# Optimal Information and Security Design

by

Nicolas Inostroza
Anton Tsoy

Working Paper No. 00105-00

Finance Theory Group

www.financetheory.com

# Optimal Information and Security Design*

Nicolas Inostroza[†] Anton Tsoy[‡]

July 11, 2023

**Abstract**

An asset owner designs an asset-backed security and a signal about its value. After privately observing the signal, he sells the security to a monopolistic liquidity supplier. Any optimal signal distribution guarantees the security sale and reveals noisy information about high valuations of the security. The optimal security is pure equity – the most informationally sensitive security. It is risky debt under external liquidity requirements akin to regulatory requirements on banks, pension funds, insurance companies. Thus, the "folk intuition" behind debt optimality as the least informationally sensitive security holds only under additional restrictions (e.g., regulatory) on security or information design.

KEYWORDS: security design, asymmetric information, information design

JEL CLASSIFICATION: D82, D86, G32

*We thank Philip Bond, Andrey Malenko, Anne-Katrin Roesler, Giulio Trigilia, Ming Yang, and seminar and conference participants at University of Rochester, University of Toronto, Finance Theory Group (Michigan), SED (Cartagena), UBC Winter Finance Conference, WFA (San Francisco).

[†]University of Toronto and FTG. Email: nicolas.inostroza@rotman.utoronto.ca

[‡]University of Toronto and FTG. Email: anton.tsoy@utoronto.ca.

# 1 Introduction

Corporations routinely sell assets or securities backed by their assets to raise liquidity. Asymmetric information is a major impediment to such sales: at the time of sale, the issuer might know more about the security than the liquidity supplier, thereby limiting the scope of trade. Extensive research in the existing literature has focused on how designing security payoffs can mitigate this inefficiency. However, in many cases, issuers possess additional tools – beyond payoff design – that provide greater flexibility in controlling the information asymmetry at the trading stage. In this paper, we introduce information design – the design of the issuer's private signals about the security before the trading stage – into the canonical problem and study the joint optimal design of both security payoffs and information.

Information design takes many forms. For example, the issuer can curb or enhance his informational advantage by distancing from or getting more involved in the management of the underlying asset. Consider, for instance, a multi-divisional company divesting one of its divisions. The general management can take a hands-off approach and give the division management lots of autonomy. At the other extreme, the division can be incorporated into the core of the company's business with its operations closely monitored. Different organization structures imply different levels of awareness of the general management about the division prospects, and hence, the extent of information asymmetry during the divestiture.

Another example is the sale of stakes by limited partners in private equity funds. Even though limited partners have access to private information about the fund's strategy and performance, they usually lack the sophistication to evaluate the fund's investment decisions, which are fully delegated to general partners. This design feature curbs their informational advantage vis-a-vis outsiders and allows them to sell their stake in the fund early if they experience a liquidity shock.

Further, mutual and hedge funds can choose the extent of their involvement in the management of companies in their portfolios. By adopting concentrated positions and engaging in activist campaigns, they can acquire enhanced knowledge. At the other extreme, assuming a passive shareholder role across numerous companies commits the fund to have only limited private information about each particular holding.

The issuer can also commit to information design by creating a structured product. For example, for a convertible bond, the conversion into equity is typically triggered when the stock price reaches a specific threshold. However, it is not uncommon to include other clauses for conversion, such as dividend payments or takeover bids if the company is a takeover target. As the management of the company is generally better informed about the likelihood of the takeover, the range of potential takeover bids, and future changes in the dividend policy, the inclusion of such clauses leads to a greater information asymmetry. Similar logic applies to other structured products as well as the design of trigger events for the execution of warrants or special dividends on preferred shares.

Information design can also be a part of the asset creation process itself. For instance, in the design of mortgage-backed securities (MBS), the issuer can strategically select the underlying pool of mortgages so that to have a larger informational advantage about the MBS value in the future. This can be done by selecting mortgages into the pool from the geographical region or the market segment in which the issuer specializes in underwriting mortgages. Alternatively, the issuer can minimize his informational advantage by creating a diverse pool of mortgages or by designing an overly complex MBS so that it is equally hard for the issuer and outside investors to value it. The same idea applies to even more complex collateralized debt obligations and securities backed by other assets, such as credit-card receivables, car loans, and student loans.

The aforementioned examples share a common feature: the issuer's ability to influence the level of future asymmetric information. This raises the question of how this ability affects classical results about which securities the issuer should issue optimally. Additionally, it prompts an examination of whether the securities commonly used by corporations to raise liquidity align with the optimal choices. This paper addresses these questions and provides answers.

**Model.** The basic setup is that of DeMarzo and Duffie (1999) and Biais and Mariotti (2005) with the design occurring before the private information is revealed to the issuer. This timing is motivated by the common practice of shelf-registration, which allows corporations to quickly react to changing conditions by registering securities well in advance of their sale. It is also relevant when the issuer is different from the seller of the security.

Formally, there are three stages: the ex-ante design stage, the trading stage, and the final stage. At the design stage, before getting any private information, the asset owner (the issuer) designs a security backed by the asset cash flows and potentially other contractible public variables and the distribution of private information about the security.

The design consists of two parts. First, the security design specifies the payoff of the security for any realization of asset cash flows and contractibles. Second, for a fixed security design, the issuer chooses the distribution of the private signal about the security value that is revealed to him at the trading stage. This choice constitutes the information design, which controls the extent of information asymmetry between the issuer and the liquidity supplier. We assume maximal flexibility in the information design: the issuer can costlessly choose any unbiased signal about the security value. This assumption provides a natural benchmark for what outcomes can be potentially attained with information design. Further, our analysis reveals which features of the signal distributions are most valuable and our results hold as long as the issuer has access to those features.

At the beginning of the trading stage, the issuer observes the signal realization. Due to liquidity costs, he discounts future asset payoffs at a higher rate than the liquidity supplier. This creates gains from trade of the security. However, efficient trade might be impeded by asymmetric information. We suppose that there is a monopolistic liquidity supplier endowed with all the bargaining power, who offers an optimal screening mechanism to the issuer, which in our setting boils down to a posted price. This assumption is realistic in applications where the security is designed to raise liquidity in crisis times when the liquidity supply is scarce and liquidity suppliers have significant market power.

**Main Results.** We first solve the information design problem for any fixed security. Any optimal signal distribution possesses two characteristic features. First, it attains perfect liquidity of the security, that is, it is optimal for the liquidity supplier to offer the price corresponding to the highest signal realization, and so, the whole issue of the security is traded with probability one.

Second, the perfect liquidity is attained by restricting the highest signal realization, which is the same across all optimal signal distributions and is generally below the maximal payoff of the security. In other words, the issuer must commit not to learn "too optimistic" information about the security. For example, for a debt security, the issuer's signal always prompts him to include a positive credit spread in his valuation. Intuitively, in order to achieve perfect liquidity, we must ensure that the price corresponding to the highest signal realization is optimal for the liquidity supplier. This is more easily achieved when the price offer, and consequently the highest signal realization, is lower.

To characterize the highest signal realization, we first introduce a class of mechanisms that we call *DD-mechanism* in recognition of the fact that they properly generalize the fully separating equilibrium in DeMarzo and Duffie (1999). It is a standard result that the space of unbiased signals about the security coincides with the space of mean-preserving contractions of the prior distribution of security payoff (Strassen's Theorem). We establish a novel characterization of the mean-preserving contraction ordering of two distributions – typically used to rank the informativeness of signals – in terms of the performance of DD-mechanisms under the two distributions. This theoretical result is instrumental in characterizing the highest signal realization in the optimal signal distribution. In particular, it allows us to replace the admissibility requirement on signal distributions with the liquidity supplier's payoff guarantee expressed in terms of DD-mechanisms.

We next turn to the central problem of joint security and information design. Following much of the literature (e.g., Nachman and Noe 1994, DeMarzo and Duffie 1999, Biais and Mariotti 2005, Axelson 2007), we focus on securities satisfying limited liability and double monotonicity. The classical results in the security design literature show that, when the private information is exogenous, costly retention of cash flows serves as a credible signal about the quality of the security and issuing debt is the optimal form of retention. The folk intuition behind these results is that debt is the least informationally sensitive security. Roughly, informationally insensitive securities are beneficial because they serve as a commitment device for the issuer not to take advantage of his future private information at the trading stage. A debt security is minimally sensitive to the issuer's private information as it promises a fixed amount, the face value, whenever possible and offers maximal downside protection when cash flows are low.

We show that this standard intuition fails when the issuer can optimally design the signal distribution: it is strictly optimal for the issuer to sell the whole asset rather than issuing any security and retaining cash flows. To prove this result, we uncover a new benefit of informationally sensitive securities. Following DeMarzo et al. (2005), we say that security $\tilde{\varphi}$ is more informationally sensitive than $\varphi$, if once we control for differences in mean payoffs, $\tilde{\varphi}$ crosses $\varphi$ from below. We establish that a more informationally sensitive security has a higher variability of payoffs, which tends to expand the set of admissible signal distributions leading to better outcomes for the issuer.

In the optimal information design, the issuer's informational advantage is crafted to achieve perfect liquidity of the security while maximizing the issuer's information rents. The restrictions imposed on the information learned by the issuer reduce the benefits of offering informationally insensitive securities. On the other hand, informationally sensitive securities hold value as they provide the issuer with greater flexibility in information design. We leverage this intuition and show that pure equity, the most informationally sensitive security, is strictly optimal.

How does the optimality of pure equity square with common practices of raising liquidity? Our result explains why in many markets, in which the adverse selection problem is potentially severe, issuers often simply liquidate assets to raise liquidity rather than design complex asset-backed securities. In the examples described above, multi-divisional firms sell entire periphery divisions in times of crisis; mutual and hedge funds liquidate their holding when facing excessive redemptions; and there is an active market for limited partners' stakes in private equity funds. Our analysis stresses that a proper information design – the ability to commit not to learn too positive private information about the asset – is a necessary condition for optimality of pure equity. As we argued above, such a commitment can be attained through providing lots of autonomy to periphery divisions, passive investment strategies of funds, or the structure of decision making in private equity funds.

