Lukacs, 1970, p. 5):[13]

$$\hat{g}(z) = \int_{\mathbb{R}} e^{iz\theta} g(\theta)\, d\theta.$$

Theorem 2. *Let $\lambda = \sum_{k=1}^{J} \lambda_k$ and let $\hat{g}(\cdot)$ be the characteristic function of the equilibrium pdf $g(\cdot)$ of excess risk exposures. If traders trade according to the trade size function (16), $\hat{g}(\cdot)$ satisfies the system*

$$\dot{\hat{g}}(z) = -\left(\frac{1}{2}\sigma_\eta^2 z^2 + 2\lambda\right)\hat{g}(z) + 2\lambda\left[\hat{g}\left(\frac{z}{2}\right)\right]^2 \tag{21}$$

for all $z \in \mathbb{R}$,

$$\hat{g}(0) = 1, \tag{22}$$

and

$$\frac{d}{dz}\hat{g}(0) = 0. \tag{23}$$

At steady state, the characteristic function admits the following explicit expression:

$$\hat{g}(z) = \prod_{k=0}^{\infty}\left(\frac{1}{1 + \frac{\sigma_\eta^2}{4^{k+1}\lambda} z^2}\right)^{2^k}. \tag{24}$$

From (24), one sees that as $\frac{\sigma_\eta}{\sqrt{\lambda}}$ goes to zero, $\hat{g}(z)$ approaches 1, which is the characteristic function of the degenerate distribution with the mass point at $\theta = 0$. This degenerate distribution would obtain if banks were to trade in a continuous Walrasian market. Thus, $\frac{\sigma_\eta}{\sqrt{\lambda}}$ can

[13] Duffie and Manso (2007), Praz (2014), Andrei and Cujean (2017), and Üslü (2019), among others, also made use of characteristic functions or Fourier transforms to deal with the convolution integral in the context of search and matching models.

be understood as a measure of misallocation resulting from the frictional structure of OTC trading. Indeed, if σ_η is larger, this means that at any instant the exogenous stochastic process of η makes ηs more dispersed in the cross section of banks, which in turn leads to a larger cross-sectional dispersion of banks' excess risk exposures, θ. On the other hand, if λ is larger, banks have more frequent opportunities to make their θs closer together, which implies a lower dispersion of θs. In the limit as λ goes to infinity, banks enjoy infinitely frequent opportunities to make their θs closer together so they successfully equalize them at $\theta = 0$, which coincides with the frictionless benchmark allocation.

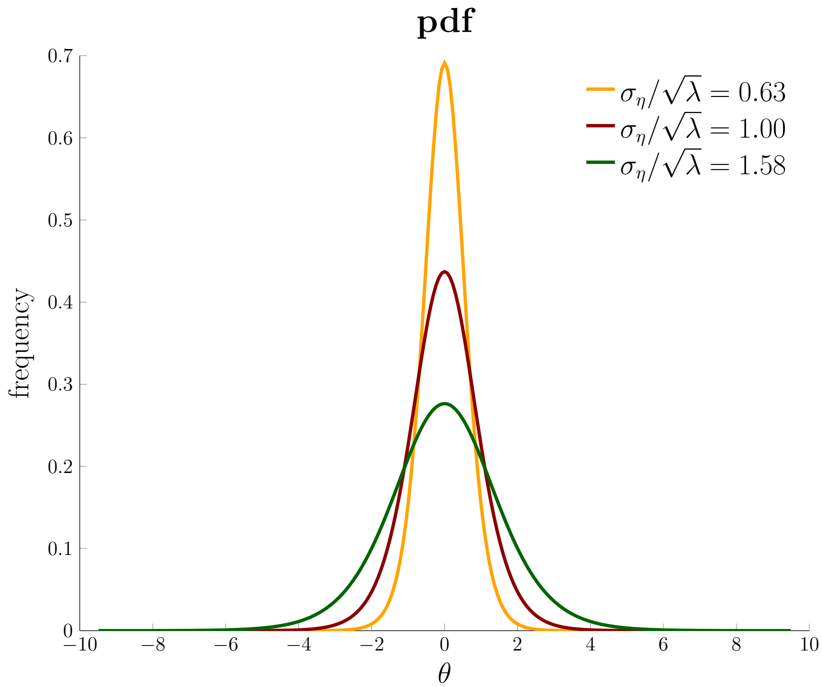

Figure 1: Equilibrium density function of banks' excess risk exposure, θ, for varying degrees of "misallocation."

Using the system (21)-(23), together with $\dot{\hat{g}}(z) = 0$, it is possible to derive recursively all moments of the stationary excess risk exposure distribution (Lukacs, 1970, p. 21):

$$\mathbb{E}[\theta^n] = i^{-n}\left[\frac{d^n}{dz^n}\hat{g}(z)\right]_{z=0}. \tag{25}$$

The following corollary reports results about the first four moments.

Corollary 3. *At steady state, banks' excess risk exposure, θ, has a symmetric, mean-zero distribution with a standard deviation of $\frac{\sigma_\eta}{\sqrt{\lambda}}$ and an excess kurtosis of $\frac{6}{7}$.*

Utilizing the same technique (25) in Section 4, we derive *in closed form* proxies for important dimensions of market il/liquidity including price dispersion, price impact, and sharp bounds for trade volume. One can also utilize the closed-form characteristic function (24) and invert it to numerically obtain the equilibrium pdf in Figure 1. Thanks to the parsimoniousness of (24), $\sigma_\eta/\sqrt{\lambda}$ alone determines the entire pdf, which is also equal to the standard deviation as stated in Corollary 3.

4 Results

In this section, we derive certain endogenous equilibrium objects that are related to market liquidity and have direct counterparts easily calculated from transaction-level data.

4.1 Trade volume

In the previous section, we have established that, as a result of search frictions, there is a *non-degenerate* distribution of excess risk exposures in the cross section of banks. According to our bilateral matching protocol, there is a measure λ_j of pairwise meetings among these banks at any instant in market j, in which each pair of banks bilaterally equalize their excess risk exposures by trading the quantity (16) stated in Theorem 1. Thus, instantaneous aggregate trading volume in market j can be calculated as

$$\mathcal{V}_j = \lambda_j \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left| q_j \left(\theta, \theta' \right) \right| g \left(\theta' \right) g \left(\theta \right) d\theta' d\theta. \tag{26}$$

By using Theorem 1 and Theorem 2, we arrive at the following proposition, which provides us with a closed-form formula for equilibrium trade volume.

Proposition 4. *Trade volume in market j in the stationary equilibrium is*

$$\mathcal{V}_j = \frac{\lambda_j}{|\psi_j|} \frac{1}{\pi} \int\limits_{\mathbb{R}_{++}} \frac{1}{z^2} \left[1 - \prod_{k=0}^{\infty} \left(\frac{1}{1 + \frac{\sigma_\eta^2}{4^{k+1}\lambda} z^2} \right)^{2^{k+1}} \right] dz. \tag{27}$$

Trade volume in market A relative to trade volume in market B is

$$\frac{\mathcal{V}_A}{\mathcal{V}_B} = \frac{\lambda_A}{\lambda_B} \frac{|\psi_B|}{|\psi_A|}. \tag{28}$$

Equation (27) shows that four parameters, σ_η, λ, λ_j, and $|\psi_j|$, together determine the trade volume in market j. The integral term is a measure of how dispersed banks' excess risk exposures are, which is increasing in $\frac{\sigma_\eta^2}{\lambda}$, i.e., how intensely banks' hedging need changes relative to how frequently they can trade in some market. The rate of meetings in market j, λ_j, has two opposing effects on trading volume. First, it has a direct positive effect, i.e., as traders meet more frequently in market j, they can exchange more of asset j in total. Second, it has an indirect negative effect through λ, i.e., as λ_j increases, the equilibrium misallocation, $\frac{\sigma_\eta}{\sqrt{\lambda}}$, decreases and this depresses the trade volume. However, the former effect dominates, and λ_j correlates positively with trading volume in market j. The systematic risk of asset j, $|\psi_j|$, has a negative impact on trade volume by decreasing the individual trade sizes. Indeed, trading an asset with large systematic risk leads to a large movement in excess risk exposures, and hence, banks trade these assets in smaller quantities when trying to equalize their excess risk exposures through bilateral trade.

One virtue of our model is to demonstrate how the cross section of trade volume is determined jointly by arbitrary combinations of asset quality (i.e., less exposure to risk) and asset liquidity (i.e., less exposure to search frictions) and also to shed light on some of the puzzling evidence documented in the empirical literature. It is common that a safer asset has also less exposure to search frictions, but there are counter-examples studied in the fixed-income literature. Geromichalos, Herrenbrueck, and Lee (2023), for example, brought attention to the case of AAA-rated vs. AA-rated US corporate bonds. Post-crisis regulations have substantially increased the difficulty of attaining the AAA rating, and so, the resulting dearth of outstanding bonds has made it more difficult to buy and sell these bonds. As a result, trade volume of AAA-rated bonds has become smaller than that of AA-rated, although AAA-rated bonds are still safer than AA-rated bonds. From (28), we see that the relative volume in market A may become smaller than one, following a sharp decline in λ_A, as long as λ_A/λ_B becomes smaller than $|\psi_A/\psi_B|$. That is to say, a safer asset A can be traded in a smaller volume (per bond) than a riskier asset B as long as buying and selling A is substantially harder compared to B. Another example was presented by Beber, Brandt, and Kavajecz (2009) in the Euro-area government bond market, which features a unique negative correlation between credit quality and liquidity across countries. Similar to the case of AAA-rated vs. AA-rated US corporate bonds, Italian government bonds have high trading volume (per bond) due to their abundance stemming from Italy's lower fiscal discipline, while lower fiscal discipline at the same time makes Italian bonds more risky.

Although (27) is an explicit expression for trade volume, it is not straightforward to study its limiting properties, especially for λ_j, because the integral cannot be computed exactly. To overcome this difficulty, we calculate sharp lower and upper bounds for trade volume using results from the recent probability theory literature.

Corollary 5. *Trade volume in market j in the stationary equilibrium satisfies the following* *inequalities:*

$$\frac{1}{4}\sqrt{\frac{7}{3}}\frac{\lambda_j}{|\psi_j|}\frac{\sigma_\eta}{\sqrt{\lambda}} \le \mathcal{V}_j \le \frac{2}{\pi}\frac{\lambda_j}{|\psi_j|}\frac{\sigma_\eta}{\sqrt{\lambda}}. \tag{29}$$

Hence,

$$\lim_{\lambda_j\to\infty}\mathcal{V}_j = \lim_{|\psi_j|\to 0}\mathcal{V}_j = \lim_{\sigma_\eta\to\infty}\mathcal{V}_j = \infty,$$
$$\lim_{\lambda_j\to 0}\mathcal{V}_j = \lim_{|\psi_j|\to\infty}\mathcal{V}_j = \lim_{\sigma_\eta\to 0}\mathcal{V}_j = 0,$$

and

$$\lim_{\lambda_k\to\infty}\mathcal{V}_j = 0$$

for all $k \in \mathcal{J}$ *such that* $k \neq j$.

Corollary 5 gives us interesting limiting results. As the systematic risk of asset j approaches zero, banks trade it in increasingly larger quantities to equalize their excess risk exposures, and hence, the trading volume of asset j approaches infinity. Vice-versa, as the systematic risk of asset j approaches infinity, its trading volume approaches zero because trading even a small quantity leads to a large change in banks' excess risk exposures. More interestingly, the effect of λ_j and λ_k for $k \neq j$ on trade volume in market j in the limit are the opposite. As λ_k for $k \neq j$ approaches infinity, banks' valuations approach the frictionless benchmark valuations and the distribution of excess risk exposures approaches the degenerate distribution in which there is no misallocation. As a result, banks do not trade asset j in the limit because trading the infinitely liquid asset k already allows them to obtain their first-best risk exposure. If λ_j approached infinity, the same effect would be observed on the equilibrium level of misallocation. However, this would not dry up the trading in market j. On the contrary, the reason why banks can achieve the degenerate distribution of excess risk exposures in this case is that they trade in market j with infinite intensity, which implies that trading volume in market j goes to infinity while volume in all other markets go to zero. This is an interesting cross-market implication of decline of search frictions in one market that would not obtain in single-asset OTC models. This is one of the implications of our model that we test in Section 5.

4.2 Price dispersion

As is typical in this class of models, different trader pairs trade at different prices because the lack of immediate access to a competing counterparty is reflected as a discount or premium in the bilaterally negotiated prices. Therefore, the law of one price does not obtain in the frictional OTC market equilibrium. An interesting equilibrium object to calculate is price dispersion, which also attracted attention in empirical research with transaction-level data from various OTC markets becoming more widely available to researchers.[14] As the measure of price dispersion, we calculate in closed form the standard deviation σ_P of the equilibrium price distribution.

Proposition 6. *Price dispersion in market j measured by the standard deviation of the stationary equilibrium price distribution is*

$$\sigma_{P_j} = \frac{1}{\sqrt{2}} \frac{r\gamma_B \sigma^2 \left|\psi_j\right|}{r + \lambda/2} \frac{\sigma_\eta}{\sqrt{\lambda}}. \tag{30}$$

Price dispersion in market A relative to price dispersion in market B is

$$\frac{\sigma_{P_A}}{\sigma_{P_B}} = \frac{|\psi_A|}{|\psi_B|}. \tag{31}$$

One advantage of our model relative to the models that restrict banks' asset positions to $\{0, 1\}$ such as Hugonnier, Lester, and Weill (2022) and Shen, Wei, and Yan (2020) is the following. In those models, the standard deviation of price is not available in closed form, but the difference between the maximum and the minimum price. From an econometric point of view, one would like a measure that takes into account the distributional effect; i.e., trades that are more likely to happen should have higher weight than trades that are less likely, in the calculation of price dispersion. Our price dispersion measure (30) takes into consideration this distributional impact.

Our price dispersion measure (30) is the product of two factors. The first factor captures the sensitivity of transaction prices in market j to banks' excess risk exposures, which decreases with λ. That λ is finite is the reason why there is a deviation from the law of one price. The second factor, common with the trade volume (29), captures the misallocation. An increase in λ reduces the equilibrium level of misallocation so banks' marginal valuations become less dispersed, so does price dispersion.

The relative price dispersion measure (31) provides very interesting cross-market comparative statics. Intuitively, an increase in the systematic risk $|\psi_A|$ of asset A increases the relative price dispersion in market A because the price of asset A becomes more sensitive to excess risk exposures when it contains more systematic risk. More surprisingly, an increase in the liquidity

[14] See, among others, Jankowitsch, Nashikkar, and Subrahmanyam (2011), Feldhütter (2012), Friewald, Jankowitsch, and Subrahmanyam (2012), Eisfeldt, Herskovic, and Liu (2023), and Pintér (2023).

λ_A does not affect the relative price dispersion in any market. This is an interesting result that could not be obtained in the comparative statics of single-asset models. In a single-asset model typically an increase in λ_A will lead to a decline in price dispersion because distortions on extensive and intensive margins alleviate. Here, these effects are present as well, but the novel cross-market effect is that an increase in λ_A leads to a decline in the price dispersion in both market A and market B by reducing misallocation and by reducing the sensitivity of prices to excess risk exposures. This happens because when trading in market B, a bank takes into account how its position in asset A can expose it to the risk of being stuck with a suboptimal portfolio due to the search frictions in market A, and vice-versa. Hence, when we look at the *relative* price dispersion, we see that the effect of an increase in λ_A work in the same way in both markets so the relative price dispersion stays unaffected.

4.3 Price impact

In search models, equilibrium price dispersion arises because banks with different marginal valuations bilaterally negotiate and then their valuation differentials translate into different realized prices. It is possible to interpret this as *price impact* due to illiquidity. Price impact arises for various reasons in market microstructure models such as strategic interaction[15] or a combination of strategic interaction and adverse selection.[16] In our model, it arises due to search frictions.