At the same time, many securities, such as MBS and other asset-backed securities, are structured as debt securities. The classical view is that this is the optimal way to raise liquidity in the presence of asymmetric information. In contrast, our result suggests that the prevalence of debt points to the presence of institutional or technological restrictions either on the information or security design. The literature imposes the extreme restriction that no information design is possible.

To reconcile this phenomenon with our theory, we present an alternative explanation for the prevalence of debt in specific markets. We examine the joint design of securities and information while imposing additional external liquidity requirements, where securities must be sold without a substantial discount on their maximum value. These requirements may arise from regulations or shareholder oversight. For instance, banks, pension funds, and insurance companies are mandated to hold sufficient high-quality liquid assets that can be quickly liquidated without significant value loss. Similarly, outside shareholders or boards of directors representing them may be concerned about management selling securities at a significant discount and may block such sales.

With these external liquidity requirements, we find that debt reemerges as the optimal security. This implies that debt is influenced by regulations or external oversight rather than being the unconstrained optimal security for raising liquidity. This formalizes the viewpoint often expressed by practitioners that debt arises due to "regulatory arbitrage," where institutional investors demand debt because regulators perceive it as sufficiently safe and liquid.

The underlying intuition for this finding is as follows: The optimal information design restricts the issuer from learning about extremely high security values, resulting in securities generally being sold at a discount to their maximum value. If this discount is substantial, it can violate the liquidity requirements and disqualify certain securities, particularly pure equity. In such a scenario, the informational insensitivity of debt becomes valuable once again, leading to its optimality.

**Related Literature** Leland and Pyle (1977), Myers and Majluf (1984), and Myers (1984) first established that in the world of asymmetric information about asset qualities, cash flow retention serves as a credible signal of asset quality and debt arises as optimal among many other securities. The folk intuition is that debt is advantageous, as it is the least sensitive to the issuer's private information. This work started an extensive literature on optimal security design under adverse selection. Most closely related to our paper are DeMarzo and Duffie (1999) and Biais and Mariotti (2005) who study security design at the ex-ante stage with an exogenous distribution of issuer's private information. Both papers show that, under general conditions, risky debt is optimal among securities satisfying limited liability and double monotonicity. Pure equity is optimal but only as a corner optimum, that is, debt with face value equal to the highest cash flow realization. Other papers showing optimality of debt include Nachman and Noe (1994), DeMarzo (2005), DeMarzo et al. (2005), Dang et al. (2013), Daley et al. (2020), Li (2022), Asriyan and Vanasco (Forthcoming), Inostroza and Figueroa (2023) among many others.

We contribute to this literature by solving the joint problem of information and security design in the by now canonical setup of DeMarzo and Duffie (1999) and Biais and Mariotti (2005). We show that generally the issuer prefers more informationally sensitive securities, because they provide more freedom in information design. In contrast to these benchmarks, the uniquely optimal security is pure equity and retention is strictly suboptimal. We further obtain debt as a constrained solution to the joint design problem, when the security must satisfy external liquidity requirements. Our characterization of optimal information design reveals an interesting parallel between this problem and the classical problem of optimal security design with exogenous private information.

There is a literature showing that informationally sensitive securities can become optimal when informational sensitivity has additional benefits to the issuer, e.g., it incentivizes information acquisition by investors (Boot and Thakor 1993, Fulghieri and Lukin 2001, Yang and Zeng 2019), it enables the aggregation of information about the optimal scale of project from informed investors (Axelson 2007), or it is complementary to public signals about the asset and allows the issuer to economize on retention (Daley et al. 2023). Our mechanism is different and to the best of our knowledge novel to the literature: informationally sensitive securities are beneficial, because they relax the constraints on the issuer's information design.

Several papers study security design with endogenous information. Yang and Zeng (2018), Yang (2020) allow for flexible information acquisition by the liquidity supplier. In Azarmsa and Cong (2020), Szydlowski (2021), the issuer additionally designs public disclosures to investors. Similarly to ours, these papers impose minimal restrictions on admissible information acquisition or disclosure policies. In contrast to our paper, the optimal security is indeterminate without either positive information acquisition costs or further financing frictions. It is debt when information acquisition is costly in Yang (2020) and depends on the kind of additional contracting frictions in Szydlowski (2021) and Azarmsa and Cong (2020). Our study of joint information and security design by the issuer is complementary to this literature.

Our paper is related to the literature on optimal information design in the monopolist

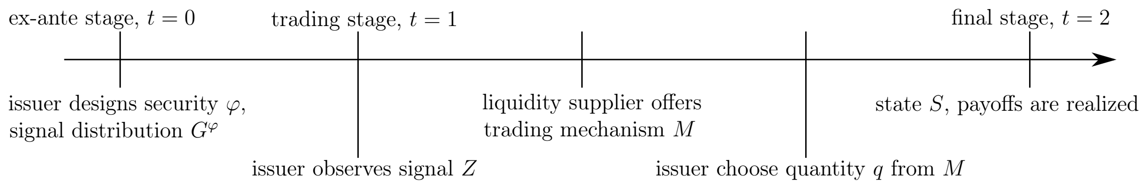



*Figure 1:* ***Timeline***

screening problem (Bergemann et al. 2015, Roesler and Szentes 2017, Glode et al. 2018).[1] Most closely related is Kartik and Zhong (2023) who also study information design with interdependent values. The two studies focus on complementary issues. While Kartik and Zhong (2023)'s central result is the characterization of the payoffs attainable across all signal distributions for fixed payoffs, ours is the solution to the joint information and security design problem. This difference motivates different sets of results. Apart from the results on optimal security design, our characterization of optimal signal distributions and mean-preserving contraction in terms of performance of DD mechanisms is novel and it enables us to establish a connection between optimal information design and the classical security design with exogenous information.[2]

The paper is organized as follows. Section 2 presents the model. Section 3 presents a simple example. Section 4 solves the information design problem. Section 5 solves the joint security and information design problem. Section 6 shows optimality of debt under external liquidity requirements. Section 7 presents extensions. Section 8 discusses empirical implications. Section 9 concludes. All omitted proofs are relegated to the Appendix and the Online Appendix.

# 2 The Model

The basic setup is that of DeMarzo and Duffie (1999) and Biais and Mariotti (2005) with the addition of information design. Figure 1 depicts the timeline. There are three stages $t \in \{0, 1, 2\}$. There is an issuer (he) owning an asset and a liquidity supplier (she). Both parties are risk-neutral. The asset generates cash flows $X$ at the final stage $t = 2$ distributed according to the c.d.f. $H$ on positive support $[\underline{x}, \overline{x}]$, $\underline{x} > 0$. At the trading stage $t = 1$, the issuer has a higher discount factor: he values future cash flows at $\delta X, \delta \in (0, 1)$, while the liquidity supplier values them at $X$. This captures the issuer's desire to free-up capital to

[1] Less related to our paper, Barron et al. (2020), Mahzoon et al. (2022) study interaction of information and contract design in the moral hazard setting.

[2] See also footnote 8 for a further discussion of the relationship between our program (16) and Proposition 2 in Kartik and Zhong (2023).

invest in alternative assets or projects, improve the liquidity position in crisis times, raise liquidity to cover redemptions (for investment funds), or focus financial resources on the core business (for multi-division companies). Since the liquidity supplier is the efficient owner of the asset, there are gains from trade.

At the ex-ante design stage $t = 0$, before receiving any private information, the issuer designs a security to be traded at $t = 1$ and a signal distribution. The security payoff $F = \varphi(S)$ is contingent on the contractible public state $S = (X, S_1, \dots, S_m)$ realized at $t = 2$, which includes cash flows $X$ and other contractible public variables, $S_1, \dots, S_m$. For example, the payoff can be benchmarked to a public index; a debt security can be converted into equity or a warrant can be exercised under certain contractible conditions; debt covenants can depend on the credit rating of the company or commonly observable management actions, such as debt issuance or dividend payments. Let $H^S$ be the c.d.f. of state $S$, and $H^\varphi$ be the c.d.f. of security payoff $F$, which is a push-forward distribution and is supported on $\left[\underline{f}^\varphi, \overline{f}^\varphi\right]$. In the case $S = X$, we have $H^S = H$, $\underline{f}^\varphi = \varphi(\underline{x})$, $\overline{f}^\varphi = \varphi(\overline{x})$, and $H^\varphi(f) = H(\varphi^{-1}(f))$, where $\varphi^{-1}(f) \equiv \sup\{x : \varphi(x) \le f\}$ is the (right-continuous) inverse of $\varphi$.

The issuer can costlessly design any unbiased signal $Z$ about the security value $F$. Without loss of generality, the signal is simply the expected value of the security, $Z = \mathbb{E}[F|Z]$. By Definition 6.D.2 of Mas-Colell, Whinston and Green (1995), $G^\varphi$ is the distribution of some unbiased signal $Z$ about $F$ if and only if $G^\varphi$ is a mean-preserving contraction of $H^\varphi$, that is, $\mathbb{E}_{G^\varphi}[Z] = \mathbb{E}_{H^\varphi}[F] \equiv \mu^\varphi$, and $G^\varphi$ second-order stochastically dominates $H^\varphi$:[3]

$$\int_{-\infty}^{y} H^\varphi(f)\, \mathrm{d}f \ge \int_{-\infty}^{y} G^\varphi(z)\, \mathrm{d}z \text{ for all } y. \tag{1}$$

We call such signal distributions *admissible* for security $\varphi$. Denote by $\mathcal{G}^\varphi$ the set of all admissible signal distributions.[4]

Commitment to the security design is often motivated in the literature by shelf-registration of securities, a common practice in finance. Our main result is strict optimality of selling the whole asset. Hence, such commitment can be obtained by simply not obtaining a license to issue securities. We discuss in the introduction various ways how the issuer can restrict his informational advantage in the future. Further, giving the issuer full flexibility in information design presents an important benchmark. As we discuss in Section 4, only commitment

[3]All integrals in this paper are Lebesgue-Stieltjes integrals for which the integration by parts formula obtains (see Theorem VI.90 in Dellacherie and Meyer 1982).