To understand the way we quantify the price impact in the equilibrium of our model, one must inspect the Nash bargained price (17). Using (15) and (16), one sees that in order to buy q units of asset j from a counterparty with current excess risk exposure of θ', a bank pays

$$P_j\left(q \,|\, \theta'\right) = \frac{u^{(j)}\left(0, \mathbb{E}\left[\mathbf{a}\right]\right)}{r} - \frac{r\gamma_B\sigma^2\psi_j}{r+\lambda/2}\left(\theta' - \psi_j q\right).$$

As can be seen, the sensitivity of the transaction price to the traded quantity is

$$\left|\frac{\partial P_j\left(q \,|\, \theta'\right)}{\partial q}\right| = \frac{r\gamma_B\sigma^2\left|\psi_j\right|^2}{r+\lambda/2}.$$

The following proposition establishes that this sensitivity is equal to (a normalized version of) the ratio of price dispersion to quantity dispersion. Thus, we quantify the price impact in the cross-section of equilibrium trades as the ratio of price dispersion to quantity dispersion.

[15] See, for example, Vayanos (1999), Rostek and Weretka (2015), Antill and Duffie (2021), and Chen and Duffie (2021).

[16] See, for example, Kyle (1985), Kyle (1989), Sannikov and Skrzypacz (2016), and Du and Zhu (2017).

Proposition 7. *Price impact in market j in the stationary equilibrium is*

$$\delta_j \equiv \frac{2\sigma_{P_j}}{\sigma_{q_j}} = \frac{r\gamma_B \sigma^2 |\psi_j|^2}{r + \lambda/2}. \tag{32}$$

Price impact in market A relative to price impact in market B is

$$\frac{\delta_A}{\delta_B} = \left|\frac{\psi_A}{\psi_B}\right|^2. \tag{33}$$

The price impact (32) is calculated using the second moment of equilibrium price and quantity distributions but, as explained above, the rationale behind it being a measure of price impact comes from the bank's problem. In particular, δ_j is equal to (the absolute value of) the sensitivity of a bank's marginal valuation for asset j to its position in asset j. Because transactions prices are equal to the trading banks' post-trade marginal valuations, δ_j thus measures how much extra a bank should pay over its counterparty's initial marginal valuation in order to buy an additional unit of asset j, just as Kyle's lambda measures how much price movement a trader's trade induces. Equation (32) shows that price impact is present due to search frictions. As search frictions vanish (i.e., $\lambda_j \to 0$ for any j), δ_j goes to 0. Importantly, price impact in one market is affected exactly the same way by the illiquidity of either markets. It is because banks use any asset to satisfy the same type of hedging need and if one market becomes more or less liquid, their reliance on that market adjust accordingly. In the end, what determines price impact is the overall illiquidity of the markets rather than the illiquidity of an individual market. As a result, (33) shows that the relative price impact is affected only by systematic risks of the asset and not by their illiquidity.

An interesting comparative statics revealed by (33) is that as the systematic risk $|\psi_A|$ of asset A increases, the relative price impact in market A increases in a convex way. Convexity arises because the systematic risk increases both the price dispersion and the reciprocal of quantity dispersion linearly. Thus, it enters the relative price impact with an exponent of 2.

5 Testing the model's implications

In this section, we empirically test the model's implications in the US corporate bond market. Corporate bonds are traded over the counter with majority of these trades being purely bilateral and subject to significant search frictions. A typical trade occurs after an investor calls over telephone, one-by-one, one or multiple dealers. Many of these dealers may not have the trading need that matches with that of the investor. And if they do, the quotes in each call are valid only for a short period of time ("as long as the breath is warm" (Bessembinder and Maxwell, 2008)), which makes it difficult to obtain multiple quotations before agreeing to a particular trade.[17] These characteristics of the corporate bond market make search frictions a prevalent component of the trading process. Considering these microstructure components, we find the corporate bond market as an appropriate laboratory to test the predictions of our model.

A natural question before we move on to describing our data sample is how the agents in our model map to the participants in the US corporate bond market. Regarding this mapping, our main approach is that of Hugonnier, Lester, and Weill (2022). That is, we consider the real-world OTC markets as fully decentralized in that all market participants are subject to the bilateral trading friction characteristic of these markets. According to this approach, both dealers and customers of the US corporate bond market map to banks in our model. Hence, in what follows, we use the full sample of transaction data containing both customer-to-dealer and interdealer trades. One caveat is, however, that banks in our model are ex-ante homogeneous agents, while dealers and customers in practice may be very different in why and how they trade. To address this concern, we repeat all our formal analyses by using the subsample of our data with only interdealer trades in Appendix D.3 and show that our main results are robust to this restriction.

5.1 Data

The data used in this study come from several sources. We obtain our bond transactions data from the enhanced version of Trade Reporting and Compliance Engine (TRACE), for the sample period from July 1, 2002 to December 31, 2021.[18] TRACE dataset covers virtually all transactions of the US corporate bond market, and reports trade price, trade size, buy/sell indicator, as well as the type of the counterparty (dealer vs. customer). We use the data filters proposed by Dick-Nielsen (2014) to eliminate erroneous entries from reversals, canceled trades, and corrected trades. We further remove the commissioned trades, the non-secondary market transactions and the transactions that are labeled as when-issued, and special price trades.

We merge the cleaned TRACE data with the Mergent Fixed Income Securities Database (FISD), to incorporate bond characteristics such as security type, offering amount, offering date, maturity date, and coupon rate. We eliminate bonds that are asset-backed, mortgage-

[17] See Bessembinder and Maxwell (2008) for a detailed review of the U.S. corporate bond market, and Feldhütter (2012) for a discussion of its appropriateness for empirical tests of search-theoretic models.

[18] Although TRACE bond data starts from July 1, 2002, our final sample period for the endogenous liquidity measures starts from the week of October 7, 2002 since we use the initial quarter as the estimation period for some of our predictors.

backed, agency-backed, equity-linked, or issued by governments or municipalities, bonds that are putable, convertible, exchangeable, preferred securities, Eurobonds/-notes, pass-through trust securities, or part of unit deals, and bonds with unusual coupons (variable rate, pay-in-kind), with sinking fund feature, or that are issued in non-USD currencies. We also remove the transactions that are priced below \$5 or above \$1,000, the transactions executed on weekends, and the transactions that occur within less than one year remaining to maturity date. We next bring the macroeconomic indicators to our data, such as GDP forecast dispersion and treasury rate, obtained from the Federal Reserve's website, as well as implied market volatility, obtained from OptionMetrics. Finally, we bring information on the credit default swaps (CDS) of the bond issuers with CDS, obtained from IHS Markit.[19]

After merging and cleaning the data, we calculate the liquidity measures and the predictors. Our objective is to construct the variables as closely as possible to the variables in the theoretical model, while keeping in mind the properties of transaction-based data. Although the corporate bond transactions data are intraday, most bonds do not trade at daily frequency. In addition, the cross section of bonds that are traded changes rapidly over time. If the liquidity measures were calculated at a high frequency (e.g., daily), we could lose illiquid bonds from our sample. If we instead calculated the liquidity measures at a lower frequency (e.g., monthly), our econometric specification could be too sluggish to capture the sensitivity of liquidity to systematic risk and to the shifts in the time-varying cross section. We therefore calculate our liquidity measures at weekly frequency to capture both the cross section of bonds and its dynamics more comprehensively.

The liquidity measures we construct include trade volume ($\mathcal{V}_j$), price dispersion (σ_{P_j}), and price impact (δ_j), which have direct counterparts in our theoretical model. We calculate these measures for each bond-week. The predictors are similarly based on the model and calculated prior to the beginning of each bond-week to avoid any time overlap between a dependent variable and a predictor.[20] We require any bond-week to have non-missing observations for liquidity measures and predictor variables to be included in our final sample. Our sampling procedure results in 4,912,241 bond-week observations of 29,446 bonds by 3,952 issuers over the sample

[19]Our sampling procedure for the CDS data is as follows. We use the senior unsecured, USD-denominated, and five-year tenor contracts. We prioritize the modified restructuring documentation clause before April 8, 2009 ("CDS Big Bang"), and no restructuring documentation clause on and after.

[20]For instance, we use the number of trades as a control for the variations in the liquidity measures due to firm-specific news events. Without making sure that the number of trades is calculated for a non-overlapping period, it could be problematic to use it as a predictor in regressions in which trade volume or price dispersion is the dependent variable. The detailed variable definitions and methodology followed in their calculations are included in Appendix A.

period from October 7, 2002 to December 31, 2021.

Table 1 presents the sample summary. The mean weekly trade volume is \$11.03 million, and median trade volume is \$2.25 million. Similarly, the weekly average number of trades has a mean of 23.07 and a median of 11.42. Inspection of quartile observations reveals that the distributions of trade volume and average number of trades are right skewed. We similarly observe highly skewed distributions in other liquidity measures and several control variables.

Table 1: Descriptive statistics

This table presents the descriptive statistics of the sample used in this study. The sample period is from October 7, 2002 to December 31, 2021. The sample includes 29,446 bonds of 3,952 issuers, and the observation unit is bond-week. The dependent variables, trade volume, price dispersion, and price impact, are calculated at weekly frequency for each bond. The predictors are calculated within the most recent quarter prior to beginning of each week. For the readily available time-series variables (e.g., treasury rate), we use the most recent weekly observation prior to beginning of the week. The table reports mean, standard deviation, 1^{st}, 25^{th}, 50^{th}, 75^{th}, and 99^{th} percentile observations for each variable. Detailed variable definitions and sources of data are provided in Appendix A.

	Mean	St. dev.	1^{st}	25^{th}	50^{th}	75^{th}	99^{th}	Obs.
Trade volume (\$mm)	11.03	29.39	0.01	0.29	2.25	10.41	119.87	4,912,241
Price dispersion	0.47	1.01	0.00	0.12	0.33	0.66	2.24	4,912,241
Price impact	22.87	113.90	0.00	0.24	1.28	10.00	304.26	4,912,241
Offering amount (\$bn)	0.65	0.67	0.01	0.25	0.50	0.80	3.00	4,912,241
Offering amt., other bonds (\$bn)	4232	1332	2135	2848	4308	5507	6100	4,912,241
Volatility beta	0.61	29.21	0.00	0.03	0.08	0.24	5.77	4,912,241
Average number of trades	23.07	38.30	1.50	5.92	11.42	24.92	171.75	4,912,241
GDP forecast dispersion	0.47	0.62	0.19	0.26	0.31	0.43	5.09	4,912,241
Treasury rate, 1 mo. (%)	1.03	1.37	0.00	0.05	0.23	1.66	5.15	4,912,241
Implied market volatility, 3 mo.	0.17	0.07	0.09	0.13	0.15	0.20	0.42	4,912,241
Average number of cancellations	0.23	0.53	0.00	0.00	0.08	0.25	2.00	4,912,241
Avg. # of cancel., other bonds	0.16	0.05	0.07	0.13	0.17	0.19	0.32	4,912,241
Average CDS depth	7.08	4.68	0.43	4.10	6.25	8.65	25.52	2,725,599

5.2 Empirical analysis

In this section, we test the implications of our model for each endogenous liquidity measure. The theoretical results regarding the determinants of trade volume ($\mathcal{V}_j$), price dispersion (σ_{P_j}), and price impact (δ_j), from Equations (29), (30), and (32), respectively, are as follows:

$$\mathcal{V}_j \sim \frac{\lambda_j}{|\psi_j|}\frac{\sigma_\eta}{\sqrt{\lambda_j+\lambda_{-j}}},$$

$$\sigma_{P_j} = \frac{1}{\sqrt{2}}\frac{r\gamma_B\sigma^2\left|\psi_j\right|}{r+\left(\lambda_j+\lambda_{-j}\right)/2}\frac{\sigma_\eta}{\sqrt{\lambda_j+\lambda_{-j}}},$$

$$\delta_j = \frac{r\gamma_B\sigma^2\left|\psi_j\right|^2}{r+\left(\lambda_j+\lambda_{-j}\right)/2}.$$

The theoretical model suggests a clear relationship for each pair of liquidity measure and predictor. For instance, it suggests that trade volume ($\mathcal{V}_j$) of a particular bond j is increasing with its offering amount (λ_j) and decreasing with its volatility beta ($|\psi_j|$) and the total offering amount of other bonds (λ_{-j}). Therefore, these equations provide the coefficient signs for each predictor that we expect to observe in the data if our search-based model of equilibrium portfolio management captures the dominant economic channels in practice. Table 2 briefly shows the empirical counterparts of the variables in our model.

Table 2: Empirical counterparts

This table presents the empirical counterparts of our variables in the theoretical model. The only variable we do not empirically measure or proxy for in this table is γ_B (risk aversion), and therefore is not included in the regressions. Detailed variable definitions are provided in Appendix A.

Variable	Empirical counterpart
$\mathcal{V}_j$	Trade volume
σ_{P_j}	Price dispersion
δ_j	Price impact
λ_j	Offering amount
λ_{-j}	Offering amount, other bonds
ψ_j	Volatility beta
σ_η	Real GDP forecast dispersion
r	Treasury rate
σ	Implied market volatility

Before proceeding to the detailed regression analysis, let us discuss our choice and construction of the key predictors, λ_j and ψ_j. We use offering amount as a proxy for asset-specific contact rates (λ_j). Because asset-specific contact rates, λ_j, are deep parameters that govern the exogenous liquidity differentials among the assets, one ideally needs to use a proxy that comes outside the sample from which the endogenous liquidity proxies are calculated. Considering this, we think offering amount is a sensible choice. Furthermore, there are theoretical and practical motivations for this choice. Theoretically, Weill (2008) shows that assets with high offering amount end up having high endogenous contact rates when *ex ante* identical investors allocate their search budget across assets. While there are no short-sale costs or restrictions in our model, shorting a corporate bond is a complicated process in practice, which involves borrowing the bond before being able to short. This may give an advantage to bonds with high offering amount in the market for borrowing corporate bonds. Thus, one may argue that interpreting the offering amount as a measure of λ_j makes sense in a model without short sale restrictions like ours because the offering amount has the role of alleviating frictions in practice, which is captured by λ_j in the model.

As a proxy for ψ_j, we construct a volatility beta measure. Traditional empirical analyses of fixed-income instruments use credit rating, coupon rate, time-to-maturity, callable bond dummy, etc., to capture various dimensions of risk such as credit risk and interest rate risk.[21] However, none of these measures provide a reasonable match for the main systematic risk measure ψ_j of our model. Because the main purpose of our empirical analysis is to test the implications of our model, we construct a novel but intuitive measure of exposure to systematic risk by following our model assumption (1). Equation (1) tells us that if the aggregate volatility is σ, the systematic volatility of asset j is equal to $\sigma\psi_j$. Therefore, to find the empirical counterpart of ψ_j, we run the following OLS regression:

$$\sigma_{j,t} = \alpha_j + \beta_j \sigma_t + \epsilon_{j,t},$$

where σ_t is implied market volatility, $\sigma_{j,t}$ is the return volatility of bond j, and the resulting coefficient β_j is what we call bond j's volatility beta.[22] It is important to note that there is no mechanical overlap between the information content of our volatility beta measure and the endogenous liquidity measures. Price dispersion and price impact calculations also require using transaction prices, but they specifically use the second moment of the *demeaned* prices. Volatility beta instead requires using mean prices to calculate bond returns, and so, the demeaning process in the calculation of price dispersion and price impact eliminates any overlap with volatility beta. Thus, any relation between volatility beta and liquidity measures captured by our regressions below is an economic relation.

Figure 2 plots the time series of cross-sectional average of each liquidity measure over the sample period at weekly frequency. In this demonstration, we partition the cross section of bonds into two subsamples with respect to their (absolute) volatility betas (i.e., above vs. below median volatility beta). A simple visual inspection reveals differences for each liquidity measure between the sample averages of low beta and high beta bonds. Relative to low beta bonds, high beta bonds have lower trade volume, higher price dispersion, and higher price impact, consistent with the model predictions.[23] We deepen this simple visual inspection with

[21]See Bessembinder, Maxwell, and Venkataraman (2006), Dick-Nielsen, Feldhütter, and Lando (2012),Hotchkiss and Jostova (2017), O'Hara and Zhou (2021), and Choi, Huh, and Shin (2024), for example.

[22]For more details, see Appendix A, which includes variable definitions. As our measure for aggregate volatility, we use implied market volatility instead of volatility index (VIX). This is because implied market volatility is a measure of volatility, while volatiliy index is a measure of price (of volatility). In any case, these two series are highly correlated (with a correlation coefficent of 0.9806), and our results are robust to using VIX instead of implied market volatility in our estimations. The Appendix Figure D.1 presents a comparison, which indicates a high degree of similarity between the two series.

[23]We find volatility beta to be an important predictor of liquidity measures. Therefore, a natural question

our formal regression analyses below in order to mainly inspect the sign of the resulting precise elasticities that capture the relation of our endogenous liquidity measures with volatility beta and other parameters.

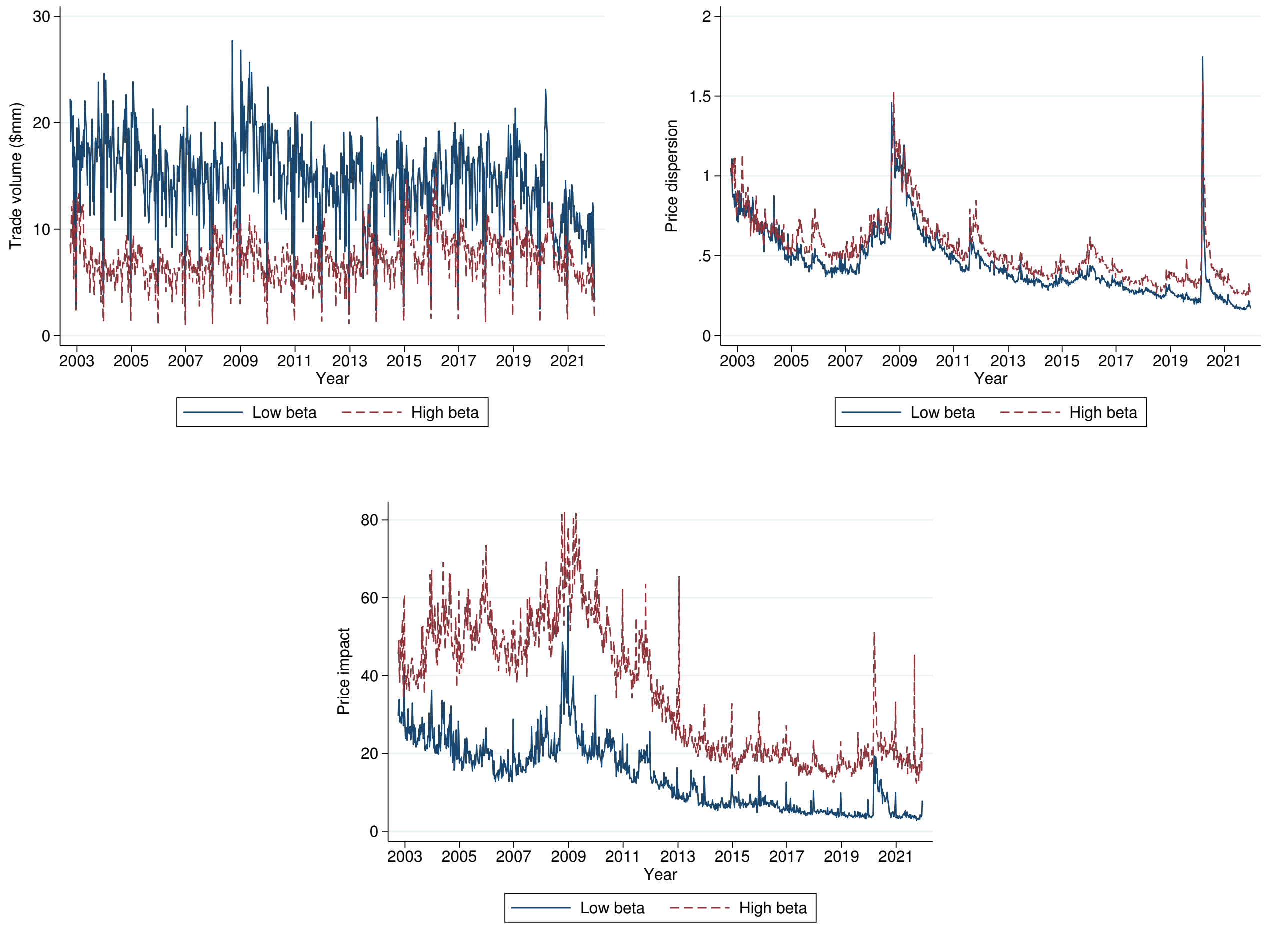

Figure 2: Time series of liquidity measures
This figure plots the cross-sectional average of liquidity measures over the sample period from October 7, 2002 to December 31, 2021, for the subsamples of bonds with low versus high (absolute) volatility betas. Calculation of liquidity measures (trade volume, price dispersion, and price impact), as well as volatility beta are described in Appendix A. Each week, we calculate the median of volatility beta and partition the cross-section of bonds into "Low beta" vs. "High beta" subsamples, based on whether a bond's volatility beta is below or above median volatility beta, respectively.

We start by noting that the theoretical formulas (29), (30), and (32) for liquidity measures all have multiplicative functional forms. In order to test their implications more accurately, we take the natural logarithm of each empirical variable.[24] We then run the linear regression

is how it relates to the traditional fixed-income risk measures. In Appendix Table D.1 we present our findings that address this question. Indeed, Table D.1 shows a significant and economically meaningful relation between volatility beta and credit risk measures as well as other main characteristics of the bond.

[24]We add 0.01 before taking logarithm of the variables that can take zero values.

specified below with the logged version of variables (log and absolute value are suppressed for simplicity).[25]

$$Liquidity_{j,t} = \alpha + \beta_1 \lambda_{j,t} + \beta_2 \psi_{j,t} + \beta_3 \, \mathrm{ANT}_{j,t} + \tau_t + \varepsilon_{j,t}, \tag{34}$$

where "$Liquidity_{j,t}$" of bond j in week t denotes the liquidity measure, trade volume ($\mathcal{V}_j$), price dispersion (σ_{P_j}), or price impact (δ_j), and τ_t denotes the time-specific intercepts for year-weeks. We run this regression separately for each measure.

Table 3 presents our findings under the baseline model (34). We directly control for the time-fixed effects in this table to isolate and focus on the cross-sectional relation of liquidity measures with the predictors. Column (1) shows our findings for trade volume ($\mathcal{V}_j$). We find that trade volume increases with the offering amount of the bond (λ_j), as suggested by the theoretical model. Specifically, one percent increase in offering amount of bond j leads to a 0.91 percent increase in trade volume of the same bond. Consistent with the theoretical results, we find that trade volume decreases with volatility beta (ψ_j, sensitivity of bond volatility to aggregate volatility).

In addition to our main cross-sectional predictors and time fixed effects, we also include the average number of trades of each bond to control for the variation in liquidity measures due to firm-specific news events. Although investors in our model trade only because of changes in their idiosyncratic hedging needs, the firm-specific news events absent in our model trigger speculative trading activity in practice (e.g., rating changes in Jankowitsch, Ottonello, and Subrahmanyam, 2018 and earnings announcements in Wei and Zhou, 2016) and this activity in turn affects bonds' realized liquidity measures. Consistent with earlier work, Table 3 presents positive and significant relation between this control variable and the liquidity measures.

In Column (2) of Table 3, we repeat our estimations for price dispersion (σ_{P_j}). We find that price dispersion is decreasing with offering amount of bond j (λ_j), exactly as predicted by the theoretical model. One percent increase in the offering amount of bond j leads to a 0.179 percent decline in the same bond's price dispersion. We also find that price dispersion is increasing with volatility beta (ψ_j), consistent with the model.

In Column (3) of Table 3, we estimate the specification for price impact (δ_j). As predicted by the model, we find that price impact is decreasing with the offering amount of bond j (λ_j). One percent increase in the offering amount of bond j leads to 1.015 percent decrease in

[25] Appendix Table D.4 presents our findings under linear functional form (i.e., not log). Note the significant increase in the model fit for all liquidity measures when we run the regressions with logged variables. The adjusted R^2s increase from 0.300, 0.163, and 0.101 in Appendix Table D.4 to 0.484, 0.216, and 0.204 in corresponding Table 4, respectively.

Table 3: Determinants of liquidity in the cross section, baseline model

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds under a log-linear functional form assumption. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript j refers to bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Offering amount (λ_j)	0.910***	-0.179***	-1.015***
	(147.78)	(-48.64)	(-125.18)
Volatility beta (ψ_j)	-0.038***	0.086***	0.110***
	(-15.10)	(51.42)	(31.88)
Average number of trades (ANT_j)	0.653***	0.721***	0.996***
	(107.21)	(142.00)	(103.32)
Intercept	-0.333***	-3.214***	-2.937***
	(-14.37)	(-192.57)	(-85.70)
Year-week FE	Y	Y	Y
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.503	0.231	0.209

price impact. Finally, consistent with the model, we find that price impact is increasing with volatility beta (ψ_j). Overall, our empirical results for the cross-sectional analysis in Table 3 are entirely consistent with the theoretical model.

We next extend our cross-sectional analysis to a test that incorporates model-informed time-series factors. Instead of using time fixed-effects for each year-week, we directly include macroeconomic indicators. More specifically, we extend our regression equation to (again, log and absolute value are suppressed for simplicity):

$$Liquidity_{j,t} = \alpha + \beta_1 \lambda_{j,t} + \beta_2 \lambda_{-j,t} + \beta_3 \psi_{j,t} + \beta_4 \,\text{ANT}_{j,t} + \beta_5 \sigma_{\eta,t} + \beta_6 r_t + \beta_7 \sigma_t + \varepsilon_{j,t}, \quad (35)$$

where $\sigma_{\eta,t}$, r_t, and σ_t denote GDP forecast dispersion, treasury rate, and implied market volatility, respectively. The contribution of this extension to our analysis is two-fold. First, this allows us to include the total offering amount of other bonds in our regression equation (35) as a proxy for λ_{-j}, which is one of the cross-sectional determinants of liquidity as seen in the theoretical formulas (29), (30), and (32). This was not possible in the earlier regression (34) because the presence of time fixed-effects would lead to a mechanical multicollinearity if we used both the offering amount of bond j and the total offering amount of other bonds as independent variables in the same regression. The second benefit of (35) is that it captures not only the asset-specific determinants of liquidity in the theoretical model but also the economy-wide determinants.

Table 4 presents the results. Compared to Table 3, there is a slight decline in adjusted

R^2s for all liquidity measures. The decline is not surprising because one naturally expects that, compared to time fixed effects, our chosen time-series variables cannot capture equally well the time-series variation in liquidity measures. That the decline is small, however, means that the extent to which they capture the time-series variation is satisfactory. Looking at the performance of the cross-sectional determinants of liquidity, our regressions indicate that our theoretical model's implications regarding λ_j and ψ_j are successfully confirmed in the data, even in a conservative full-sample analysis like ours.[26] Turning to λ_{-j}, we find that while the signs of the slope coefficient of λ_{-j} in the trade volume regression and the price dispersion regression are consistent with our theory the same sign in the price impact regression is inconsistent.[27] The success of the implications regarding λ_{-j} is, therefore, more ambiguous. One can always blame the offering amount of all other bonds as being an imperfect proxy for λ_{-j}, and so, argue that the model is not given the best chance to be consistent with the data. We however take the view of these inconsistencies highlighting some of the strong assumptions our model makes. For example, as a reasonable starting point, banks in our model are assumed to be homogeneous in terms of their access to all J markets. This assumption does not leave any room for clientele effects, which are very likely to be present in the corporate bond market. Thus, in reality, in the calculation of λ_{-j}s, investors of different bonds may focus on different smaller universes of bonds instead of the full sample that we use. An empirical introspection may suggest that repeating our analysis with different bond subsamples containing bonds that are more likely to be substitutes from the investors' viewpoint would give the model a better chance to work. We do not take this route because our model does not inform us about how we could choose those subsamples.

Overall, our *cross-sectional* results from empirical tests of liquidity are mostly consistent with the implications of the theoretical model, both with time fixed effects and with a model-informed set of macroeconomic indicators. Because our model is developed mainly to obtain precise cross-sectional implications, we interpret this empirical consistency as pointing to the success and usefulness of the search-theoretic approach in uncovering the determinants of endogenous liquidity differentials across OTC assets. For the sake of completeness, we discuss the non-cross-sectional predictors of Table 4 in Appendix E.

[26]An alternative to using the full sample would be to consider only dealer-to-dealer trades, which may be seen as more consistent with our model's assumption of ex-ante homogeneous agents. In Appendix Table D.2, we repeat our analyses but instead use only the dealer-to-dealer trades when calculating our liquidity measures, and we find that our main results are robust to this restriction.

[27]Interestingly, the sign of λ_{-j} in the Appendix Table D.4 (where we run the regression assuming a linear functional form) is consistent with the theory in all three regressions. However, we base our conclusions on Table 4, as it more accurately corresponds to the functional forms in the theoretical model.

Table 4: Determinants of liquidity in the cross section and over time, full model

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds under a log-linear functional form assumption. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Offering amount (λ_j)	0.910***	-0.177***	-1.013***
	(147.94)	(-47.94)	(-125.20)
Offering amount, other bonds (λ_{-j})	-0.575***	-0.361***	0.268***
	(-16.27)	(-17.75)	(9.97)
Volatility beta (ψ_j)	-0.027***	0.096***	0.110***
	(-8.69)	(43.95)	(32.53)
Average number of trades (ANT_j)	0.653***	0.722***	0.994***
	(106.44)	(146.31)	(104.07)
GDP forecast dispersion (σ_η)	-0.110***	-0.186***	-0.098***
	(-5.95)	(-12.73)	(-8.57)
Treasury rate (r)	0.012	-0.019***	-0.030***
	(1.64)	(-5.01)	(-6.59)
Implied market volatility (σ)	-0.087**	0.562***	0.727***
	(-2.33)	(20.05)	(25.56)
Intercept	4.210***	0.620***	-3.959***
	(15.36)	(4.17)	(-18.66)
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.484	0.216	0.204

5.3 Proxying for search frictions

One challenge we face, that is also prevalent in the literature, is the difficulty of proxying for search frictions (i.e., inverse of asset-specific contact rates). Search frictions relate to many factors and manifest in many forms largely unobserved by the econometrician, which involve being able to find a counterparty willing to trade a given asset, at a sufficient quantity, at a desirable price, in a timely manner, among other considerations. Although one can cleanly model search frictions in theory, its measurement in practice is challenging due to limitations with the available data.

In our main analyses, we use the variable offering amount as a proxy for (the inverse of) search frictions, and explain our motivations for doing so in detail in Section 5.2. However, one may argue that the variable offering amount could wear many hats and its relations with our liquidity measures could be confounded by other factors than what is due to search frictions. To address this concern, we propose an alternative proxy for search frictions: trade cancellations.

We use the number of trade cancellations a given bond has during a time period as an alternative proxy for (the inverse of) search frictions. We conjecture that a trade cancellation is more likely to occur if it is easy to find an equivalent or better trade on the bond (i.e., cancellations *reveal* that search frictions are milder). In contrast, if it is difficult to find a trade that is at least as good as the current trade, then the current trade would be less likely to get canceled.