[4]Generally, both $G^\varphi$ and $H^\varphi$ can have mass points. For instance, if $S = X$ and $\varphi$ is a debt security with face value $D < \overline{x}$, then $H^\varphi$ has a mass point at $D$. Even if $\varphi$ does not create any mass points in $H^\varphi$, the signal distribution $G^\varphi$ chosen by the issue is only requited to be right-continuous, and so, can have mass points.

to certain signal features is sufficient to attain the maximal payoff.

At the trading stage $t = 1$, the issuer observes a realization $z$ of signal $Z$ drawn from $G^{\varphi}$. We call $z$ the issuer type. The issuer can obtain liquidity from the liquidity supplier by selling the whole or part of the security issue. We suppose that the liquidity supplier is monopolistic, which allows us to focus on the key trade-offs between liquidity of the security $\varphi$ and the issuer's information rents. It is a realistic assumption if the issuer designs the security to raise liquidity during crisis periods when liquidity is scarce and the liquidity supplier has a monopoly power.[5]

The liquidity supplier offers a mechanism that specifies the quantity of the security to be traded, $q \in [0, 1]$, and the corresponding transfer to the issuer, $T(q) \in \mathbb{R}^{+}$. The issuer chooses a quantity from this mechanism and has an option not to trade and keep the security. Given a signal realization $z$, the issuer's expected payoff from choosing to trade quantity $q$ in exchange for $T(q)$ equals $\delta\mathbb{E}[X - qF|z] + T(q)$. Without loss of generality, we focus on direct mechanisms $M$ which ask the issuer to truthfully report his type $z \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]$ and induce the allocation $q(\tilde{z}) \in [0, 1]$ and transfer $\tau(\tilde{z}) \geq 0$ as a function of the report $\tilde{z}$. Let $\mathcal{M}$ be the set of all such mechanisms. A mechanism is *incentive compatible* if

$$z \in \arg \max_{\tilde{z} \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]} \left\{\delta\mathbb{E}[X|z] + \tau(\tilde{z}) - \delta q(\tilde{z}) z\right\}, \text{ for all } z \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]. \tag{2}$$

The issuer can always keep the security, and hence his value from trade cannot be below $\delta\mathbb{E}[X|z]$. A mechanism is *individually rational* if

$$\tau(z) - \delta q(z) z \geq 0, \text{ for all } z \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]. \tag{3}$$

Given signal distribution $G^{\varphi}$, the liquidity supplier solves

$$\Pi(G^{\varphi}) \equiv \max_{M \in \mathcal{M}} \pi(M|G^{\varphi}) \text{ subject to conditions (2) and (3),}$$

where $\pi(M|G^{\varphi}) \equiv \int_{\underline{f}^{\varphi}}^{\overline{f}^{\varphi}} (zq(z) - \tau(z))\, \mathrm{d}G^{\varphi}(z)$ is her profit from the mechanism $M$.

By Proposition 1 in Biais and Mariotti (2005), it is optimal for the liquidity supplier to offer a posted price mechanism, for any given $G^{\varphi}$. Specifically, the liquidity supplier is willing to buy the whole issue of security $\varphi$ at a fixed price $p$. Issuer types $z \leq p/\delta$ sell the security, types above $p/\delta$ hold the security. Thus, it is without loss of optimality to focus on posted price mechanisms satisfying $q(z) = \mathbf{1}\{z \leq p/\delta\}$ and $\tau(z) = p \times \mathbf{1}\{z \leq p/\delta\}$ for some

[5]In Section 7, we relax this assumption and show how our results are modified in the extension of the model where the liquidity supplier is competitive in normal times, but monopolistic in the crisis times.

$p$, where $\mathbf{1}\{\cdot\}$ is the indicator function. For any $p \geq \underline{f}^{\varphi}/\delta$, the liquidity supplier's profit then is

$$\pi\left(p|G^{\varphi}\right) \equiv \int_{\underline{f}^{\varphi}}^{p/\delta}(z-p)\,\mathrm{d}G^{\varphi}(z) = \left(\mathbb{E}\left[Z|Z \leq p/\delta\right]-p\right)G^{\varphi}(p/\delta),$$

and the issuer's expected information rent (before he learns his type) equals

$$v\left(p|G^{\varphi}\right) \equiv \int_{\underline{f}^{\varphi}}^{p/\delta}(p-\delta z)\,\mathrm{d}G^{\varphi}(z) = \left(p-\delta\mathbb{E}\left[Z|Z \leq p/\delta\right]\right)G^{\varphi}(p/\delta).$$

For any $G^{\varphi}$, the liquidity supplier's set of optimal posted prices is $p^{*}\left(G^{\varphi}\right) \equiv \arg\max_{p}\pi\left(p|G^{\varphi}\right)$.

We suppose that, when indifferent between several $p \in p^{*}\left(G^{\varphi}\right)$, the liquidity supplier chooses the most preferred price for the issuer. Thus, the issuer's expected payoff from choosing the signal structure $G^{\varphi}$ and security $\varphi$ equals

$$\max_{p\in p^{*}(G^{\varphi})}\int_{\underline{f}^{\varphi}}^{\overline{f}^{\varphi}}\left(\delta\mathbb{E}\left[X|z\right]+(p-\delta z)\,\mathbf{1}\left\{z \leq p/\delta\right\}\right)\mathrm{d}G^{\varphi}(z) = \max_{p\in p^{*}(G^{\varphi})}\delta\mathbb{E}\left[X\right]+v\left(p|G^{\varphi}\right),$$

where the equality follows from the law of iterated expectations. We denote $V\left(G^{\varphi}\right) \equiv \max_{p\in p^{*}(G^{\varphi})} v\left(p|G^{\varphi}\right)$.

Observe that once the signal distribution $G^{\varphi}$ is chosen, the security design $\varphi$ enters into the issuer's and the liquidity supplier's objectives only through $G^{\varphi}$. This is intuitive, because the signal $Z$ summarizes all the relevant private information about the security. However, the security design plays an important role, as it affects the set of admissible signal distribution, $\mathcal{G}^{\varphi}$. This fact creates a non-trivial interconnection between the security and information design.

For a given security design $\varphi$, the issuer's optimal information design program is

$$\mathbf{V}\left(\varphi\right) \equiv \max_{G^{\varphi}\in\mathcal{G}^{\varphi}} V\left(G^{\varphi}\right). \tag{4}$$

Our results about information design do not require any further structure on $S$ or $\varphi$. In fact, we can take the prior distribution of the security payoff, $H^{\varphi}$, as a primitive.

We impose more structure in the security design problem. Following the literature, we suppose that $S = X$ and $\varphi(X)$ is *monotone*, that is, right-continuous and weakly increasing in $X$. We say $\varphi$ satisfies *limited liability* if $\varphi\left(X\right) \in [0, X]$, and *double monotonicity* if both $\varphi\left(X\right)$ and $X - \varphi\left(X\right)$ are both weakly increasing in $X$. These two assumptions are standard in the security design literature. We denote the set of such securities by $\Phi$. The optimal

|  | $F=\underline{x}$ | $F=D$ |
|---|---|---|
| B | $\tau/2$ | 0 |
| G | $(1-\tau)\,2$ | $1/2$ |

(a) Revealing bad news

|  | $F=\underline{x}$ | $F=D$ |
|---|---|---|
| B | $1/2$ | $(1-\tau)/2$ |
| G | 0 | $\tau/2$ |

(b) Revealing good news

*Table 1:* ***Signal distributions***

*Tables describe joint distribution of signals $Z \in \{B, G\}$ and security values $F \in \{\underline{x}, D\}$. Parameter $\tau$ controls the precision of signals with $\tau = 0$ corresponding to uninformative signals and $\tau = 1$ corresponding to perfectly revealing signals.*

security design problem is

$$\max_{\varphi\in\Phi} \mathbf{V}(\varphi)\,. \tag{5}$$

# 3 Simple Example

In this section, we present a simple example, which hints at our main results and highlights the need for a general model.

Suppose $X$ takes two equally likely values $\underline{x}$ and $\overline{x}$, with $\overline{x} > \underline{x}$. Assume $(1-\delta)\underline{x}/2 > \mu-\delta\overline{x}$ meaning that if the issuer perfectly learns $X$ and sells the asset, then the liquidity supplier prefers to target type $\underline{x}$ by offering $p = \delta\underline{x}$ (and trade with probability $1/2$) rather than make a pooling offer $\delta\overline{x}$ and trade with both types. Thus, if the signal is either perfectly revealing or uninformative, the issuer's information rents are zero.

We impose further restrictions. First, we only consider debt securities $\varphi(X) = \min\{X, D\}$, $D \in [\underline{x}, \overline{x}]$. Second, we consider one of two binary signals with values G and B described in Table 1. At the ex-ante design stage, the issuer chooses the debt face value $D$ and the signal precision $\tau$.

**Revealing bad news:** Consider first the signal distribution in Table 1a. Signal B perfectly reveals "bad news" that the security value $F = \underline{x}$, while signal G leads to the posterior probability of $F = D$ equal to $1/(2-\tau)$. The issuer gets positive information rents only if the liquidity supplier prefers the pooling offer $p = \delta\mathbb{E}[\varphi(X)\,|\mathrm{G}]$ to the screening offer $p = \delta\underline{x}$ that is only accepted by B-type (with probability $\tau/2$). Thus, it is necessary that

$$(1-\delta)\,\underline{x}\tau/2 \le \mu^{\varphi} - \delta\mathbb{E}[\varphi(X)\,|\mathrm{G}]\,. \tag{6}$$

In this case, the issuer's expected payoff equals $\delta(\mathbb{E}[\varphi(X)\,|\mathrm{G}] - \mu^{\varphi})$. Since $\mathbb{E}[\varphi(X)\,|\mathrm{G}] = \underline{x}+(D-\underline{x})/(2-\tau)$, making the signal more precise (by increasing $\tau$) increases the issuer's expected payoff but tightens the constraint (6). Thus, optimal $\tau^*(D)$ is the highest $\tau$ that makes

(6) bind (unless $\tau^* = 1$). We can compute explicitly $\tau^*(D) = 2 - 4\delta \left(1 + \sqrt{1 + \frac{8\delta(1-\delta)\underline{x}}{D-\underline{x}}}\right)^{-1}$ and corresponding issuer's expected payoff $\tau^*(D)\delta\left(D - \underline{x}\right) / \left(4 - 2\tau^*(D)\right)$, which is strictly increasing in $D$. We conclude that, in this case, there are no benefits of retention and the issuer sells the whole asset: $D = \overline{x}$. For example, for $\delta = 3/4$, $\underline{x} = 1$, and $\overline{x} = 3$, the issuer's maximal payoff is $\approx 0.41$ attained by selling the whole asset and setting $\tau^* \approx 0.71$.