For this test, we utilize the trade cancellations reported in TRACE. In addition to the transactions that eventually settle, the TRACE data also reports the transactions that get canceled. The timeline for a typical transaction includes an execution time, a report time, and a settlement date. Most trades get reported within minutes of transaction (with the introduction of TRACE and post-trade transparency requirements), and eventually settle if no further modification occurs. However, the counterparties of the trade are allowed 20 business days to cancel the trade.[28] When preparing the trade cancellations, we use the same data filters as we do for corporate bond trades, with the exception that we keep the cancellation entries only.[29] We then count the number of cancellations reported for each bond in the most recent 12 weeks prior to the current week.[30]

Table 5 presents our findings. In Panel A of Table 5, we run the regression with the baseline model as we do in Table 3 but use the number of cancellations instead of offering amount to proxy for search frictions. We find that that as the number of cancellations increases, the trade volume also increases. Consistent with our conjecture, more cancellations suggest that it is easy to find a desirable substitute trade on the bond which results in a higher trade volume. Moreover, we find that as the number of cancellations increases, price dispersion and price impact decrease. This is again consistent with the conjecture that higher number of cancellations implies that it is easier to find a substitute trade, which increases the liquidity on the bond.[31]

In Panel B of Table 5, we run the regression with the full model as we do in Table 4 but use the number of cancellations instead of offering amount (and average number of cancellations in

[28]For more information, see "TRACE OTC Corporate Bonds and Agency Debt User Guide" at `https://www.finra.org/filing-reporting/trace/documentation`.

[29]Interdealer trades are typically reported by both buying and selling dealers, and as such, data filters proposed by Dick-Nielsen (2014) drop one side of the interdealer trade entries (buy side) to avoid double counting. However, interdealer trade cancellations are not often reported by both sides. We similarly drop the buy side of interdealer cancellation entries for consistency. It is not clear if an ideal filter exists for interdealer cancellation entries, but if we err, it would be against double counting them.

[30]A cancellation entry includes the execution time and settlement date of the original trade, but has a different report time, and it could arrive minutes or days after the the original trade. Therefore, we use the report times of cancellation entries as their timestamp when bringing cancellation entries to our main data.

[31]Appendix Table D.3 shows our results if we also include offering amount and offering amount of the other bonds to the relevant specifications. Our results are robust to these alternative specifications.

other bonds instead of offering amount of the other bonds) to proxy for search frictions. Our results for the number of cancellations are unchanged, which are entirely consistent with our conjecture of cancellations as a proxy for search frictions as well as our results with the offering amount variable. In terms of the number of cancellations in other bonds, our findings largely confirm our previous findings except for the relation between average number of cancellations of other bonds and the price dispersion liquidity measure. Overall, our findings in Panel B of Table 5 mostly confirm our previous results.

Table 5: Proxying for search frictions, trade cancellations

This table presents the relation between trade cancellations and the liquidity in corporate bonds. Panel A presents the results under the baseline model, where bond-level controls except OA_j & OA_{-j} (offering amount & offering amount, other bonds) and year-week fixed effects collectively represent the predictors used in Table 3 except offering amount. Panel B presents the results under the full model, where bond-level controls except OA_j & OA_{-j} (offering amount & offering amount, other bonds) and market-level controls collectively represent the predictors used in Table 4 except offering amount and offering amount of the other bonds. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

Panel A: Baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.303***	-0.040***	-0.320***
	(68.23)	(-19.41)	(-53.31)
Bond-level controls except OA_j & OA_{-j}	Y	Y	Y
Year-week FE	Y	Y	Y
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.366	0.215	0.082

Panel B: Full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.305***	-0.027***	-0.306***
	(66.94)	(-12.49)	(-51.65)
Average number of cancellations$_{-j}$	-0.384***	0.579***	0.786***
	(-9.23)	(22.55)	(20.95)
Bond-level controls except OA_j & OA_{-j}	Y	Y	Y
Market-level controls	Y	Y	Y
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.346	0.201	0.074

6 Liquidity across asset classes

A key feature of our theoretical model is its characterization of liquidity in the cross-section of assets. Therefore, our empirical analyses mainly focus on testing the relations that come out of our model in the cross-section of assets, with the corporate bond market as our laboratory. However, an interesting question arises when we consider the availability of assets from different asset classes on the same entity. How does the liquidity of an asset from a different class on the same entity relate to the liquidity of the current asset?

6.1 Theory

An investor could satisfy her hedging need via an asset different than a bond, if such an asset on the same entity is available for trade. A credit default swap (CDS) is one such asset. A CDS is a derivative contract on a reference bond that requires periodic premium payments and pays off in the event of a default. Buying a CDS contract on the bond is implicitly equivalent to a short position on the bond, and vice versa. Therefore, the availability of CDS expands the bank's choice set when it comes to satisfying its hedging need, and makes it easier to take position on an entity when it is difficult to find the bond of that entity.

We extend a two-asset version of our baseline model by taking as given the liquidity of one asset (CDS) and endogenizing the liquidity of the other asset (bond). Namely, we let investors choose their λ_b upon birth before their time-varying states are realized. Thus, an investor chooses her λ_b to maximize

$$\mathbb{E}\left[V\left(\eta, \boldsymbol{a}, \lambda_b \,|\, \lambda_b^*, \lambda_c\right)\right] - \chi\left(\lambda_b\right),$$

where λ_b^* denotes the other agents' contact rate in the bond market, λ_c denotes the exogenous contact rate in the CDS market, the expectation is taken considering that $(\eta, \boldsymbol{a})$ realizes from the economy's steady state distribution, and $\chi(\cdot)$ is assumed to be strictly increasing and twice differentiable. Using Theorem 1 and Corollary 3, the problem can be written as

$$\max_{\lambda_b \geq 0} \frac{\gamma_B \sigma^2 \sigma_\eta^2}{2r + \lambda_c + \lambda_b}\left(-1 + \frac{\lambda_c + \lambda_b}{2\left(\lambda_c + \lambda_b^*\right)} - \frac{r}{\lambda_c + \lambda_b}\right) - \chi\left(\lambda_b\right). \tag{36}$$

By using the solution of this optimization problem, we obtain the following proposition.

Proposition 8. *There exists a unique symmetric equilibrium. Assume pure variable costs with constant marginal cost:* $\chi(\lambda_b) = \chi_0 \lambda_b$. *Then, the equilibrium contact rate in the bond market is*

$$\lambda_b^* = \sqrt{\frac{\gamma_B}{\chi_0}} \sigma \sigma_\eta - \lambda_c$$

in the limit as r *approaches zero.*

Proposition 8 implies that when the marginal cost is constant and there is no fixed cost, investors endogenously choose their total contact rate, $\lambda_b^* + \lambda_c$, to be $\sqrt{\frac{\gamma_B}{\chi_0}}\sigma\sigma_\eta$ in equilibrium. That is, they want to trade faster if they are more risk averse, if there is more aggregate or idiosyncratic risk, and if choosing a higher contact rate is less costly. However, they do not care if they satisfy their trading need by trading bond or trading CDS. Then, if it is easier to trade CDS contracts (larger λ_c), investors endogenously choose a lower contact rate in the bond market (smaller λ_b^*). In the next subsection, we empirically test this prediction of our extended model.

6.2 Empirics

We next bring the insights from Proposition 8 to our empirical tests in the cross section of US corporate bonds. Proposition 8 predicts that if a CDS is easier to trade, the bond on the same entity must be difficult to trade. Thus, we use the ease of trading CDS as a proxy for (the inverse of) λ_j. Specifically, we utilize the CDS data available from IHS Markit. We use the composite CDS depth of five-year contracts as our measure of how easy to trade a CDS contract.[32] For any given day, the CDS depth variable represents the number of unique CDS quote contributors on that day.[33]

Table 6 presents our findings, both for the baseline and the full model in Panels A and B, respectively. Both Panels A and B confirm that as the CDS depth increases, trade volume on the bond decreases while price dispersion on the bond and price impact on the bond increase. Thus, our empirical findings for all three bond liquidity measures are exactly as predicted by our theoretical model. Our findings are also consistent with Oehmke and Zawadowski (2017), who argue that bond and CDS markets are alternative trading venues for hedging and speculation.

Another interesting result of Proposition 8 and Table 6 relates to our discussion of search friction proxies. Namely, our theory and empirics imply that one may consider CDS depth as a search friction proxy for the bond. The idea is that, as it gets easier to find a CDS counterparty, investors may look for the CDS on the bond's issuer to satisfy their hedging need, which in turn increases the difficulty of finding the bond.[34] Thus, consistently, we observe in Table 6

[32]CDS depth is a commonly used measure of CDS liquidity in the literature (see, for example, Qiu and Yu (2012), Feldhütter, Hotchkiss, and Karakaş (2016), Lee, Naranjo, and Velioglu (2018), among others).

[33]Note that the minimum non-missing value this variable can take in a given day is two, since Markit requires at least two contributors for each quote.

[34]It could be the other way around as well, that is, the ease of finding the bond might reduce the trading activity in the CDS. We do not make a claim on the direction, but just on the correlation. To see this, note that one could swap the subscripts of b and c in Subsection 6.1, and the results of Proposition 8 would still hold.

that offering amount and CDS depth have the exact opposite signs as the two predictors of endogenous liquidity measures. While offering amount is a positive proxy for λ_j, CDS depth serves a negative proxy.

Overall, our findings in this cross-asset-class analysis lend significant support to our theoretical model, and point to usefulness of the CDS depth variable to understand bond-specific search frictions.

Table 6: Liquidity across assets, bonds and CDS

This table presents the relation between CDS depth of a bond's issuer and liquidity of the bond. Panel A presents the results under the baseline model, where offering amount, bond-level controls except OA_j & OA_{-j} (offering amount & offering amount, other bonds), and year-week fixed effects collectively represent the predictors used in Table 3. Panel B presents the results under the full model, where offering amount, bond-level controls except OA_j (offering amount), and market-level controls collectively represent the predictors used in Table 4. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

Panel A: Baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average CDS depth$_j$	-0.151***	0.095***	0.255***
	(-15.61)	(16.19)	(19.10)
Offering amount	0.898***	-0.154***	-0.969***
	(116.43)	(-33.62)	(-96.62)
Bond-level controls except OA_j & OA_{-j}	Y	Y	Y
Year-week FE	Y	Y	Y
Observations	2,725,599	2,725,599	2,725,599
Adjusted R^2	0.519	0.226	0.197

Panel B: Full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average CDS depth$_j$	-0.148***	0.096***	0.256***
	(-15.03)	(16.24)	(19.77)
Offering amount	0.896***	-0.154***	-0.966***
	(116.36)	(-33.75)	(-96.91)
Bond-level controls except OA_j	Y	Y	Y
Market-level controls	Y	Y	Y
Observations	2,725,599	2,725,599	2,725,599
Adjusted R^2	0.501	0.212	0.192

7 Conclusion

We develop a search-theoretic model to study the impact of heterogeneity in asset characteristics on their endogenous liquidity differentials. Thanks to the tractability of our model, we derive natural theoretical counterparts for various measures of market liquidity easily calculated from transaction-level data. We find that the alleviation of search frictions of one asset may lead to opposite observations regarding other assets' liquidity depending on which liquidity measure is used. Based on data from the US corporate bond market, our empirical tests indicate significant support for the search-and-bargaining framework, which uncovers the determinants of endogenous liquidity differentials across OTC assets.

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A Variable definitions

Variable	Type	Description	Source
Dependent variables			
Trade volume ($\mathcal{V}_j$)	\$mm	Weekly total of trade sizes (in par value amount) of the bond, in million dollars.	TRACE
Price dispersion (σ_{P_j})	Decimal	The square root of weekly second moment of demeaned bond prices. Demeaned prices for each bond-day are calculated as the difference between bond price and daily mean bond price. We follow Jankowitsch, Nashikkar, and Subrahmanyam (2011) and Feldhütter (2012) in our definition of price dispersion.	TRACE
Price impact (δ_j)	Decimal	The ratio of weekly price dispersion to square root of weekly second moment of trade sizes of the bond, multiplied by two.	TRACE
Predictors			
Offering amount$_j$ (λ_j)	\$bn	Offering amount (par value) of the bond j, in billion dollars.	FISD
Offering amount$_{-j}$ (λ_{-j})	\$bn	Total offering amount (par value) of the other bonds, in billion dollars. Calculated as the summation of the offering amount of the unique bonds that have been traded other than bond j, over the 12 weeks prior to the beginning of current week.	FISD, TRACE
Volatility beta (ψ_j)	Decimal	The sensitivity of weekly volatility of bond returns to implied volatility. For each bond, we first calculate the weekly standard deviation of daily returns, where daily bond returns are calculated based on trade size weighted average of bond prices (clean price plus accrued interest, and coupons if any). Then, for each bond-quarter, we regress weekly volatility of bond returns on implied volatility. We further take the absolute value of the coefficient estimate of this regression, to more accurately test the theoretical model. This volatility beta is the sensitivity measure of bond volatility to systematic volatility. For any given bond-week, we use the volatility beta from the most recent quarter prior to the beginning of week.	FISD, OptionMetrics, TRACE
Average number of trades	Decimal	The average of the weekly number of trades of bond j, calculated over the trailing 12 weeks prior to the beginning of current week. If we do not observe any trade on a given day, we assume that the number of trades is zero for that day.	TRACE
GDP forecast dispersion (σ_η)	Decimal	The dispersion measure D3 for one quarter ahead real GDP level from the Survey of Professional Forecasters. It is the logarithmic difference between the 75$^{\text{th}}$ percentile and the 25$^{\text{th}}$ percentile of the one quarter ahead forecasts of real GDP level, multiplied by 100.	FED
Treasury rate (r)	Pct.	One-month Treasury bill rate. We take the average of daily Treasury rates to have Treasury rates at weekly frequency.	FED
Implied market volatility (σ)	Decimal	The implied volatility of Chicago Board of Options Exchange (CBOE) S&P 500 index European call option. For any given day, we use the implied volatility of the European call option that is closest to being at-the-money, and then with days to expiry that is closest to 91 days ($\sim$3 months). We take the average of daily implied volatility levels to have implied volatility levels at weekly frequency.	OptionMetrics
Average number of cancellations$_j$	Decimal	The average of the weekly number of trade cancellations of bond j, calculated over the trailing 12 weeks prior to the beginning of current week. If we do not observe any trade cancellation on a given day, we assume that the number of trade cancellations is zero for that day.	TRACE
Average number of cancellations$_{-j}$	Decimal	The average of the average number of cancellations of the bonds other than bond j.	TRACE
Average CDS depth$_j$	Decimal	The average of the weekly summation of composite CDS depth of bond j's issuer, calculated over the trailing 12 weeks prior to the beginning of current week. CDS depth is the number of distinct CDS dealers providing quotes for the issuer's CDS on a given day. If we do not observe CDS depth on a given day, we assume that CDS depth is zero for that day.	Markit

B Optimization

In this appendix, we study the stochastic control problem faced by an individual bank with the reduced-form linear-quadratic utility, $u(\eta, \mathbf{a})$, in the search-theoretic equilibrium of Section 2. Following closely the steps in Duffie, Gârleanu, and Pedersen (2005), Vayanos and Weill (2008), and Üslü (2019), we define the bank's problem and provide HJB equations and an optimality verification argument.

B.1 Bank's problem

We fix a probability space $(\Omega, \mathcal{F}, \Pr)$ and a filtration $\{\mathcal{F}_t, t \geq 0\}$ of sub-σ-algebras satisfying the usual conditions (see Protter, 2004). Bank i can be of either one of the $J+1$-dimensional continuum of types denoted by $(\eta, \mathbf{a}) \in \mathcal{T} \equiv \mathbb{R} \times \mathbb{R}^J$. Shocks to the hedging need type η are governed by a diffusion process B^i with constant volatility σ_η. The arrival times of potential counterparties are counted by J independent adapted counting processes denoted by N_j^i with constant intensity $2\lambda_j$ for all $j \in \mathcal{J}$. The details of these independent diffusion and counting processes are as described in Section 2.

Starting with initial type $(\eta_0, \mathbf{a}_0)$ and initial wealth W_0, bank i chooses a feasible trading strategy $\{\mathbf{a}_t\}_{t\in[0,\infty)} = \{(a_{jt}, a_{-jt})\}_{t\in[0,\infty)}$ and an adapted consumption and wealth process $\{(c_t, W_t)\}_{t\in[0,\infty)}$ subject to the following feasibility conditions. First, the portfolio $\mathbf{a}_t$ must remain constant during the inter- and intra-arrival times of the counting processes N_j^i, $j \in \mathcal{J}$. Second, when the bank is in state $(\eta, a_j, a_{-j}) \in \mathcal{T}$ and when the process N_{jt}^i jumps, the bank transitions into the state $(\eta, a_j + q_{jt}[(\eta, \mathbf{a}), (\eta', \mathbf{a}')]) \in \mathcal{T}$, where the trade quantity, $q_{jt}[(\eta, \mathbf{a}), (\eta', \mathbf{a}')]$, is bargained with the countarparty of type $(\eta', \mathbf{a}')$ who is drawn according to the joint cdf, $\Phi_t(\eta', \mathbf{a}')$, of hedging need types and asset positions.[35]

First, we describe a bank's indirect utility at time t from its traders' remaining lifetime consumption. As is typical, the arguments of this indirect utility function are the bank's current wealth W_t, its current type $(\eta_t, \mathbf{a}_t)$, and time t. Mathematically, the indirect utility is

$$J(W_t, \eta_t, \mathbf{a}_t, t) = \sup_{C,\mathbf{a}} \mathbb{E}_t \int_0^\infty e^{-rs} dC_{t+s} \tag{37}$$

[35]Because, in our reduced-form environment, banks have effectively quasi-linear preferences with the effective utility being linear in consumption and linear-quadratic in asset positions, terms of trade are independent of wealth levels, as will be clear shortly.

subject to

$$dW_t = rW_t dt - dC_t + u(\eta_t, \mathbf{a}_t)dt - \sum_{j=1}^{J} P_{jt}\left[(\eta_{t-}, \mathbf{a}_{t-}), (\eta_t', \mathbf{a}_t')\right] da_{jt}, \tag{38}$$

$$da_{jt} = \begin{cases} q_{jt}\left[(\eta_{t-}, \mathbf{a}_{t-}), (\eta_t', \mathbf{a}_t')\right] & \text{if } (\eta_t', \mathbf{a}_t') \text{ is contacted in market } j \\ 0 & \text{if no contact in market } j, \end{cases}$$

where

$$\begin{aligned} &\{q_{jt}\left[(\eta, \mathbf{a}), (\eta', \mathbf{a}')\right], P_{jt}\left[(\eta, \mathbf{a}), (\eta', \mathbf{a}')\right]\} = \\ &\arg\max_{q,P} \Big\{ \left[J(W - qP, \eta, a_j + q, a_{-j}, t) - J(W, \eta, \mathbf{a}, t)\right]^{\frac{1}{2}} \\ &\qquad\qquad \left[J(W' + qP, \eta', a_j' - q, a_{-j}', t) - J(W', \eta', \mathbf{a}', t)\right]^{\frac{1}{2}} \Big\}, \end{aligned}$$

subject to

$$\begin{aligned} &J(W - qP, \eta, a_j + q, a_{-j}, t) \geq J(W, \eta, \mathbf{a}, t), \\ &J(W' + qP, \eta', a_j' - q, a_{-j}', t) \geq J(W', \eta', \mathbf{a}', t). \end{aligned}$$

where $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot \mid \mathcal{F}_t]$ is the conditional expectation with respect to the filtration $\mathcal{F}$, $\{C_t\}_{t\in[0,\infty)}$ is a cumulative consumption process, $\{(\eta_t, \mathbf{a}_t)\}_{t\in[0,\infty)}$ is a $\mathcal{T}$-valued type process induced by the feasible trading strategy $\{\mathbf{a}_t\}_{t\in[0,\infty)}$, and the benefit $u(\eta_t, \mathbf{a}_t)$ has a similar holding benefit/cost interpretation as in Üslü (2019). The main difference is that our specification is for a multi-asset environment while Üslü's specification has a single asset. Accordingly, Üslü's specification is a special case of ours when the number of assets is equal to one, $m_1 = \delta$, $\eta = 0$, and $r\gamma_B\sigma^2\psi_1^2 = \kappa$.

Note that (37) and (38) imply that the indirect utility is linear in wealth, i.e., $J(W_t, \eta_t, \mathbf{a}_t, t) = W_t + V(\eta_t, \mathbf{a}_t, t)$, where

$$V(\eta_t, \mathbf{a}_t, t) = \sup_{\mathbf{a}} \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)} u(\eta_s, \mathbf{a}_s)\, ds - e^{-r(s-t)} \sum_{j=1}^{J} P_{js}\left[(\eta_{s-}, \mathbf{a}_{s-}), (\eta_s', \mathbf{a}_s')\right] da_{js}\right]. \tag{39}$$

Finally, to guarantee the global optimality of the trading strategy induced by the martingale (39), we impose the transversality condition

$$\lim_{t\to\infty} e^{-rt} V(\eta, \mathbf{a}, t) = 0 \tag{40}$$

for all $(\eta, \mathbf{a}) \in \mathcal{T}$ and the condition

$$\mathbb{E}\left[\int_0^T \left(e^{-rs} V(\eta_s, \mathbf{a}_s, s)\right)^2 ds\right] < \infty \tag{41}$$

for any $T > 0$, for any initial bank type $(\eta_0, \mathbf{a}_0)$, any feasible trading strategy $\{\mathbf{a}_t\}_{t\in[0,\infty)}$, and the associated type process $\{(\eta_t, \mathbf{a}_t)\}_{t\in[0,\infty)}$. These conditions will allow us to complete the usual verification argument for stochastic control.

B.2 HJB equations

To further characterize V, q_j, and P_j, we focus on a particular bank i and a particular time t and let τ_j be an exponential random variable that represents the next (stopping) time at which bank i meets another bank in market j for $j \in \mathcal{J}$, and let $\tau = \min\{\tau_1, \tau_2, ..., \tau_J\}$. Then,

$$\begin{aligned}
V(\eta_t, \mathbf{a}_t, t) = \mathbb{E}_t \Bigg[& \int_t^\tau e^{-r(s-t)} u(\delta_s, a_s)\, ds \\
& + \sum_{j=1}^J e^{-r(\tau_j - t)} \mathbb{I}_{\{\tau_j = \tau\}} \int_{\mathbb{R}^J}\int_{\mathbb{R}} \Big\{ V(\eta_{\tau_j}, a_{j\tau_j} + q_{j\tau_j}\left[(\eta_{\tau_j}, \mathbf{a}_{\tau_j}), (\eta', \mathbf{a}')\right], a_{-j\tau_j}) \\
& - q_{j\tau_j}\left[(\eta_{\tau_j}, \mathbf{a}_{\tau_j}), (\eta', \mathbf{a}')\right] P_{j\tau_j}\left[(\eta_{\tau_j}, \mathbf{a}_{\tau_j}), (\eta', \mathbf{a}')\right] \Big\} \Phi_{\tau_j}(d\eta', d\mathbf{a}') \Bigg]. \quad (42)
\end{aligned}$$

Assuming sufficient regularity for Ito's lemma for Lévy processes to hold, we differentiate the both sides of (42) with respect to time argument t and suppress it:

$$\begin{aligned}
\dot{V}(\eta, \mathbf{a}) = {} & rV(\eta, \mathbf{a}) - u(\eta, \mathbf{a}) - \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta, \mathbf{a}) \\
& - \sum_{j=1}^J 2\lambda_j \int_{\mathbb{R}^J}\int_{\mathbb{R}} \{ V(\eta, a_j + q_j[(\eta, \mathbf{a}), (\eta', \mathbf{a}')], a_{-j}) - V(\eta, \mathbf{a}) \\
& - q_j[(\eta, \mathbf{a}), (\eta', \mathbf{a}')] P_j[(\eta, \mathbf{a}), (\eta', \mathbf{a}')] \} \Phi(d\eta', d\mathbf{a}'). \quad (43)
\end{aligned}$$

A stationary value function must satisfy $\dot{V}(\eta, \mathbf{a}) = 0$. Hence, after using the price implied by the Nash bargaining procedure, (43) implies the HJB equation (10) of Section 3.

B.3 Optimality verification

In order to verify the sufficiency of the HJB equation (10) for individual optimality, we consider any initial bank type $(\eta_0, \mathbf{a}_0)$, any feasible trading strategy $\{\mathbf{a}_t\}_{t\in[0,\infty)}$, and the associated type process $\{(\eta_t, \mathbf{a}_t)\}_{t\in[0,\infty)}$. Without loss of generality, we assume that the wealth process is $W_t = 0$ for all $t \geq 0$. Then, the resulting cumulative consumption process $\{C_t^{\mathbf{a}}\}_{t\in[0,\infty)}$ satisfies

$$dC_t^{\mathbf{a}} = u(\eta_t, \mathbf{a}_t)dt - \sum_{j=1}^J P_{jt}\left[(\eta_{t-}, \mathbf{a}_{t-}), (\eta_t', \mathbf{a}_t')\right] da_{jt}. \quad (44)$$

At any time $T > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-rs} dC_s^{\mathbf{a}} + e^{-rT} V(\eta_T, \mathbf{a}_T)\right] \\
&= \mathbb{E}\left[\int_0^T e^{-rs} dC_s^{\mathbf{a}} + V(\eta_0, \mathbf{a}_0) + \int_0^T d\left(e^{-rs} V(\eta_s, \mathbf{a}_s)\right)\right] \\
&= \mathbb{E}\left[V(\eta_0, \mathbf{a}_0) + \int_0^T e^{-rs} dC_s^{\mathbf{a}} + \int_0^T \left(-re^{-rs} V(\eta_s, \mathbf{a}_s)\right) ds + \int_0^T e^{-rs} d\left(V(\eta_s, \mathbf{a}_s)\right)\right] \\
&= \mathbb{E}\Bigg[V(\eta_0, \mathbf{a}_0) + \int_0^T e^{-rs}\Bigg(dC_s^{\mathbf{a}} - rV(\eta_s, \mathbf{a}_s) + \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta_s, \mathbf{a}_s) \\
&\qquad + \sum_{j=1}^{J} \left(V(\eta_s, a_{js} + q_{js}[(\eta_{s-}, \mathbf{a}_{s-}), (\eta_s', \mathbf{a}_s')], a_{-js}) - V(\eta_s, \mathbf{a}_s)\right) dN_{js}\Bigg)\Bigg], \quad (45)
\end{aligned}$$

where N_{js}s are counting processes that govern the arrivals of potential counterparties in markets for $j \in \mathcal{J}$. Note that any side payment to effect a transaction at an arrival time of N^j is reflected by $C^{\mathbf{a}}$ according to (44).

We next calculate the stochastic integrals containing the counting processes. The condition (41) implies that

$$\int_0^T |V(\eta_s, \mathbf{a}_s) - V(\eta_s, a_{js-}, a_{-js})| \, ds \leq \sup_{s, s' \in [0,T]} |V(\eta_{s'}, \mathbf{a}_{s'}) - V(\eta_s, \mathbf{a}_s)| \, T < \infty.$$

Using Corollary $C4$ of Brémaud (1981, p. 235),

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-rs}\left(V(\eta_s, a_{js} + q_{js}[(\eta_{s-}, \mathbf{a}_{s-}), (\eta_s', \mathbf{a}_s')], a_{-js}) - V(\eta_s, \mathbf{a}_s)\right) dN_{js}\right] \\
&= \mathbb{E}\left[\int_0^T e^{-rs}\left\{2\lambda_j \int_{\mathbb{R}^J}\int_{\mathbb{R}} \left(V(\eta_s, a_{js} + q_{js}[(\eta_{s-}, \mathbf{a}_{s-}), (\eta_s', \mathbf{a}_s')], a_{-js}) - V(\eta_s, \mathbf{a}_s)\right) \Phi_s(d\eta_s', d\mathbf{a}_s')\right\} ds\right].
\end{aligned}$$

Combining this equality with (45),

$$\mathbb{E}\left[\int_0^T e^{-rs}dC_s^{\mathbf{a}} + e^{-rT}V(\eta_T, \mathbf{a}_T)\right] = \mathbb{E}\left[V(\eta_0, \mathbf{a}_0) + \int_0^T e^{-rs}dC_s^{\mathbf{a}}\right.$$

$$+\int_0^T e^{-rs}\left(-rV(\eta_s, \mathbf{a}_s) + \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta_s, \mathbf{a}_s)\right.$$

$$\left.\left.+\sum_{j=1}^J 2\lambda_j \int_{\mathbb{R}^J}\int_{\mathbb{R}} \left(V\left(\eta_s, a_{js} + q_{js}[(\eta_{s-}, \mathbf{a}_{s-}),(\eta_s', \mathbf{a}_s')], a_{-js}\right) - V(\eta_s, \mathbf{a}_s)\right)\Phi_s(d\eta_s', d\mathbf{a}_s')\right) ds\right]$$

$$\leq \mathbb{E}\left[V(\eta_0, \mathbf{a}_0) + \sup_C \left\{\int_0^T e^{-rs}dC_s + \int_0^T e^{-rs}\left(-rV(\eta_s, \mathbf{a}_s) + \frac{1}{2}\sigma_\eta^2 V_{\eta\eta}(\eta_s, \mathbf{a}_s)\right.\right.\right.$$

$$\left.\left.\left.+\sum_{j=1}^J 2\lambda_j \int_{\mathbb{R}^J}\int_{\mathbb{R}} \left(V\left(\eta_s, a_{js} + q_{js}[(\eta_{s-}, \mathbf{a}_{s-}),(\eta_s', \mathbf{a}_s')], a_{-js}\right) - V(\eta_s, \mathbf{a}_s)\right)\Phi_s(d\eta_s', d\mathbf{a}_s')\right) ds\right\}\right]$$

$$= V(\eta_0, \mathbf{a}_0).$$

This, in turn, means that

$$V(\eta_0, \mathbf{a}_0) \geq \mathbb{E}\left[\int_0^{\tau^n} e^{-rt}dC_t^{\mathbf{a}}\right] + \mathbb{E}\left[e^{-r\tau^n}V(\eta_{\tau^n}, \mathbf{a}_{\tau^n})\right],$$

at any future meeting date τ^n, $n \in \mathbb{N}$. Then, we let $n \to \infty$ and use the transversality condition (40), which allow us to obtain $V(\eta_0, \mathbf{a}_0) \geq J(C^{\mathbf{a}})$. Since $V(\eta_0, \mathbf{a}_0) = J(C^*)$, where C^* is the consumption process associated with the candidate equilibrium strategy, optimality has been verified.