**Revealing good news:** Consider now the signal distribution in Table 1b. Signal G perfectly reveals "good news" that $F = D$. The issuer gets positive information rents only if the liquidity supplier makes a pooling offer $p = \delta D$ equal to the security value for type $G$. The liquidity supplier prefers to do so rather than make screening offer $\mathbb{E}\left[\varphi\left(X\right)|\mathrm{B}\right]$ accepted with probability $1/2 + \left(1 - \tau\right)/2$ if and only if

$$(1 - \delta)\,\mathbb{E}\left[\varphi\left(X\right)|\mathrm{B}\right](1/2 + (1 - \tau)/2) \leq \mu^{\varphi} - \delta D. \tag{7}$$

Then, the issuer's expected payoff is $\delta\left(D - \mu^{\varphi}\right) = \delta\left(D - \underline{x}\right)/2$ and is independent of the signal precision $\tau$. Increasing informativeness of signal $\tau$ decreases the payoff from making a screening offer $\mathbb{E}\left[\varphi\left(X\right)|\mathrm{B}\right]$, as it lowers both $\mathbb{E}\left[\varphi\left(X\right)|\mathrm{B}\right]$ and probability of its acceptance. Hence, $\tau = 1$ is optimal for any $D$, that is, the issuer perfectly learns the value of security. Plugging it into (7), we get $D \leq \delta\underline{x}/(2\delta - 1)$. Thus, the optimal $D^* = \delta\underline{x}/(2\delta - 1) < \overline{x}$ (by $(1 - \delta)\underline{x}/2 > \mu - \delta\overline{x}$). We conclude that retention is optimal and the optimal signal is perfectly revealing. For example, for $\delta = 3/4$, $\underline{x} = 1$, and $\overline{x} = 3$, the issuer's maximal payoff is 0.1875 attained by issuing debt $D^* \approx 1.5$ and $\tau^* = 1$.

This simple example suggests several potential lessons. First, for some signal distributions, retention of cash flows is suboptimal and the issuer simply sells the whole asset. Second, this is not generally the case. For other signal distributions, the issuer optimally retains asset cash flows. Third, retention is suboptimal when the issuer gets a noisy signal about high valuations of security. These insights are confirmed by our general results. Yet, as we show below, two-signal distributions are generally suboptimal, and it is therefore not clear which security is optimal for the optimal signal distribution. Further, two-signal distributions are too rigid to get insights about which security "shapes" are preferred by the issuer. To study these and other issues we next turn to the general model.

# 4 Information Design

In this section, we fix security $\varphi$ and solve the information design program (4).

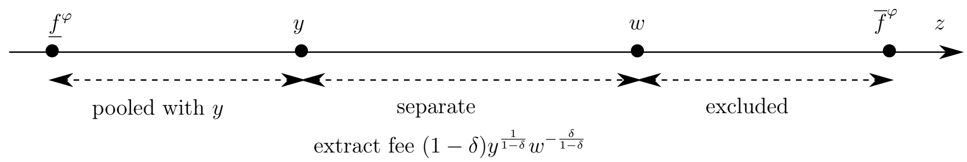



*Figure 2:* ***Allocation in DeMarzo-Duffie mechanisms***

**Characterization of Mean-Preserving Contraction.** We first characterize the mean-preserving contraction ordering in terms of the performance of DD-mechanisms defined as follows. Mechanism $M_{y,w}^{DD} = (q(\cdot), \tau(\cdot))$ is a *DD-mechanism* parametrized by $(y, w), 0 \leq y \leq w$, if

$$q(z) = \begin{cases} 1, \\ (y/z)^{1/(1-\delta)}, \\ 0, \end{cases} \quad \tau(z) = \begin{cases} y - (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)}, & z \leq y, \\ y^{1/(1-\delta)} z^{-\delta/(1-\delta)} - (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)}, & z \in (y, w], \\ 0, & z > w. \end{cases} \tag{8}$$

DD-mechanisms properly generalize the separating equilibrium in DeMarzo and Duffie (1999). To see this, fix a signal distribution $G$ with support $[l, u]$ and consider first the DD-mechanism $M_{l,u}^{DD}$. The quantity schedule in $M_{l,u}^{DD}$ coincides with that in the separating equilibrium of DeMarzo and Duffie (1999) (their Proposition 2), but due to competitive pricing of securities in their paper, $M_{l,u}^{DD}$ has a lower transfer schedule that fully extracts information rents from the highest issuer type $u$. For $y \neq l$ and/or $w \neq u$, DD-mechanisms augment this construction in that (i) all types below $y$ pool and trade the whole security ($q(z) = 1, z \leq y$); (ii) all types in $[y, w]$ separate as in the separating equilibrium with support $[y, w]$; (iii) all types above $w$ are excluded from trade; (iv) the mechanism fully extracts information rents from type $w$ (see Figure 2). DD-mechanisms have the following properties.

**Lemma 1.** *Mechanism $M_{y,w}^{DD}$ is incentive-compatible and individually rational. The expected profit of the liquidity supplier from $M_{y,w}^{DD}$ under $G^{\varphi}$ equals*

$$\pi\left(M_{y,w}^{DD} \middle| G^{\varphi}\right) \equiv (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} G^{\varphi}(w) - \underbrace{(y - \mathbb{E}_{G^{\varphi}}[Z | Z \leq y]) G^{\varphi}(y)}_{\text{lemons term}}, \tag{9}$$

*Further, $\pi\left(M_{y,w}^{DD} \middle| G^{\varphi}\right)$ is weakly less than the expected profit from the optimal posted price mechanisms under $G^{\varphi}$,* $\max_p \pi(p|G^{\varphi})$.

To understand (9), suppose that the liquidity supplier wants to implement the separating equilibrium allocation for types in $[y, w]$ specified in equation (8). She can do so by paying the fair price for quantities traded by these types minus some fixed fee as long as the fee does not violate (3). The information rents in the separating equilibrium are the smallest for the highest separated type, $w$. Thus, the maximal fee that the liquidity supplier can charge while satisfying (3) for all $z \in [y, w]$ equals $f_{y,w} \equiv (1-\delta)\, wq\,(w) = (1-\delta)\, y^{1/(1-\delta)} w^{-\delta/(1-\delta)}$. If distribution $G^{\varphi}$ is supported on $[y, w]$, then $f_{y,w}$ is the expected profit of the liquidity supplier.

Generally, distribution $G^{\varphi}$ can assign positive probability to types below $y$ and above $w$, and we need to make two adjustments to compute the liquidity supplier's profit. First, the liquidity supplier gets the fee $f_{y,w}$ only with probability $G^{\varphi}(w)$. Second, with probability $G^{\varphi}(y)$, the liquidity supplier buys the whole security issue from types $z \leq y$ and gets $\mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]$ in expectation rather than $y$. We thus subtract the term $\left(y - \mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]\right) G^{\varphi}(y)$, which we call the "lemons term." These two adjustments give us (9).

DD-mechanisms are generally suboptimal and are dominated by posted prices, yet, we next show that they are instrumental in describing admissible signal distributions. We denote by $u(G^{\varphi})$ and $l(G^{\varphi})$ the highest and the lowest signal $z$ in the support of $G^{\varphi}$.

**Theorem 1.** *Suppose distributions $G^{\varphi}$ and $\tilde{G}$ on $\mathbb{R}_+$ have the same mean $\mu$. Let $l = l\,(G^{\varphi})$, $\tilde{l} = l(\tilde{G})$, $u = u\,(G^{\varphi})$, and $\tilde{u} = u(\tilde{G})$. Then, $\tilde{G}$ is a mean-preserving contraction of $G^{\varphi}$ if and only if $[\tilde{l}, \tilde{u}] \subseteq [l, u]$ and*

$$\pi\left(M^{DD}_{y,\tilde{u}}\middle|\tilde{G}\right) \geq \pi\left(M^{DD}_{y,\tilde{u}}\middle|G^{\varphi}_{\tilde{u}}\right) \text{ for } y \in [\tilde{l}, \tilde{u}), \tag{10}$$

*where*

$$G^{\varphi}_{\tilde{u}}(z) \equiv G^{\varphi}(z)\mathbf{1}\left\{z < \tilde{u}\right\} + \mathbf{1}\left\{z \geq \tilde{u}\right\}. \tag{11}$$

Theorem 1 establishes that $\tilde{G}$ is a mean-preserving contraction of $G^{\varphi}$ if and only if DD-mechanisms perform better under $\tilde{G}$ than under the properly modified $G^{\varphi}$. To develop intuition, we prove Theorem 1 under the stronger condition that $u = \tilde{u}$, in which case $G^{\varphi}_{\tilde{u}}$ in (11) coincides with $G^{\varphi}$. By Lemma 1, $\pi\left(M^{DD}_{y,\tilde{u}}\middle|G^{\varphi}\right) = (1-\delta)y^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)} - \left(y - \mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]\right) G^{\varphi}(y)$, $y < \tilde{u}$, and an analogous expression holds for $\tilde{G}$. The first term does not vary across the two distributions. Thus, inequalities (10) are equivalent to $\left(y - \mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]\right) G^{\varphi}(y) \geq \left(y - \mathbb{E}_{\tilde{G}}\left[Z|Z \leq y\right]\right) \tilde{G}(y)$ for all $y < \tilde{u}$. Integrating by parts, this expression is equivalent to

$$\int_{-\infty}^{y} G^{\varphi}(z)\mathrm{d}z \geq \int_{-\infty}^{y} \tilde{G}(z)\mathrm{d}z \text{ for all } y < \tilde{u}, \tag{12}$$

Since both distributions have the same mean, this condition is equivalent to $\tilde{G}$ being a mean-preserving contraction of $G^{\varphi}$.