C Proofs

C.1 Proof of Theorem 1

Conjecture

$$V(\eta, \mathbf{a}) = D + \mathbf{E}^T\mathbf{a} + F\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) + M\eta \tag{46}$$

for D, $\mathbf{E}$, F, and M to be determined. Take the derivative with respect to $\mathbf{a}$:

$$\frac{\partial V}{\partial \mathbf{a}}(\eta, \mathbf{a}) = \mathbf{E} + 2F(\eta\boldsymbol{\psi} + \Psi\mathbf{a}).$$

The marginal valuation for asset j is, then,

$$V^{(j)}(\eta, \mathbf{a}) = E_j + 2F\psi_j \left(\eta + \sum_{k=1}^{J} \psi_k a_k\right).$$

Using the FOC (7) for Nash bargaining,

$$q_j\left[(\eta, \mathbf{a}), (\eta', \mathbf{a}')\right] = \frac{\eta' - \eta + \sum_{k=1}^{J} \psi_k (a'_k - a_k)}{2\psi_j}. \tag{47}$$

(46) implies

$$\begin{aligned}
&V(\eta, a_j + q_j(\mu, \mu'), a_{-j}) - V(\eta, a_j, a_{-j}) + V\left(\eta', a'_j - q_j(\mu, \mu'), a'_{-j}\right) - V\left(\eta', a'_j, a'_{-j}\right) \\
&= -2qF\psi_j \left[\eta' - \eta - \psi_j q + \sum_{k=1}^{J} \psi_k (a'_k - a_k)\right].
\end{aligned}$$

Using (47),

$$\begin{aligned}
&V(\eta, a_j + q_j(\mu, \mu'), a_{-j}) - V(\eta, a_j, a_{-j}) + V\left(\eta', a'_j - q_j(\mu, \mu'), a'_{-j}\right) - V\left(\eta', a'_j, a'_{-j}\right) \\
&= -\frac{1}{2}F\left[\eta' - \eta + \sum_{k=1}^{J} \psi_k (a'_k - a_k)\right]^2 \\
&= -\frac{1}{2}F\left[\eta' + \boldsymbol{\psi}^T\mathbf{a}' - \left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right)\right]^2 \\
&= -\frac{1}{2}F\left[\left(\eta' + \boldsymbol{\psi}^T\mathbf{a}'\right)^2 - 2\left(\eta' + \boldsymbol{\psi}^T\mathbf{a}'\right)\left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right) + \left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right)^2\right] \\
&= -\frac{1}{2}F\left[(\eta')^2 + 2\eta'\boldsymbol{\psi}^T\mathbf{a}' + (\mathbf{a}')^T\Psi\mathbf{a}' - 2\left(\eta' + \boldsymbol{\psi}^T\mathbf{a}'\right)\left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right) + \eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right].
\end{aligned}$$

Then, we are ready to set up the equation that will determine the undetermined coefficients using the HJB (11):

$$\begin{aligned}
&r\left[D + \mathbf{E}^T\mathbf{a} + F\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) + M\eta\right] = \mathbf{m}^T\mathbf{a} - \frac{1}{2}r\gamma_B\sigma^2\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) + \sigma_\eta^2 F \\
&- \frac{1}{2}\lambda F \int_{\mathbb{R}}\int_{\mathbb{R}^J} \left[(\eta')^2 + 2\eta'\boldsymbol{\psi}^T\mathbf{a}' + (\mathbf{a}')^T\Psi\mathbf{a}' - 2\left(\eta' + \boldsymbol{\psi}^T\mathbf{a}'\right)\left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right)\right. \\
&\qquad\qquad \left. + \eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right]\Phi(d\mathbf{a}', d\eta').
\end{aligned}$$

Letting $\mathbb{E}[\cdot]$ denote the cross-sectional mean and noticing that $\mathbb{E}[\eta'] = 0$,

$$\begin{aligned}
&r\left[D + \mathbf{E}^T\mathbf{a} + F\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) + M\eta\right] = \mathbf{m}^T\mathbf{a} - \frac{1}{2}r\gamma_B\sigma^2\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right) + \sigma_\eta^2 F \\
&- \frac{1}{2}\lambda F\left\{\mathbb{E}\left[(\eta')^2 + 2\eta'\boldsymbol{\psi}^T\mathbf{a}' + (\mathbf{a}')^T\Psi\mathbf{a}'\right] - 2\left(\eta + \boldsymbol{\psi}^T\mathbf{a}\right)\boldsymbol{\psi}^T\mathbb{E}[\mathbf{a}'] + \eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right\}.
\end{aligned}$$

Thus, the coefficients solve

$$
\begin{aligned}
rD &= \sigma_\eta^2 F - \frac{1}{2}\lambda F \mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}'\right] \\
r\mathbf{E} &= \mathbf{m} + \lambda F \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right] \boldsymbol{\psi} \\
rF &= -\frac{1}{2} r \gamma_B \sigma^2 - \frac{1}{2}\lambda F \\
rM &= \lambda F \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
D &= \frac{\gamma_B \sigma^2}{2r+\lambda}\left(-\sigma_\eta^2 + \frac{\lambda}{2}\mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}'\right]\right) \\
\mathbf{E} &= \frac{1}{r}\mathbf{m} - \lambda \frac{\gamma_B \sigma^2}{2r+\lambda} \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right] \boldsymbol{\psi} \\
F &= -\frac{r\gamma_B \sigma^2}{2r+\lambda} \\
M &= -\lambda \frac{\gamma_B \sigma^2}{2r+\lambda} \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right].
\end{aligned}
$$

Putting together,

$$
\begin{aligned}
V(\eta, \mathbf{a}) &= \frac{\gamma_B \sigma^2}{2r+\lambda}\left(-\sigma_\eta^2 + \frac{\lambda}{2}\mathbb{E}\left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}'\right]\right) \\
&+ \left(\frac{1}{r}\mathbf{m} - \lambda \frac{\gamma_B \sigma^2}{2r+\lambda} \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right] \boldsymbol{\psi}\right)^T \mathbf{a} - \frac{r\gamma_B \sigma^2}{2r+\lambda}\left(\eta^2 + 2\eta \boldsymbol{\psi}^T \mathbf{a} + \mathbf{a}^T \Psi \mathbf{a}\right) - \lambda \frac{\gamma_B \sigma^2}{2r+\lambda} \boldsymbol{\psi}^T \mathbb{E}\left[\mathbf{a}'\right] \eta,
\end{aligned}
$$

which is Equation (14) of Theorem 1. By taking the derivative with respect to $\mathbf{a}$, one obtains (15). (47) is equal to (16). Substituting into the Nash bargaining price (8), one obtains (17). Since $V(\eta, \cdot)$ stated above is negative definite for all $\eta \in \mathbb{R}$, (16) and (17) constitute the unique solution to the Nash bargaining problem. By construction, $V(\eta, \mathbf{a})$ given by (14) is the unique quadratic solution to the HJB equation (11). Finally, it is a matter of algebra to show that the value function we have constructed satisfies the transversality conditions (40) and (41).

C.2 Proof of Lemma 2

The dynamics of the composite type θ for a given bank i is

$$
d\theta_t = \sigma_\eta dB_t^i + \sum_{j=1}^{J}\left[\theta_{t-} + q_j\left(\theta_{t-}, \theta_t'\right)\psi_j\right] dN_t^j - \sum_{j=1}^{J}\theta_{t-} dN_t^j, \tag{48}
$$

where N^j is an independent Poisson process with jump intensity $2\lambda_j$ for $j \in \mathcal{J}$ and θ_t', the counterparty's composite type, is a random draw from the pdf $g(t, \theta')$.

Define

$$H(t,\theta_0,\theta) \equiv \Pr[\theta_t \le \theta \,|\, \theta_0]$$

and

$$h(t,\theta_0,\theta) \equiv \frac{\partial}{\partial\theta} H(t,\theta_0,\theta).$$

In equilibrium, the dynamics of the cross-sectional pdf of composite types, $g(t,\theta)$, is generated by (48):

$$g(t+s,\theta) = \int_{\mathbb{R}} g(t,\xi)\, h(s,\xi,\theta)\, d\xi.$$

It follows that for any $s > 0$,

$$\frac{1}{s}\left[g(t+s,\theta) - g(t,\theta)\right] = \frac{1}{s}\int_{\mathbb{R}} \left[g(t,\xi) - g(t,\theta)\right] h(s,\xi,\theta)\, d\xi. \tag{49}$$

Taking the limit in (49) as $s \to 0$ and applying the Ito's lemma for Lévy processes on the RHS leads to

$$\frac{\partial g(t,\theta)}{\partial t} = \frac{1}{2}\sigma_\eta^2 \frac{\partial^2 g(t,\theta)}{\partial\theta^2} + \sum_{j=1}^{J} 2\lambda_j \left[\frac{\partial}{\partial\theta}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{I}_{\left\{q_j\left(\tilde{\theta},\theta'\right)\psi_j \le \theta - \tilde{\theta}\right\}} g(t,\theta')\, g\left(t,\tilde{\theta}\right) d\theta' d\tilde{\theta}\right] - \sum_{j=1}^{J} 2\lambda_j g(t,\theta).$$

This second-order partial differential equation (PDE) satisfied by the densities at dates $t > 0$ generated by Lévy processes is called the *Kolmogorov forward equation.*[36]

Since (16) provides us with explicit expression for trade sizes, we can get rid of indicator function inside the integral:

$$\frac{\partial g(t,\theta)}{\partial t} = \frac{1}{2}\sigma_\eta^2 \frac{\partial^2 g(t,\theta)}{\partial\theta^2} + \sum_{j=1}^{J} 2\lambda_j \left[\frac{\partial}{\partial\theta}\int_{\mathbb{R}}\int_{-\infty}^{2\theta-\theta'} g(t,\theta')\, g\left(t,\tilde{\theta}\right) d\theta' d\tilde{\theta}\right] - \sum_{j=1}^{J} 2\lambda_j g(t,\theta).$$

One can calculate the derivate inside the square bracket using Leibniz rule:

$$\frac{\partial g(t,\theta)}{\partial t} = \frac{1}{2}\sigma_\eta^2 \frac{\partial^2 g(t,\theta)}{\partial\theta^2} + \sum_{j=1}^{J} 4\lambda_j \left[\int_{\mathbb{R}} g(t,\theta')\, g(t,2\theta-\theta')\, d\theta'\right] - \sum_{j=1}^{J} 2\lambda_j g(t,\theta).$$

By defining $\lambda \equiv \sum_{j=1}^{J} \lambda_j$ and suppressing ts, one obtains Equation (18) of the lemma. Equation (19) obtains because $g(\theta)$ is a pdf. Equation (20) is implied by the market-clearing conditions and the fact that η does not have a drift.

[36] For a reference, see Guttorp (1995, p. 133) or Stokey (2009, p. 50).

C.3 Proof of Theorem 2

We first calculate the characteristic function of the second term on the RHS of (18):

$$\int_{\mathbb{R}} 4\lambda \left[\int_{\mathbb{R}} g(\theta') g(2\theta - \theta') d\theta' \right] e^{iz\theta} d\theta = 4\lambda \int_{\mathbb{R}} \left[g(\theta') \int_{\mathbb{R}} g(2\theta - \theta') e^{iz\theta} d\theta \right] d\theta'$$

$$= 4\lambda \int_{\mathbb{R}} \left[g(\theta') \int_{\mathbb{R}} g(2\theta - \theta') e^{i\frac{z}{2}(2\theta - \theta')} d(2\theta - \theta') \right] \frac{1}{2} e^{i\frac{z}{2}\theta'} d\theta'$$

$$= 4\lambda \int_{\mathbb{R}} \left[g(\theta') \hat{g}\left(\frac{z}{2}\right) \right] \frac{1}{2} e^{i\frac{z}{2}\theta'} d\theta' = 2\lambda \hat{g}\left(\frac{z}{2}\right) \int_{\mathbb{R}} g(\theta') e^{i\frac{z}{2}\theta'} d\theta' = 2\lambda \left[\hat{g}\left(\frac{z}{2}\right) \right]^2.$$

That if $\hat{g}(z)$ is the characteristic function of $g(\theta)$, $(-iz)^n \hat{g}(z)$ is the characteristic function of $\frac{\partial^n}{\partial \theta^n} g(\theta)$ implies that the characteristic function of the first term on the RHS of (18) is

$$-\frac{1}{2}\sigma_\eta^2 z^2 \hat{g}(z).$$

Putting together and using the linearity, differentiability, and integrability of the characteristic function, Equation (21) of the theorem obtains.

To obtain Equation (22) and (23), we apply the identities satisfied by all characteristic functions

$$\hat{g}(0) = \int_{\mathbb{R}} g(\theta) d\theta$$

and

$$\hat{g}'(0) = i \int_{\mathbb{R}} \theta g(\theta) d\theta$$

to Equation (19) and (20), respectively.

To derive the last equation of the theorem, note that, at steady state, (21) implies

$$\hat{g}(z) = \frac{1}{1 + \frac{\sigma_\eta^2 z^2}{4\lambda}} \left[\hat{g}\left(\frac{z}{2}\right) \right]^2, \tag{50}$$

which also implies

$$\hat{g}\left(\frac{z}{2}\right) = \frac{1}{1 + \frac{\sigma_\eta^2 z^2}{4^2\lambda}} \left[\hat{g}\left(\frac{z}{4}\right) \right]^2,$$

Substituting into (50),

$$\hat{g}(z) = \frac{1}{1 + \frac{\sigma_\eta^2 z^2}{4\lambda}} \left(\frac{1}{1 + \frac{\sigma_\eta^2 z^2}{4^2\lambda}} \right)^2 \left[\hat{g}\left(\frac{z}{4}\right) \right]^4.$$

Evaluating (50) at $\frac{z}{4}$ and substituting into the previous equality,

$$\hat{g}\left(z\right)=\frac{1}{1+\frac{\sigma_{\eta}^{2}z^{2}}{4\lambda}}\left(\frac{1}{1+\frac{\sigma_{\eta}^{2}z^{2}}{4^{2}\lambda}}\right)^{2}\left(\frac{1}{1+\frac{\sigma_{\eta}^{2}z^{2}}{4^{3}\lambda}}\right)^{4}\left[\hat{g}\left(\frac{z}{8}\right)\right]^{8}.$$

Repeating the same procedure, one can induce Equation (24) of the theorem. What remains to show is that the RHS of (24) does not vanish. Rewrite (24):

$$\hat{g}\left(z\right)=\lim_{K\to\infty}\prod_{k=0}^{K}\left[\zeta\left(k,z\right)\right]^{2^{k}},\tag{51}$$

where

$$\zeta\left(k,z\right)\equiv\frac{1}{1+\frac{\sigma_{\eta}^{2}}{4^{k+1}\lambda}z^{2}}.$$

Note that $\zeta\left(k,\cdot\right)$ is the characteristic function of a Laplace distribution for all $k\in\{0,1,2,...\}$, which means it is an infinitely divisible characteristic function for all k (Lukacs, 1970, p. 109). Then, Corollary to Theorem 5.3.3 of Lukacs (1970) implies that $\left[\zeta\left(k,\cdot\right)\right]^{2^{k}}$ is an infinitely divisible characteristic function for all k as well because 2^{k} is a positive real number (p. 111). Theorem 5.3.2 of Lukacs (1970) states that the product of a finite number of infinitely divisible characteristic functions is an infinitely divisible characteristic function (p. 109). Thus,

$$\prod_{k=0}^{K}\left[\zeta\left(k,z\right)\right]^{2^{k}}$$

is an infinitely divisible characteristic function. Then, from Theorem 5.3.3 of Lukacs (1970), the limit (51) is an infinitely divisible characteristic function because it is the limit of a sequence of infinitely divisible characteristic functions (p. 110). Finally, Theorem 5.3.1 of Lukacs (1970) implies that the RHS of (24) does not vanish because $\hat{g}\left(z\right)\neq0$ for all $z\in\mathbb{R}$ holds for any infinitely divisible characteristic function (p. 108).

C.4 Proof of Proposition 4

Substituting (16) into (26),

$$\mathcal{V}_{j}=\frac{\lambda_{j}}{2\left|\psi_{j}\right|}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\theta'-\theta\right|g\left(\theta'\right)g\left(\theta\right)d\theta'd\theta.$$

Written in a more compact way,

$$\mathcal{V}_{j}=\frac{\lambda_{j}}{2\left|\psi_{j}\right|}\mathbb{E}\left[\left|\theta'-\theta\right|\right].\tag{52}$$

Thus, we need to calculate the first absolute moment of $\theta' - \theta$. Note that the characteristic function of $\theta' - \theta$ is $\hat{g}(z)\,\hat{g}(-z)$ because θ' and θ are independently distributed due to random matching. Also, using the fact that $\hat{g}(\cdot)$ is an even function, the characteristic function of $\theta' - \theta$ is $[\hat{g}(z)]^2$.