Intuitively, $\tilde{G}$ being a mean-preserving contraction of $G^{\varphi}$ means that the extent of adverse selection is less severe under $\tilde{G}$, which is captured by a smaller lemons term. We argue that this improves the liquidity supplier's profit from DD-mechnamisms under $\tilde{G}$. As we discussed above, the expected profit from DD-mechanisms equals to the expected fee, $f_{y,w}G(w)$, minus the lemons term. If $w = \tilde{u} = u$, then only the lemons term depends on the distribution $G$ (see Lemma 1). Thus, DD-mechanisms bring higher expected profits to the liquidity supplier, when this lemons term is smaller. In words, this occurs when the liquidity supplier does not lose much from ignoring the differences in values between $y$ and types below $y$. In turn, the second-order stochastic dominance orders distributions by the size of the lemons term (c.f. (12)), which gives us Theorem 1.[6]

In the general case of $u \neq \tilde{u}$, the comparison of expected profits from DD-mechanisms depends on the difference $f_{y,u}G^{\varphi}(u) - f_{y,\tilde{u}}G^{\varphi}(\tilde{u})$ along with the difference in lemons terms. Then, we can modify the distribution $G^{\varphi}$ to $G^{\varphi}_{\tilde{u}}$ by shifting the probability mass from $z > \tilde{u}$ to $z = \tilde{u}$ (cf. equation (11)). Distributions $\tilde{G}$ and $G^{\varphi}_{\tilde{u}}$ have the same maximum of the support, and by the argument above, DD-mechanisms yielding higher profits under $\tilde{G}$ than under $G^{\varphi}_{\tilde{u}}$ is equivalent to $\tilde{G}$ being a mean-preserving contraction of $G^{\varphi}$. The full proof of Theorem 1 formalizes this logic.

**Optimal Information Design** We first introduce a class of distributions that play a special role in our analysis. We say that the signal distribution over $[l, u]$ is a *constant-profit distribution* if it takes the form:

$$G^{CP}_{l,u}(z) = \begin{cases} 0 & , z < l, \\ (z/u)^{\delta/(1-\delta)} & , z \in [l, u]. \end{cases} \tag{13}$$

Let $\mathcal{G}^{CP}$ be the class of all constant profit distributions. The distribution $G^{CP}_{l,u}$ has a mass point of $(l/u)^{\delta/(1-\delta)}$ at $l$ and a continuous density on $(l, u]$. In the information design literature (Roesler and Szentes 2017, Kartik and Zhong 2023), this class of distributions have the special feature that they make the liquidity supplier indifferent between offering any posted price. Our next lemma formalizes this idea in our environment.

**Lemma 2.** *Consider any mechanism $M$ satisfying (2) and (3), and such that there is no distortion at the bottom, $q(l) = 1$, and no rents at the top, $\tau(u) = \delta q(u)u$. Let $\pi^{CP}_{l,u}$ be the*

[6] DD-mechanisms reduce the difference in profits across distributions to the differences in lemon terms. Not all mechanisms have this property. For example, posted prices do not work this way. For a price $p$, the liquidity supplier's profit is $(\mathbb{E}_{G^{\varphi}}[Z|Z \leq p/\delta] - p)\, G^{\varphi}(p/\delta) = (\mathbb{E}_{G^{\varphi}}[Z|Z \leq p/\delta] - p/\delta)\, G^{\varphi}(p/\delta) + p\,(1 - 1/\delta)\, G^{\varphi}(p/\delta)$. While we can still single out the lemons term, the remaining term still depends on distribution $G^{\varphi}$.

*maximal profit of the liquidity supplier under $G^{CP}_{l,u}$. Then,*

$$\pi\left(M\middle|G^{CP}_{l,u}\right)=\pi^{CP}_{l,u}\equiv(1-\delta)\,l^{1/(1-\delta)}u^{-\delta/(1-\delta)}. \tag{14}$$

*In particular, $\pi\left(M^{DD}_{y,u}\middle|G^{CP}_{l,u}\right)=\pi^{CP}_{l,u}$ for any $y\in[l,u]$.*

Lemma 2 implies that the liquidity supplier weakly prefers to not screen the issuer and buy the whole issue of the security at the highest price of $\delta u$. This is the most preferred outcome for the issuer as it entails the highest price and trade with certainty. Further, it establishes a useful fact that, for any DD-mechanism $M^{DD}_{y,u}, y\in[l,u]$, the expected profit under distribution $G^{CP}_{l,u}$ is the same and equals $\pi^{CP}_{l,u}$.

We can alternatively parameterize the distribution $G^{CP}_{l,u}$ by its mean $\mu$ and the liquidity supplier's expected profit $\pi$: given $\mu$ and $\pi$, [7]

$$u=(\mu-\pi)/\delta \text{ and } l=(\pi/(1-\delta))^{1-\delta}((\mu-\pi)/\delta)^{\delta}. \tag{15}$$

We write $\tilde{G}^{CP}_{\mu,\pi}$ to refer to a constant-profit distribution parameterized by its mean, $\mu$, and the expected profit of the liquidity supplier, $\pi$. The next lemma establishes that for any signal distribution, we can find an admissible constant-profit distribution that dominates it for the issuer.

**Lemma 3.** *For any security $\varphi$ and any signal distribution $G^{\varphi}\in\mathcal{G}^{\varphi}$, let $\mu$ be the expected security payoff under $G^{\varphi}$ and $\pi=\Pi(G^{\varphi})$ be the expected profit of the liquidity supplier under $G^{\varphi}$. Then, (i) $\tilde{G}^{CP}_{\mu,\pi}$ is a mean-preserving contraction of $G^{\varphi}$; (ii) $\tilde{G}^{CP}_{\mu,\pi}\in\mathcal{G}^{\varphi}$; (iii) $V\left(\tilde{G}^{CP}_{\mu,\pi}\right)=(1-\delta)\,\mu-\pi\geq V(G^{\varphi})$.*

The proof leverages the fact that under constant-profit distributions, DD-mechanisms (typically suboptimal) become optimal screening mechanisms. To illustrate the idea behind the proof, we provide here the argument under the additional assumption that under $G^{\varphi}$, the liquidity supplier offers price $p=\delta u(G^{\varphi})$, and so, $\pi=\mu-\delta u(G^{\varphi})$. To simplify notations, we write $\tilde{G}^{\varphi}$ in place of $\tilde{G}^{CP}_{\mu,\pi}$ and denote $l=l(G^{\varphi}), \tilde{l}=l\left(\tilde{G}^{\varphi}\right), u=u(G^{\varphi})$, and $\tilde{u}=u\left(\tilde{G}^{\varphi}\right)$. By (15), $u=\tilde{u}=(\mu-\pi)/\delta$. Since $\pi$ is the maximal expected profit of the liquidity supplier under $G^{\varphi}$, $\pi\geq\pi\left(M^{DD}_{y,\tilde{u}}\middle|G^{\varphi}\right), y\in[l,\tilde{u})$. In particular, $\pi\geq\pi\left(M^{DD}_{l,\tilde{u}}\middle|G^{\varphi}\right)=(1-\delta)\,l^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)}$, which combined with (15) implies $\tilde{l}=(\pi/(1-\delta))^{1-\delta}\tilde{u}^{\delta}\geq l$, and so, $[\tilde{l},\tilde{u}]\subseteq[l,u]$. Further, by Lemma 2, $\pi$ is the profit from DD-mechanisms $M^{DD}_{y,\tilde{u}}, y\in[\tilde{l},\tilde{u}]$, hence, $\pi\left(M^{DD}_{y,\tilde{u}}\middle|\tilde{G}^{\varphi}\right)=\pi\geq\pi\left(M^{DD}_{y,\tilde{u}}\middle|G^{\varphi}\right)$, $y\in[\tilde{l},\tilde{u})$. Thus, by Theorem 1, $\tilde{G}^{\varphi}$ is a mean-

[7] Since the total surplus is below $(1-\delta)\,\mu$, $\pi\leq(1-\delta)\,\mu$. Hence, (15) imply that $l\leq\mu\leq u$.

preserving contraction of $G^\varphi$, and so, $\tilde{G}^\varphi \in \mathcal{G}^\varphi$. Finally, $V(\tilde{G}^\varphi) = \delta(\tilde{u} - \mu) = V(G^\varphi)$, which completes the proof.

By Lemma 3, it is without loss of optimality to restrict attention to admissible constant-profit distributions, $G^{CP}_{l,u} \in \mathcal{G}^\varphi$. Moreover, under $G^{CP}_{l,u}$, the liquidity supplier optimally offers $\delta u$ that is always accepted, which is also preferred by the issuer whose expected payoff equals $\delta(u - \mu^\varphi)$. Thus, we can find an optimal signal distribution by looking for a constant-profit distribution $G^{CP}_{l,u}$ that solves[8]

$$\max_{l,u:\underline{f}^\varphi \le l \le u \le \overline{f}^\varphi} \left\{ \delta(u - \mu^\varphi) : G^{CP}_{l,u} \in \mathcal{G}^\varphi \right\}. \tag{16}$$

The next theorem characterizes all optimal signal distributions.

**Theorem 2.** *For any security $\varphi$, let $u^\varphi$ be a solution to*

$$\max_{u \in \left[\underline{f}^\varphi, \overline{f}^\varphi\right]} \delta(u - \mu^\varphi) \text{ s.t. } \mu^\varphi - \delta u \ge \pi\left(M^{DD}_{y,u} \middle| H^\varphi_u\right), y \in \left[\underline{f}^\varphi, u\right). \tag{17}$$

*Then, $\mathbf{V}(\varphi)$ equals the value of the program (17) and an admissible signal distribution $G^\varphi \in \mathcal{G}^\varphi$ is optimal for $\varphi$ if and only if (i) trade occurs with probability one under $G^\varphi$; (ii) $u(G^\varphi) = u^\varphi$. Further, $G^{CP}_{l^\varphi,u^\varphi}$ is an optimal signal distribution for $\varphi$, where $l^\varphi = u^\varphi \left(\frac{\mu^\varphi/u^\varphi - \delta}{1-\delta}\right)^{1-\delta}$.*

Theorem 2 uncovers two distinct features of optimal information designs. First, the optimal signal distribution must ensure *perfectly liquidity*, i.e., the full issue of the security is always sold to the liquidity supplier. Second, the issuer prefers not to learn "too optimistic" information about the security value, i.e., his signal $Z$ is below certain $u^\varphi$, which is generally less than the highest payout of the security $\overline{f}^\varphi$. Importantly, the constant-profit distribution $G^{CP}_{l^\varphi,u^\varphi}$ in Theorem 1 is only one solution to the information design problem, and may not be unique.[9] Theorem 1 shows that any signal distribution with highest signal realization $u^\varphi$ that attains perfect liquidity maximizes the issuer's payoff. This means that the issuer does not necessarily need full flexibility in choosing signal distributions.

To see why perfect liquidity is optimal suppose that under some $G^\varphi$, the liquidity supplier offers $p^* < \delta u(G^\varphi)$, which screens out issuer types above $z^* \equiv p^*/\delta$. This means that the

[8] That the information design program reduces to program (16) was first established in Proposition 2 in Kartik and Zhong (2023). We prove it independently using a different technique that relies on special properties of DD-mechanisms, in particular, novel Theorem 1. This technique enables us to establish the connection between the information design and the classical security design with exogenous information in Theorem 2, which is also new to the information design literature.

[9] For example, for any security $\varphi$ that satisfies the constraints in program (17), the prior distribution $H^\varphi$ is an optimal signal distribution. Indeed, $H^\varphi$ is admissible and attains $\delta\left(\overline{f}^\varphi - \mu^\varphi\right)$ the maximal issuer's payoff in program (17). However, in general, the optimal signal distribution is different from the prior distribution $H^\varphi$.

issuer learns private information about types above $z^*$, but does not profit from it. The issuer can alternatively try to lower the highest signal to $u < u(G^\varphi)$ by "redistributing private information" away from the top of the signal distribution until the security becomes perfectly liquid. If the resulting highest signal $u$ is above $z^*$, then the issuer is better off, because he now trades with probability one at a higher price of $\delta u$. The proof of Lemma 3 establishes that such a "redistribution of private information" is always possible by using constant-profit distributions.

However, Theorem 2 says more. It shows that perfect liquidity prevails at *any* optimal signal distribution. To see this, suppose that $G^\varphi \in \mathcal{G}^\varphi$ is optimal for $\varphi$, and let $\pi = \Pi(G^\varphi)$. Then, $V(G^\varphi) \leq (1-\delta)\mu^\varphi - \pi = V(\tilde{G}^{CP}_{\mu^\varphi,\pi})$, with strict inequality if and only if trade occurs with probability less than one under $G^\varphi$. By Lemma 3, $\tilde{G}^{CP}_{\mu^\varphi,\pi}$ is admissible, and so, $V(G^\varphi) = V(\tilde{G}^{CP}_{\mu^\varphi,\pi})$ by optimality of $G^\varphi$. Thus, the security is perfectly liquid under any optimal $G^\varphi$ and the issuer's maximal payoff equals $\delta(u(G^\varphi) - \mu^\varphi)$, and so, all optimal signal distributions must have the same highest signal, call it $u^\varphi$. Leveraging the fact that $\tilde{G}^{CP}_{\mu^\varphi,\pi}$ is admissible, we obtain the characterization of $u^\varphi$ in (17).

Theorem 2 confirms our earlier insight from the example in Section 3 that getting a noisy signal about high valuations is preferable to the issuer, but also shows that binary signals are generally suboptimal. Intuitively, the issuer tries to extract maximal information rents without violating the perfect liquidity of the security. While binary signals might be too coarse as a class, the constant-profit distributions are just right for this task as they keep the liquidity supplier just indifferent between charging any price in their support.

Theorem 2 reveals an interesting parallel between how the issuer designs information and how he designs securities with exogenous information. When the issuer's signal is $X$ itself, Biais and Mariotti (2005) shows that the optimal security solves

$$\max_{\varphi\in\Phi} \delta\left(\overline{f}^\varphi - \mu^\varphi\right) \text{ s.t. } \mu^\varphi - \delta\overline{f}^\varphi \geq \pi\left(\delta y | H^\varphi\right), y \in \left[\underline{f}^\varphi, \overline{f}^\varphi\right]. \tag{18}$$

The constraint states that the optimal security is perfectly liquid: the liquidity supplier prefers offering $\delta\overline{f}^\varphi$ and always buying the security to making a screening posted price offer $\delta y, y \in \left[\underline{f}^\varphi, \overline{f}^\varphi\right)$. Both this constraint and that in (17) take the form of a payoff guarantee to the liquidity supplier. However, the latter is weaker, because DD-mechanism are suboptimal and $\left[\underline{f}^\varphi, u\right) \subset \left[\underline{f}^\varphi, \overline{f}^\varphi\right]$. As we show below, the issuer benefits from both of these relaxations.

Perfect liquidity is attained differently with the design of information versus securities. With exogenous information, the perfect liquidity is attained by designing a debt security. This way, the issuer's signal is bounded at the top by the debt face value and has a point mass there, which discourages the liquidity supplier from screening (Biais and Mariotti 2005).

Similarly, with information design, the issuer's signal is bounded by $u^{\varphi}$, but rather than having a point mass at $\overline{f}^{\varphi}$, it redistributes more flexibly the probability mass from above $u^{\varphi}$ not to create screening incentives for the liquidity supplier.

# 5 Security Design

In this section, we solve the joint information and security design problem. Following the literature, we assume that the contractible information $S$ is the asset payoff $X$ itself.

**Informationally Sensitive Securities.** Our first result formalizes the idea that the issuer weakly prefers more informationally sensitive securities, because they give him "more freedom" in information design.

We say that security $\tilde{\varphi}$ is *more informationally sensitive* than $\varphi$ if there exists $x^* \in [\underline{x}, \overline{x}]$ such that $\tilde{\varphi}(x) - \mu^{\tilde{\varphi}} \leq \varphi(x) - \mu^{\varphi}$ for $x < x^*$ and $\tilde{\varphi}(x) - \mu^{\tilde{\varphi}} \geq \varphi(x) - \mu^{\varphi}$ for $x > x^*$. This definition is similar to that in DeMarzo et al. (2005). In words, once we control for differences in means, $\tilde{\varphi}$ crosses $\varphi$ from below at some $x^*$. Thus, informational sensitivity captures differences in shape of securities. For example, convex securities like call option are more informationally sensitive than standard equity (i.e., $\varphi(X) = \alpha X, \alpha \in [0,1]$), which in turn is more informationally sensitive than concave securities like debt. Following Fagiuoli et al. (1999), $\tilde{\varphi}(X)$ dominates $\varphi(X)$ in the *convex order* (denoted $\tilde{\varphi} \succeq_{\text{cvx}} \varphi$) if $\mathbb{E}_H [\phi(\tilde{\varphi}(X))] \geq \mathbb{E}_H [\phi(\varphi(X))]$ for any convex function $\phi$; whereas $\tilde{\varphi}(X)$ dominates $\varphi(X)$ in the *dilation order* (denoted $\tilde{\varphi} \succeq_{\text{dil}} \varphi$) if $\tilde{\varphi} - \mu^{\tilde{\varphi}} \succeq_{\text{cvx}} \varphi - \mu^{\varphi}$.

**Theorem 3.** *Suppose that securities $\varphi$ and $\tilde{\varphi}$ are monotone and $\tilde{\varphi}$ is more informationally sensitive than $\varphi$. Then, $\tilde{\varphi} \succeq_{\text{dil}} \varphi$ and*

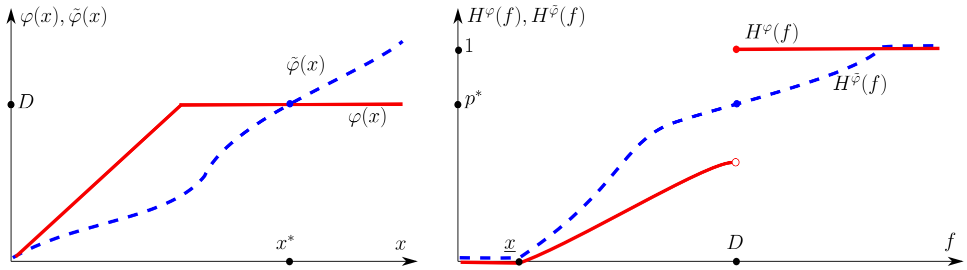
$$\int_{-\infty}^{y} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f - \Delta_{\mu} H^{\tilde{\varphi}}(y + \Delta_{\mu}), \text{ for all } y, \tag{19}$$

*where $\Delta_{\mu} = \mu^{\tilde{\varphi}} - \mu^{\varphi}$. In particular, if $\mu^{\tilde{\varphi}} = \mu^{\varphi}$, then $\tilde{\varphi} \succeq_{\text{cvx}} \varphi$, and*

$$\int_{-\infty}^{y} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f, \text{ for all } y, \tag{20}$$

*and $\mathbf{V}(\tilde{\varphi}) \geq \mathbf{V}(\varphi)$.*

Let us start with the simpler case where $\mu^{\tilde{\varphi}} = \mu^{\varphi}$. Then, a more informationally sensitive security $\tilde{\varphi}$ induces more variability of security payoffs in the sense that $H^{\varphi}$ second-order stochastically dominates $H^{\tilde{\varphi}}$ (inequalities (20)). This gives the issuer more freedom in the




(a) **Securities** $\varphi$ **and** $\tilde{\varphi}$ (b) **C.d.f.s** $H^{\varphi}$ **and** $H^{\tilde{\varphi}}$

*Figure 3:* ***Illustration of Theorem 3***

choice of signal distributions: the issuer can choose all the signal distributions for $\tilde{\varphi}$ as for $\varphi$, but potentially strictly more. Formally, $\mathcal{G}^{\varphi} \subseteq \mathcal{G}^{\tilde{\varphi}}$ (from (1)). Thus, among securities with the same average payoff, the issuer weakly prefers more informationally sensitive ones.