Corollary 3.3 of Pinelis (2018) implies that

$$\mathbb{E}[|\theta' - \theta|] = \frac{2}{\pi}\int_{0+}^{\infty} \frac{1 - [\hat{g}(z)]^2}{z^2} dz.$$

Substituting into (52) and using (24), one obtains Equation (27) of the proposition. It is straightforward to obtain (28) from (27).

C.5 Proof of Corollary 3 and 5

In the probability theory literature, some sharper bounds for first absolute moments have recently been developed than usual Hölder-Lyapunov inequalities could provide. For the upper bound, we use Theorem 6 of Ushakov (2011):

$$\mathbb{E}[|\theta' - \theta|] \le \frac{4}{\pi}\sqrt{var[\theta]} \tag{53}$$

because θ' and θ are independently distributed due to random matching. And, for the lower bound we use Corollary 2.3 of Berger (1997):

$$\frac{\left\{\mathbb{E}\left[(\theta' - \theta)^2\right]\right\}^{\frac{3}{2}}}{\left\{\mathbb{E}\left[(\theta' - \theta)^4\right]\right\}^{\frac{1}{2}}} \le \mathbb{E}[|\theta' - \theta|].$$

Again using the fact that θ and θ' are independently distributed and $\mathbb{E}[\theta] = 0$, this can be re-written as

$$\frac{2\left(\mathbb{E}[\theta^2]\right)^{\frac{3}{2}}}{\left\{\mathbb{E}[\theta^4] + 3\left(\mathbb{E}[\theta^2]\right)^2\right\}^{\frac{1}{2}}} \le \mathbb{E}[|\theta' - \theta|]. \tag{54}$$

Thus, we need higher order usual moments of θ to be able to calculate the bounds for trade volume. Using (21) and (25) and equating $\dot{\hat{g}}(z) = 0$, one easily obtains the moments reported in Corollary 3:

$$\mathbb{E}\left[\theta^2\right] = \frac{\sigma_\eta^2}{\lambda}$$
$$\mathbb{E}\left[\theta^3\right] = 0$$
$$\mathbb{E}\left[\theta^4\right] = \frac{27}{7}\frac{\sigma_\eta^4}{\lambda^2}.$$

Substituting into (54) and (53),

$$\frac{1}{2}\sqrt{\frac{7}{3}}\frac{\sigma_\eta}{\sqrt{\lambda}} \leq \mathbb{E}\left[|\theta' - \theta|\right] \leq \frac{4}{\pi}\frac{\sigma_\eta}{\sqrt{\lambda}}.$$

Combining with (52), one obtains Equation (29) of Corollary 5. Then, the limiting results follow by Squeeze Theorem.

C.6 Proof of Proposition 6

$$\sigma^2_{P_j} \equiv \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left\{P_j\left(\theta, \theta'\right) - \mathbb{E}\left[P_j\left(\theta'', \theta'''\right)\right]\right\}^2 g\left(\theta'\right) g\left(\theta\right) d\theta' d\theta,$$

where

$$\mathbb{E}\left[p_j\left(\theta'', \theta'''\right)\right] = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} P_j\left(\theta'', \theta'''\right) g\left(\theta'''\right) g\left(\theta''\right) d\theta''' d\theta'' = \frac{1}{r}\frac{\partial u}{\partial a_j}\left(0, \mathbb{E}\left[\mathbf{a}'\right]\right).$$

The last equality follows from (17) and $\mathbb{E}\left[\theta\right] = 0$. Thus,

$$\sigma^2_{P_j} = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left\{P_j\left(\theta, \theta'\right) - \frac{1}{r}\frac{\partial u}{\partial a_j}\left(0, \mathbb{E}\left[\mathbf{a}'\right]\right)\right\}^2 g\left(\theta'\right) g\left(\theta\right) d\theta' d\theta.$$

Using (17),

$$\begin{aligned}
\sigma^2_{P_j} &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left(-\frac{r\gamma_B\sigma^2\psi_j}{r + \lambda/2}\frac{\theta + \theta'}{2}\right)^2 g\left(\theta'\right) g\left(\theta\right) d\theta' d\theta \\
&= \left(\frac{1}{2}\frac{r\gamma_B\sigma^2\psi_j}{r + \lambda/2}\right)^2 \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left(\theta + \theta'\right)^2 g\left(\theta'\right) g\left(\theta\right) d\theta' d\theta \\
&= 2\left(\frac{1}{2}\frac{r\gamma_B\sigma^2\psi_j}{r + \lambda/2}\right)^2 \mathbb{E}\left[\theta^2\right].
\end{aligned}$$

Using the second moment derived in the earlier proof C.5 and taking the square-root of both sides, Equation (30) of the proposition follows. It is straightforward to obtain (31) from (30).

C.7 Proof of Proposition 7

$$\sigma^2_{q_j} \equiv \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left\{q_j\left(\theta, \theta'\right) - \mathbb{E}\left[q_j\left(\theta'', \theta'''\right)\right]\right\}^2 g\left(\theta'\right) g\left(\theta\right) d\theta' d\theta,$$

where

$$\mathbb{E}\left[q_j\left(\theta'', \theta'''\right)\right] = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} q_j\left(\theta'', \theta'''\right) g\left(\theta'''\right) g\left(\theta''\right) d\theta''' d\theta'' = 0.$$

The last equality follows from (16) and $\mathbb{E}[\theta] = 0$. Thus,

$$\sigma_{q_j}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} [q_j(\theta, \theta')]^2 g(\theta') g(\theta) d\theta' d\theta.$$

Using (16),

$$\begin{aligned}\sigma_{q_j}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\theta' - \theta}{2\psi_j} \right)^2 g(\theta') g(\theta) d\theta' d\theta \\ &= \left(\frac{1}{2\psi_j} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} (\theta' - \theta)^2 g(\theta') g(\theta) d\theta' d\theta \\ &= 2 \left(\frac{1}{2\psi_j} \right)^2 \mathbb{E}\left[\theta^2\right].\end{aligned}$$

Using the second moment derived in the earlier proof C.5 and taking the square-root of both sides,

$$\sigma_{q_j} = \sqrt{2} \frac{\sigma_\eta}{|\psi_j| \sqrt{\lambda}}.$$

Combining with (30), one obtains Equation (32) of the proposition. It is straightforward to obtain (33) from (32).

C.8 Proof of Proposition 8

Equation (14) implies

$$\begin{aligned}V(\eta, \boldsymbol{a}, \lambda_b \,|\, \lambda_b^*, \lambda_c) = \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \left(-\sigma_\eta^2 + \frac{\lambda_b + \lambda_c}{2} \mathbb{E}_{\lambda_b^*} \left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}' \right] \right) \\ - (\lambda_b + \lambda_c) \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \boldsymbol{\psi}^T \mathbb{E}[\mathbf{a}'] \eta + \left(\frac{1}{r} \mathbf{m} - (\lambda_b + \lambda_c) \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \boldsymbol{\psi}^T \mathbb{E}[\mathbf{a}'] \boldsymbol{\psi} \right)^T \mathbf{a} \\ - \frac{r \gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \left(\eta^2 + 2\eta \boldsymbol{\psi}^T \mathbf{a} + \mathbf{a}^T \Psi \mathbf{a} \right).\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}\left[V(\eta, \boldsymbol{a}, \lambda_b \,|\, \lambda_b^*, \lambda_c)\right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^J} \left\{ \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \left(-\sigma_\eta^2 + \frac{\lambda_b + \lambda_c}{2} \mathbb{E}_{\lambda_b^*} \left[(\eta')^2 + 2\eta' \boldsymbol{\psi}^T \mathbf{a}' + (\mathbf{a}')^T \Psi \mathbf{a}' \right] \right) \right. \\ &- (\lambda_b + \lambda_c) \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \boldsymbol{\psi}^T \mathbb{E}_{\lambda_b^*}[\mathbf{a}'] \eta + \left(\frac{1}{r} \mathbf{m} - (\lambda_b + \lambda_c) \frac{\gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \boldsymbol{\psi}^T \mathbb{E}_{\lambda_b^*}[\mathbf{a}'] \boldsymbol{\psi} \right)^T \mathbf{a} \\ &\left. - \frac{r \gamma_B \sigma^2}{2r + \lambda_b + \lambda_c} \left(\eta^2 + 2\eta \boldsymbol{\psi}^T \mathbf{a} + \mathbf{a}^T \Psi \mathbf{a} \right) \right\} \Phi(d\mathbf{a}, d\eta).\end{aligned}$$

And, in equilibrium,

$$\begin{aligned}
&\mathbb{E}_{\lambda_b^*}\left[\mathbf{a}'\right] = \mathbf{0}, \\
&\mathbb{E}_{\lambda_b^*}\left[(\eta')^2 + 2\eta'\boldsymbol{\psi}^T\mathbf{a}' + (\mathbf{a}')^T\Psi\mathbf{a}'\right] = \mathbb{E}_{\lambda_b^*}\left[\left(\theta' + \boldsymbol{\psi}^T\mathbb{E}_{\lambda_b^*}\left[\mathbf{a}'\right]\right)^2\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left[V\left(\eta, \boldsymbol{a}, \lambda_b \mid \lambda_b^*, \lambda_c\right)\right] = \iint\limits_{\mathbb{R}\ \mathbb{R}^J} &\left\{\frac{\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(-\sigma_\eta^2 + \frac{\lambda_b+\lambda_c}{2}\mathbb{E}_{\lambda_b^*}\left[\theta'\right]\right)\right. \\
&\left. + \frac{1}{r}\mathbf{m}^T\mathbf{a} - \frac{r\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right)\right\}\Phi\left(d\mathbf{a}, d\eta\right).
\end{aligned}$$

Using the second moment derived in the earlier proof C.5, one obtains

$$\begin{aligned}
&\mathbb{E}\left[V\left(\eta, \boldsymbol{a}, \lambda_b \mid \lambda_b^*, \lambda_c\right)\right] = \iint\limits_{\mathbb{R}\ \mathbb{R}^J}\left\{\frac{\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(-\sigma_\eta^2 + \frac{\lambda_b+\lambda_c}{2}\frac{\sigma_\eta^2}{\lambda_c+\lambda_b^*}\right)\right. \\
&\left. + \frac{1}{r}\mathbf{m}^T\mathbf{a} - \frac{r\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right)\right\}\Phi\left(d\mathbf{a}, d\eta\right) \\
&= \frac{\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(-\sigma_\eta^2 + \frac{\lambda_b+\lambda_c}{2}\frac{\sigma_\eta^2}{\lambda_c+\lambda_b^*}\right) \\
&\qquad + \frac{1}{r}\mathbf{m}^T\mathbb{E}_{\lambda_b}\left[\mathbf{a}\right] - \frac{r\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\mathbb{E}_{\lambda_b}\left[\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right].
\end{aligned}$$

Because the system is ergodic,

$$\begin{aligned}
&\mathbb{E}_{\lambda_b}\left[\mathbf{a}\right] = \mathbf{0}, \\
&\mathbb{E}_{\lambda_b}\left[\eta^2 + 2\eta\boldsymbol{\psi}^T\mathbf{a} + \mathbf{a}^T\Psi\mathbf{a}\right] = \mathbb{E}_{\lambda_b}\left[\left(\theta + \boldsymbol{\psi}^T\mathbb{E}_{\lambda_b}\left[\mathbf{a}\right]\right)^2\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[V\left(\eta, \boldsymbol{a}, \lambda_b \mid \lambda_b^*, \lambda_c\right)\right] \\
&= \frac{\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(-\sigma_\eta^2 + \frac{\lambda_b+\lambda_c}{2}\frac{\sigma_\eta^2}{\lambda_c+\lambda_b^*}\right) - \frac{r\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\mathbb{E}_{\lambda_b}\left[\theta^2\right] \\
&\qquad = \frac{\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\left(-\sigma_\eta^2 + \frac{\lambda_b+\lambda_c}{2}\frac{\sigma_\eta^2}{\lambda_c+\lambda_b^*}\right) - \frac{r\gamma_B\sigma^2}{2r+\lambda_b+\lambda_c}\frac{\sigma_\eta^2}{\lambda_c+\lambda_b},
\end{aligned}$$

which implies (36).

The first order condition of the optimization problem (36) is

$$\begin{aligned}
&-\frac{\gamma_B\sigma^2\sigma_\eta^2}{\left(2r+\lambda_c+\lambda_b\right)^2}\left(-1 + \frac{\lambda_c+\lambda_b}{2\left(\lambda_c+\lambda_b^*\right)} - \frac{r}{\lambda_c+\lambda_b}\right) \\
&\qquad + \frac{\gamma_B\sigma^2\sigma_\eta^2}{2r+\lambda_c+\lambda_b}\left(\frac{1}{2\left(\lambda_c+\lambda_b^*\right)} + \frac{r}{\left(\lambda_c+\lambda_b\right)^2}\right) - \chi'\left(\lambda_b\right) \leq 0
\end{aligned}$$

and with equality if $\lambda_b > 0$. The second order condition is

$$
\begin{aligned}
&-2\frac{\gamma_B\sigma^2\sigma_\eta^2}{(2r+\lambda_c+\lambda_b)^3}\left(1+\frac{r}{\lambda_c+\lambda_b^*}+\frac{r}{\lambda_c+\lambda_b}\right)-2\frac{\gamma_B\sigma^2\sigma_\eta^2}{(2r+\lambda_c+\lambda_b)^2}\frac{r}{(\lambda_c+\lambda_b)^2} \\
&\qquad -\frac{\gamma_B\sigma^2\sigma_\eta^2}{2r+\lambda_c+\lambda_b}\left(\frac{1}{2\left(\lambda_c+\lambda_b^*\right)^2}+\frac{2r}{(\lambda_c+\lambda_b)^3}\right)-\chi''\left(\lambda_b\right)\leq 0.
\end{aligned}
$$

Because the second order condition always holds with strict inequality, there is a unique optimum λ_b given any λ_b^*. Then, the first order condition evaluated at $\lambda_b = \lambda_b^*$ pins down the equilibrium:

$$
\frac{\gamma_B\sigma^2\sigma_\eta^2}{2\left(\lambda_c+\lambda_b^*\right)}\left(\frac{1}{2r+\lambda_c+\lambda_b^*}+\frac{1}{\lambda_c+\lambda_b^*}\right)-\chi'\left(\lambda_b^*\right)\leq 0.
$$

The left hand side is strictly decreasing in λ_b^*, which implies the uniqueness of the equilibrium. Finally, by using the assumption $\chi\left(\lambda_b\right)=\chi_0\lambda_b$, the proof is complete.

Supplement to "Liquidity in the Cross Section of OTC Assets"

This online appendix contains discussions and additional empirical results omitted from the printed manuscript.