In the general case, when $\varphi$ and $\tilde{\varphi}$ have different expected payoffs, while higher informational sensitivity of security gives more freedom in information design, a security with higher expected payoff constrains it. Indeed, when $H^{\tilde{\varphi}}$ first-order stochastically dominates $H^{\varphi}$, $\int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y} H^{\tilde{\varphi}}(f)\,\mathrm{d}f$ for all $y$, and so, the constraint (1) is tighter. Theorem 3, and in particular, inequalities (19), quantifies the trade-off between these two forces, which will be central in establishing the strict optimality of pure equity below.

Theorem 3 is based on the observation that higher informational sensitivity of $\tilde{\varphi}$ implies that it dominates $\varphi$ in the dilation ordering. Then, by letting $\phi(z) = \max\{y - z, 0\}$ for some $y$, and noting that $\mathbb{E}[\phi(\varphi(X))] = \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f$, after some simplification we obtain inequalities (19). In the case of $\mu^{\tilde{\varphi}} = \mu^{\varphi}$, (19) become (20), which together with $\mu^{\tilde{\varphi}} = \mu^{\varphi}$ imply that $\mathcal{G}^{\varphi} \subseteq \mathcal{G}^{\tilde{\varphi}}$. Thus, constraints in program (4) are weaker for $\tilde{\varphi}$ than for $\varphi$, and so, $\mathbf{V}(\tilde{\varphi}) \geq \mathbf{V}(\varphi)$.

To establish $\tilde{\varphi} \succeq_{\mathrm{dil}} \varphi$, observe first that it is without loss of generality to assume that $\mu^{\tilde{\varphi}} = \mu^{\varphi}$ and show that $\tilde{\varphi} \succeq_{\mathrm{cvx}} \varphi$. Then, by Lemma 2.1 in Fagiuoli et al. (1999) , it is sufficient to show

$$\int_0^p (H^{\varphi})^{-1}(q)\,\mathrm{d}q \geq \int_0^p \left(H^{\tilde{\varphi}}\right)^{-1}(q)\,\mathrm{d}q, \text{ for } p \in [0,1], \tag{21}$$

where $(H^{\varphi})^{-1}(q) \equiv \sup\{f : H^{\varphi}(f) \leq q\}$. The general proof is in the Appendix, and here, we demonstrate the argument for the special case when $\varphi$ is debt with face value $D$ and $H^{\tilde{\varphi}}$ is continuous at $f = D$ (Figure 3a). If $\tilde{\varphi}$ is more informationally sensitive than $\varphi$, then $H^{\tilde{\varphi}}(f) \geq H^{\varphi}(f)$ for $f \leq D$ and $H^{\tilde{\varphi}}(f) < H^{\varphi}(f) = 1$ for $f \in \left(D, \overline{f}^{\tilde{\varphi}}\right)$ (Figure 3b). Since $H^{\tilde{\varphi}}$ is continuous at $f = D$, there is $p^*$ such that $\left(H^{\tilde{\varphi}}\right)^{-1}(q) \leq (H^{\varphi})^{-1}(q)$ for $q \in [0, p^*]$

and $\left(H^{\tilde{\varphi}}\right)^{-1}(q) \geq \left(H^{\varphi}\right)^{-1}(q) = D$ for $q \in [p^*, 1]$. The former implies inequalities (21) for $p \in [0, p^*]$. The latter together with $\mu^{\tilde{\varphi}} = \mu^{\varphi}$ implies that, for $p \in [p^*, 1]$,

$$\begin{aligned}\int_0^p \left(H^{\varphi}\right)^{-1}(q)\,\mathrm{d}q &= \int_0^1 \left(H^{\varphi}\right)^{-1}(q)\,\mathrm{d}q - \int_p^1 \left(H^{\varphi}\right)^{-1}(q)\,\mathrm{d}q \\ &= \underbrace{\mu^{\varphi}}_{=\mu^{\tilde{\varphi}}} - \int_p^1 \underbrace{\left(H^{\varphi}\right)^{-1}(q)}_{\leq \left(H^{\tilde{\varphi}}\right)^{-1}(q)}\mathrm{d}q \geq \mu^{\tilde{\varphi}} - \int_p^1 \left(H^{\tilde{\varphi}}\right)^{-1}(q)\,\mathrm{d}q = \int_0^p \left(H^{\tilde{\varphi}}\right)^{-1}(q)\,\mathrm{d}q,\end{aligned}$$

which establishes (21).

**Optimality of Pure Equity.** We next restrict attention to securities $\varphi \in \Phi$ satisfying limited liability and double-monotonicity and solve program (5). We first present an auxiliary result stating that, if possible, it is weakly optimal to add safe debt to any security.

**Lemma 4.** *For any security $\varphi \in \Phi$ such that $\varphi(X) < \underline{x}$ with positive probability, there is $\varepsilon > 0$ such that $\tilde{\varphi}(X) = \varphi(X) + \varepsilon$ belongs to $\Phi$ and is weakly preferred by the issuer. Further, $\varphi(X) > \underline{x}$ with positive probability for any optimal security $\varphi$.*

Analogous results often appear in the security design literature with exogenous private information (DeMarzo and Duffie 1999, Biais and Mariotti 2005). There, pledging a safe payoff of $\varepsilon$ does not give the liquidity supplier extra incentives to screen the issuer. Hence, by switching to security $\tilde{\varphi}$, the issuer gives up $\varepsilon$ of future asset payoff, which he values at $\delta\varepsilon$, but also increases the security price by $\delta\varepsilon$. This intuition from models with exogenous private information is carried to our model by noticing that if $G^{\varphi} \in \mathcal{G}^{\varphi}$, then a translation of $G^{\varphi}$ by $\varepsilon$ belongs to $\mathcal{G}^{\tilde{\varphi}}$.

We can now prove strict optimality of pure equity.

**Theorem 4.** *Suppose $H$ admits a continuous density on its support. Then, pure equity $\varphi(X) = X$ is the unique optimal security design solving the problem (5).*

Recall that Theorem 3 provides a general comparison of securities without imposing any restrictions on the distribution $H$ and only requiring monotonicity of $\varphi$. When considering securities in $\Phi$, we have additional structure. Within $\Phi$, call options $\varphi(X) = \max\{0, X - K\}, K \in [\underline{x}, \overline{x}]$, are the most informationally sensitive in that holding the expected payoff of the security fixed, the call option crosses from below any security $\varphi \in \Phi$. In a similar way, debt securities $\varphi(X) = \min\{X, D\}, D \in [\underline{x}, \overline{x}]$, are the least informationally sensitive securities in $\Phi$. Further, pure equity, $\varphi(X) = X$, is the most informationally sensitive security among all securities in $\Phi$.

Weak optimality of pure equity follows directly from our previous results. Indeed, by Theorem 3, in search of an optimal security, we can restrict attention to most informationally sensitive securities – call options $\varphi(X) = \max\{0, X - K\}$. Let $K^*$ be the lowest strike price among all optimal call options. By Lemma 4, any call option $\varphi(X) = \max\{0, X - K\}$ with $K > 0$ is weakly dominated by $\varphi(X) + \varepsilon$, which in turn is dominated by a more informationally sensitive call option $\tilde{\varphi}(X) = \max\left\{0, X - \tilde{K}\right\}$ with $\tilde{K} < K$ such that $\mu^{\tilde{\varphi}} = \mu^{\varphi}$ (by Theorem 3). Thus, $K^* = 0$, and so, pure equity (which is the call option with the strike price of 0) is weakly optimal.

The key challenge in establishing Theorem 4 is showing the strict optimality of pure equity. This result is not a priori obvious, as there are countervailing forces. While increasing informational sensitivity relaxes the information design constraints, pledging more cash flows tightens them. To see this more explicitly, let us use Theorem 2 to re-write problem (5):

$$\max_{\varphi \in \Phi, u \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]} \delta\left(u - \mu^{\varphi}\right) \text{ s.t. } \mu^{\varphi} - \delta u \geq \pi\left(M^{DD}_{y,u} \middle| H^{\varphi}_{u}\right), y \in \left[\underline{f}^{\varphi}, u\right). \tag{22}$$

Increasing informational sensitivity relaxes the constraints by lowering the lemons term in $\pi\left(M^{DD}_{y,u} \middle| H^{\varphi}_{u}\right)$ (cf. equation (9)). In turn, pledging more cash flows increases the liquidity supplier's payoff through the increase in $\mu^{\varphi}$, which relaxes the constraints in (22). However, it also leads to a reduction in $H^{\varphi}$ due to the FOSD shift of $\varphi(X)$, which increases profits from DD-mechanisms (through the lemons term in equation (9)), and hence, tightens the constraints. Further, pledging more cash flows increases $\mu^{\varphi}$, and hence, lowers the issuer's payoff.

To prove Theorem 4, we show that the issuer can obtain a strict gain by properly changing both the informational sensitivity of the security and pledging more cash flows. Specifically, Theorem 3 and Lemma 4 imply that it is without loss of optimality to focus on securities of the form $\varphi(x) = \underline{x} + \max\{0, x - K\}, K \in [\underline{x}, \overline{x})$, that are combinations of safe debt $\underline{x}$ and a call option with strike price $K$. The next lemma shows that, unless $K = \underline{x}$, we can construct $\tilde{\varphi} \in \Phi$ that is strictly preferred by the issuer. Since $K = \underline{x}$ corresponds to pure equity, we get the desired conclusion of Theorem 4.