Semih Üslü[1] **Güner Velioğlu**[2]

D Additional empirical results

D.1 Market volatility measures

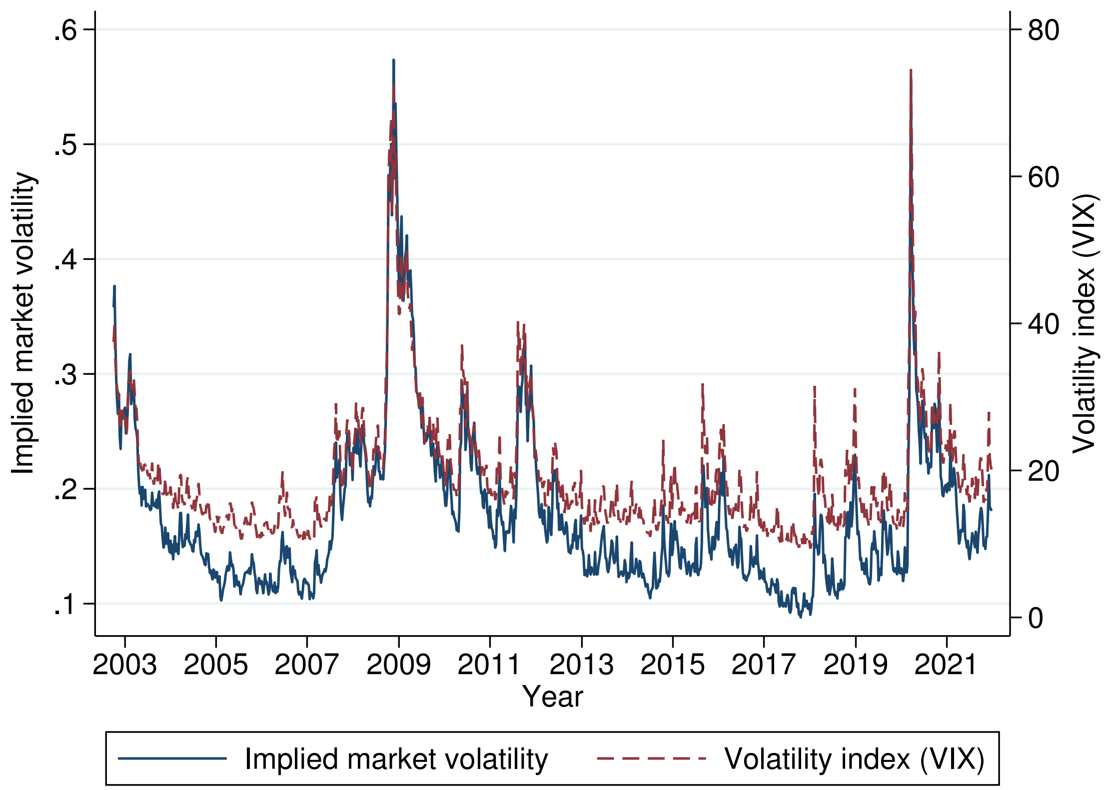

Figure D.1: Time series of market volatility measures

This figure plots two market volatility measures over the sample period from October 7, 2002 to December 31, 2021. Implied market volatility is the implied volatility of CBOE S&P 500 index European call option (see Appendix A for more details). Volatility index (VIX) is CBOE volatility index, obtained from the Federal Reserve's website. We take the average of daily VIX levels to have VIX levels at weekly frequency.

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D.2 Volatility beta

Table D.1: Relation of volatility beta with other risk measures

This table presents the relation of bond volatility beta with other risk measures. Bond rating is bond's credit rating provided by S&P, Moody's, or Fitch, in availability order, where letter ratings are converted to numbers from 1 (AAA) to 22 (D). Coupon rate is bond's coupon rate in percentages. Years to maturity is the number of years left to bond's maturity date. Bond age is the number of years since bond's offering date. Callable dummy equals one if bond is callable, and equals zero otherwise. Offering amount is the issuance size of the bond. Detailed descriptions of volatility beta and average number of trades are provided in Appendix A. We take the logarithm of each variable except the callable dummy. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

	(1) Volatility beta (ψ_j)	(2) Volatility beta (ψ_j)	(3) Volatility beta (ψ_j)
Bond rating	0.339***	0.337***	0.401***
	(21.45)	(20.67)	(37.50)
Coupon rate		0.129***	0.088***
		(6.10)	(5.73)
Years to maturity		0.466***	0.399***
		(66.60)	(78.20)
Bond age			0.124***
			(24.37)
Callable dummy			-0.116***
			(-11.14)
Offering amount			-0.348***
			(-84.90)
Average number of trades			-0.378***
			(-69.55)
Intercept	-3.220***	-4.286***	-3.672***
	(-93.46)	(-113.24)	(-119.06)
Year-week FE	Y	Y	Y
Observations	4,889,790	4,889,381	4,889,381
Adjusted R^2	0.155	0.213	0.389

D.3 Interdealer trades

Table D.2: Determinants of liquidity in the cross section and over time, interdealer trades

This table presents our main findings when the liquidity measures are instead calculated based on interdealer trades only, under a log-linear functional form assumption as in our main results. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

Panel A: Baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Offering amount (λ_j)	0.522***	-0.142***	-0.678***
	(95.16)	(-40.72)	(-92.00)
Volatility beta (ψ_j)	-0.010***	0.108***	0.127***
	(-5.19)	(65.88)	(40.93)
Average number of trades (ANT$_j$)	0.878***	0.473***	0.476***
	(151.08)	(120.43)	(63.63)
Intercept	-2.831***	-2.617***	-0.291***
	(-135.36)	(-195.41)	(-10.37)
Year-week FE	Y	Y	Y
Observations	4,206,079	4,206,079	4,206,079
Adjusted R^2	0.401	0.198	0.130

Panel B: Full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Offering amount (λ_j)	0.522***	-0.139***	-0.675***
	(95.09)	(-39.43)	(-90.87)
Offering amount, other bonds (λ_{-j})	-0.373***	-0.508***	0.019
	(-12.28)	(-24.82)	(0.66)
Volatility beta (ψ_j)	-0.013***	0.120***	0.140***
	(-5.00)	(54.71)	(40.11)
Average number of trades (ANT$_j$)	0.870***	0.476***	0.484***
	(144.99)	(123.94)	(64.65)
GDP forecast dispersion (σ_η)	-0.018	-0.186***	-0.226***
	(-1.21)	(-11.61)	(-12.72)
Treasury rate (r)	0.036***	-0.008**	-0.035***
	(6.10)	(-2.03)	(-6.99)
Implied market volatility (σ)	-0.023	0.649***	0.770***
	(-0.74)	(21.07)	(22.55)
Intercept	0.273	2.621***	0.699***
	(1.16)	(18.10)	(3.03)
Observations	4,206,079	4,206,079	4,206,079
Adjusted R^2	0.387	0.179	0.122

Panel C: Trade cancellations, baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.206***	-0.021***	-0.233***
	(65.15)	(-12.29)	(-55.07)
Bond-level controls except OA$_j$ & OA$_{-j}$	Y	Y	Y
Year-week FE	Y	Y	Y
Observations	4,206,079	4,206,079	4,206,079
Adjusted R^2	0.357	0.185	0.068

Panel D: Trade cancellations, full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.213***	-0.006***	-0.226***
	(65.46)	(-3.37)	(-53.71)
Average number of cancellations$_{-j}$	-0.420***	0.656***	0.951***
	(-13.60)	(24.64)	(29.70)
Bond-level controls except OA$_j$ & OA$_{-j}$	Y	Y	Y
Market-level controls	Y	Y	Y
Observations	4,206,079	4,206,079	4,206,079
Adjusted R^2	0.343	0.167	0.061

Panel E: CDS depth, baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average CDS depth$_j$	-0.079***	0.057***	0.155***
	(-9.52)	(10.59)	(13.30)
Offering amount	0.508***	-0.135***	-0.663***
	(69.43)	(-31.47)	(-69.95)
Bond-level controls except OA$_j$ & OA$_{-j}$	Y	Y	Y
Year-week FE	Y	Y	Y
Observations	2,354,299	2,354,299	2,354,299
Adjusted R^2	0.407	0.201	0.128

Panel F: CDS depth, full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average CDS depth$_j$	-0.080***	0.060***	0.164***
	(-9.58)	(10.93)	(14.26)
Offering amount	0.507***	-0.134***	-0.660***
	(69.30)	(-31.46)	(-69.90)
Bond-level controls except OA$_j$	Y	Y	Y
Market-level controls	Y	Y	Y
Observations	2,354,299	2,354,299	2,354,299
Adjusted R^2	0.393	0.181	0.118

D.4 Trade cancellations and offering amount as search friction proxies

Table D.3: Proxying for search frictions, trade cancellations and offering amount

This table presents the relation between trade cancellations and the liquidity in corporate bonds, with offering amount included in the specifications. Panel A presents the results under the baseline model, where offering amount, bond-level controls except OA_j & OA_{-j} (offering amount & offering amount, other bonds), and year-week fixed effects collectively represent the predictors used in Table 3. Panel B presents the results under the full model, where offering amount, offering amount of the other bonds, bond-level controls except OA_j & OA_{-j} (offering amount & offering amount, other bonds), and market-level controls collectively represent the predictors used in Table 4. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

Panel A: Baseline model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.172***	-0.014***	-0.173***
	(68.05)	(-7.78)	(-44.94)
Offering amount	0.872***	-0.176***	-0.976***
	(148.85)	(-47.46)	(-123.41)
Bond-level controls except OA_j & OA_{-j}	Y	Y	Y
Year-week FE	Y	Y	Y
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.511	0.231	0.215

Panel B: Full model

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Average number of cancellations$_j$	0.177***	-0.009***	-0.169***
	(69.25)	(-4.71)	(-44.29)
Average number of cancellations$_{-j}$	-0.081*	0.371***	0.323***
	(-1.75)	(14.17)	(12.18)
Offering amount	0.871***	-0.174***	-0.975***
	(147.53)	(-46.43)	(-122.97)
Offering amount, other bonds	-0.355***	-0.171***	0.193***
	(-8.33)	(-7.98)	(6.76)
Bond-level controls except OA_j & OA_{-j}	Y	Y	Y
Market-level controls	Y	Y	Y
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.493	0.219	0.210

D.5 Linear functional form

Table D.4: Determinants of liquidity in the cross section and over time, linear functional form

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds under a linear functional form assumption. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript j refers to bond j, and the subscript $-j$ refers to all other bonds except bond j. Detailed variable definitions are provided in Appendix A. In this table only, all variables are winsorized at the 1% and 99% levels. The standard errors are double clustered by bond and week, and the t-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

	(1) Trade volume ($\mathcal{V}_j$)	(2) Price dispersion (σ_{P_j})	(3) Price impact (δ_j)
Offering amount (λ_j)	12.3451***	-0.1309***	-21.0816***
	(52.20)	(-24.34)	(-34.63)
Offering amount, other bonds (λ_{-j})	-0.0014***	-0.0001***	-0.0028***
	(-17.10)	(-20.72)	(-19.90)
Volatility beta (ψ_j)	0.0481	0.0202***	5.6668***
	(1.59)	(15.65)	(36.12)
Average number of trades (ANT$_j$)	0.1804***	0.0047***	0.0570***
	(42.82)	(52.14)	(10.10)
GDP forecast dispersion (σ_η)	-1.0561***	-0.0515***	-1.4109***
	(-7.62)	(-7.85)	(-7.70)
Treasury rate (r)	0.2004***	-0.0045**	0.1263
	(3.06)	(-2.35)	(1.19)
Implied market volatility (σ)	-1.6136	1.5006***	54.8467***
	(-1.06)	(16.45)	(19.10)
Intercept	4.6767***	0.4446***	33.1316***
	(8.77)	(17.67)	(35.16)
Observations	4,912,241	4,912,241	4,912,241
Adjusted R^2	0.300	0.163	0.101

In this section, we alternatively run the following regression assuming a linear functional form:

$$Liquidity_{j,t} = \alpha + \beta_1\lambda_{j,t} + \beta_2\lambda_{-j,t} + \beta_3\psi_{j,t} + \beta_4 ANT_{j,t} + \beta_5\sigma_{\eta,t} + \beta_6 r_t + \beta_7\sigma_t + \varepsilon_{j,t},$$

where "$Liquidity_{j,t}$" of bond j in week t denotes the liquidity measure; trade volume ($\mathcal{V}_j$), price dispersion (σ_{P_j}), or price impact (δ_j). We run this regression separately for each measure.

An important limitation of this naïve specification is that it does not account for the proportionality between the liquidity measures and the predictors as uncovered by our theoretical formulas. In other words, the specified linear functional form does not capture the multiplicative relation between the predictors in jointly determining the liquidity measures, and so, may be inconsistent with the actual data generating processes. While being mindful of this possibility, we present the results of this estimation in Table D.4. The adjusted R^2s are 0.300, 0.163, and 0.101 in Table D.4, which are, as expected, low in comparison to 0.484, 0.216, and 0.204 of

Table 4, respectively. Although the coefficients have similar signs in both tables, we base our main conclusions on Table 4, which more accurately corresponds to the functional form of the theoretical model.

E Non-cross-sectional predictors of liquidity

For completeness of our Table 4 discussion, we next consider the non-cross-sectional predictors in Table 4, while noting the high degree of overlap in the information content of the empirical counterparts of these variables. For example, although we can isolate σ and σ_η in theory, this is difficult in practice. As such, we view these variables as jointly controlling for the fuzzy macroeconomic factors, rather than distinctly identifying economic channels and sharply corresponding to the variables in theory.

As part of our non-cross-sectional predictors in Table 4, we use the implied market volatility, the GDP forecast dispersion among the professional forecasters, and the treasury rate as proxies for the aggregate volatility (σ), the hedging need dispersion (σ_η) among investors in our model, and the investors' discount rate (r), respectively. As predicted by our model, the implied market volatility is strongly and positively related with price dispersion and price impact. According to the model, the implied market volatility is not supposed to affect trade volume because it only scales up and down all investors' exposure to the aggregate systematic risk endowment, while what matters for trade volume is only the dispersion in their exposure to systematic risk, which is not affected by σ, but by σ_η. However, Table 4 indicates that both implied market market volatility and GDP forecast dispersion reduce trade volume, instead of GDP forecast dispersion only. We suspect that this is because the implied market volatility and the GDP forecast dispersion have overlapping information content while σ and σ_η are easily distinguished in our model by construction. In terms of price dispersion, the GDP forecast dispersion has the opposite sign of what is predicted by the model. Another failure of GDP forecast dispersion shows up in the price impact regression. That is, while the model predicts no relation between price impact and the GDP forecast dispersion, its regression coefficient turns out to be negative and significant. Therefore, only the empirical relationship between trade volume and GDP forecast dispersion is consistent with our model's prediction among the predictions for σ_η.

The last and perhaps the least curious case is the implications regarding the treasury rate, which we designate as a proxy for investors' discount rate. The implications of discount rate have typically very natural interpretations in this class of models. As discount rate increases, investors care more for quick trading and so they start trading as if frictions got more severe.[3] Accordingly, as discount rate increases, the sensitivity of investors' marginal valuation to their excess risk exposure increases in our model, and this leads to larger price dispersion and price impact. However, signs of the coefficients of treasury rate reported in the price dispersion and price impact regressions in Table 4 are exactly the opposite of the signs predicted by our theory. This is not a very curious case because treasury rate is a highly contaminated proxy for discount rate due to time-varying liquidity premia of the treasury securities.[4] Indeed, it is very possible that the time-variation of liquidity premia is stronger than the time-variation of investors' discount rate, and so, a larger treasury rate typically captures a smaller liquidity premium rather than a larger discount rate. With this interpretation in mind, the regression coefficients become less confusing, because it is natural to think that times of larger liquidity premia are associated with large price dispersion and price impact in the OTC markets.

To sum up, our *cross-sectional* results from empirical tests of liquidity are mostly consistent with the implications of the theoretical model, both with time fixed effects and with a model-informed set of macroeconomic indicators. Because our model is developed mainly to obtain precise cross-sectional implications, we interpret this empirical consistency as pointing to the success and usefulness of the search-theoretic approach in uncovering the determinants of endogenous liquidity differentials across OTC assets. There is no question, however, that there is an obvious need for determining better proxies for aggregate parameters such as investors' hedging need dispersion and discount rate. Because these are typically important parameters in many dynamic search models, we believe our results point to a room for improvement in future research that takes seriously testing this class of models.

[3] Indeed, the limiting cases as frictions vanish and as discount rate goes to zero typically coincide with each other.

[4] See Lagos (2010) for a discussion.