**Lemma 5.** *Consider $\varphi(x) = \underline{x} + \max\{0, x - K\}, K \in (\underline{x}, \overline{x})$, and $u \in [\underline{x}, \underline{x} + \overline{x} - K]$ that satisfies*

$$\mu^{\varphi} - \delta u \geq \pi\left(M^{DD}_{y,u} \middle| H^{\varphi}_{u}\right), \ y \in \left[\underline{f}^{\varphi}, u\right). \tag{23}$$

*There exists a more informationally sensitive security $\tilde{\varphi} \in \Phi$ and $\tilde{u} \leq \overline{f}^{\tilde{\varphi}}$ such that $\mu^{\tilde{\varphi}} > \mu^{\varphi}$, $\delta\left(\tilde{u} - \mu^{\tilde{\varphi}}\right) > \delta\left(u - \mu^{\varphi}\right)$, and conditions (23) hold for $\tilde{\varphi}$ and $\tilde{u}$. Thus, $\mathbf{V}(\tilde{\varphi}) > \mathbf{V}(\varphi)$.*

The rough idea behind Lemma 5 is that increasing the informational sensitivity of the

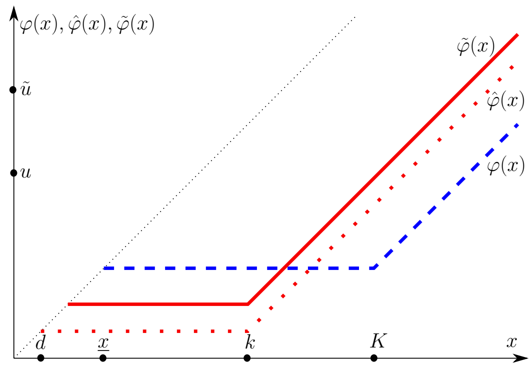



*Figure 4:* ***Illustration of Lemma 5***

security relaxes the constraint in (22) sufficiently so that, even if we also increase the security payoff, we can choose a signal distribution that is strictly preferred by the issuer. Formally,

*Proof.* Fix $\Delta > 0$ and consider security $\hat{\varphi}(x) = d + \max\{0, x - k\}$, where $k = K - \Delta$ and $d = \underline{x} - \int_k^K (1 - H(x))\,\mathrm{d}x$. Security $\hat{\varphi}$ is a combination of the safe debt $d$ and the call option with a lower strike $k < K$. By construction, $\hat{\varphi}$ is more informationally sensitive than $\varphi$ and $\mu^{\hat{\varphi}} = \mu^{\varphi}$. Let us perturb security $\hat{\varphi}$ to $\tilde{\varphi} = \hat{\varphi} + \Delta_\mu$ by increasing its payoff by $\Delta_\mu > 0$ and increase $u$ to $\tilde{u} = u + \Delta_u$, where $\Delta_u = \varepsilon$ and $\Delta_\mu = \varepsilon - \varepsilon^2$ for some $\varepsilon > 0$ (see Figure 4). This way, the issuer's payoff increases by $\delta(\Delta_u - \Delta_\mu) = \delta\varepsilon^2$, and it only remains to verify inequalities (23) for $\tilde{\varphi}$ and $\tilde{u}$. Denote

$$L(y|\varphi, u) \equiv \mu^{\varphi} - \delta u - (1-\delta)\, y^{1/(1-\delta)} u^{-\delta/(1-\delta)} + \left(y - \mathbb{E}_{H^{\varphi}}[Z|Z \leq y]\right) H^{\varphi}(y). \qquad (24)$$

Let $\tilde{y}$ be the point where $L(\cdot|\tilde{\varphi}, \tilde{u})$ attains a global minimum over $[d, \tilde{u}]$. This means that $L_y(\tilde{y}|\tilde{\varphi}, \tilde{u}) = 0$ whenever $\tilde{y} \in (d, \tilde{u})$, whereas $L_y(\tilde{y}|\tilde{\varphi}, \tilde{u}) \leq 0$ if $\tilde{y} = \tilde{u}$, $L_y(\tilde{y}|\tilde{\varphi}, \tilde{u}) \geq 0$ if $\tilde{y} = d$. Consider first the case where $\tilde{y} \in [\underline{x}, \tilde{u})$ and let $\gamma \equiv \tilde{y}/\tilde{u} < 1$. Then, $L_y(\tilde{y}|\tilde{\varphi}, \tilde{u}) = 0$ implies that $H^{\tilde{\varphi}}(\tilde{y}) = (\tilde{y}/\tilde{u})^{\delta/(1-\delta)}$. Theorem 3 implies that, for any $y \geq d$,

$$\int_{-\infty}^{y} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f - \Delta_\mu H^{\tilde{\varphi}}(y + \Delta_\mu) \geq \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f - \Delta_\mu H^{\tilde{\varphi}}(y) + o(\varepsilon),$$

where the last inequality is by $H^{\tilde{\varphi}}(y + \Delta_\mu) = H(y + \Delta_\mu + k - d)$ for $y \geq d$ (by construction of $\tilde{\varphi}$) and $H$ admitting a continuous density on $[\underline{x}, \overline{x}]$. Thus, using (24) and the convexity of

$(1-\delta)\,\tilde{y}^{1/(1-\delta)}u^{-\delta/(1-\delta)}$ in $u$,

$$\begin{aligned} L(\tilde{y}|\tilde{\varphi},\tilde{u}) - L(\tilde{y}|\varphi,u) &= \underbrace{\Delta_\mu - \delta\Delta_u}_{=(1-\delta)\varepsilon+o(\varepsilon)} \underbrace{-(1-\delta)\tilde{y}^{1/(1-\delta)}\left(\tilde{u}^{-\delta/(1-\delta)} - u^{-\delta/(1-\delta)}\right)}_{\geq\varepsilon\delta(\tilde{y}/\tilde{u})^{1/(1-\delta)}} \\ &\quad + \underbrace{\int_{-\infty}^{\tilde{y}} \left(H^{\tilde{\varphi}}(f) - H^{\varphi}(f)\right)\mathrm{d}f}_{\geq -\varepsilon H^{\tilde{\varphi}}(\tilde{y})+o(\varepsilon)} \\ &\geq \varepsilon\left(1-\delta+\delta\,(\tilde{y}/\tilde{u})^{1/(1-\delta)} - H^{\tilde{\varphi}}(\tilde{y})\right) + o(\varepsilon) \\ &= \varepsilon\left(1-\gamma^{\delta/(1-\delta)} - \delta\left(1-\gamma^{1/(1-\delta)}\right)\right) + o(\varepsilon)\,. \end{aligned}$$

Since function $1 - t^{\delta/(1-\delta)} - \delta\left(1 - t^{1/(1-\delta)}\right)$ is strictly decreasing for $t < 1$ and equals 0 at $t = 1$, $1 - \gamma^{\delta/(1-\delta)} - \delta\left(1-\gamma^{1/(1-\delta)}\right) > 0$. Thus, by choosing $\varepsilon$ sufficiently small, we conclude that $L(\tilde{y}|\tilde{\varphi},\tilde{u}) \geq L(\tilde{y}|\varphi,u) \geq 0$. Finally, in Appendix, we consider the remaining cases $\tilde{y} = \tilde{u}$ and $\tilde{y} \in [d, \underline{x})$, which completes the proof. □

**Discussion.** We postpone the discussion of empirical implications till Section 8, and here, we focus on purely theoretical issues. Theorem 4 is in stark contrast to classical results in the literature. The literature on markets with asymmetric information, starting from Leland and Pyle (1977), emphasizes the importance of cash flow retention as a credible signal of asset quality. The security design literature, which is described in the introduction, robustly establishes that debt is often the optimal form of retention. Most strikingly, Biais and Mariotti (2005) shows that, in our setup, if the issuer perfectly learns $X$ at $t = 1$, then for any distribution of cash flows $H$, a debt security is optimal within $\Phi$. Note that there is no contradiction between these two results, because pure equity is a special case of debt with face value $\overline{x}$. The difference is that in Biais and Mariotti (2005) pure equity arises only under special conditions on $H$, while with optimal information design, pure equity is strictly optimal for any $H$.

The standard intuition is that debt serves as a commitment device for the issuer not to take advantage of their future private information when trading with the liquidity supplier. A debt security pays a fixed face value whenever possible and offers maximal downside protection when cash flows are below the face value. In other words, debt is not sensitive to the issuer's private information most of the times, and when it is, the liquidity supplier receives the maximal payout feasible. This insensitivity of debt to private information is crucial in mitigating the lemons' problem and increasing its liquidity.

In contrast, when the issuer can optimally design the signal distribution alongside the payoff structure of the security, the informational insensitivity of debt is no longer necessary.

In fact, any form of retention becomes strictly suboptimal. This is because information design already commits the issuer not to learn too optimistic information about the security and guarantees its perfect liquidity, making security design redundant for these purposes. In turn, more informationally sensitive securities give the issuer more flexibility in information design. If the issuer retains cash flows by offering some security, there is always room to increase its informational sensitivity, which, in conjunction with pledging more cash flows to the security, leads to a strict improvement for the issuer. As a result, the optimal security is pure equity.

# 6 Optimality of Debt under Liquidity Requirements

In this section, we microfound debt based on external liquidity requirements on securities.

**Liquidity Requirements.** Suppose the issuer can only offer securities satisfying two additional external liquidity requirements: (i) the whole security is always sold at $t = 1$; (ii) for a fixed $\rho \in [0, \delta]$, the security price satisfies $p \geq \rho \overline{f}^{\varphi}$.[10]

Such liquidity requirements capture shareholder or regulatory oversight often encountered in practice. The corporation's shareholders (or board members representing them) can be concerned that insiders sell securities at a large discount. If they believe that the security price is much lower than the true value, say below $\rho\varphi(x)$, they might block the sale. If shareholders do not have the insiders' private information, they can impose the floor on the price $\rho \overline{f}^{\varphi}$, which guarantees that the security is never sold below a fraction $\rho$ of its true value.

Another context in which these liquidity requirements naturally arise is the design of mortgage-backed securities or collateralized loan obligations. We need to modify the model in the spirit of "learning-by-holding" in Plantin (2009). Suppose that at $t = 0$, the issuer designs the security and the information that will be privately revealed to the security holder, say a bank, at $t = 1$. For example, investors in CLOs and MBSs receive proprietary information about the asset pool and its performance from the asset-pool manager and underwriter.[11] In period 1, the bank observes signal $Z$ and trades the security if hit by a liquidity shock. If the bank is competitive, then the issuer can extract all information rents, $\mathbf{V}(\varphi)$.

If the bank is subject to liquidity requirements imposed by regulators, it might have a strong preference for high-quality liquid assets. For example, Basel III qualifies assets as such

[10] Observe that $\rho > \delta$ is not sustainable, as offering $\delta \overline{f}^{\varphi}$ dominates any price $p > \delta \overline{f}^{\varphi}$ for the liquidity supplier.

[11] Under Regulation AB, the SEC imposes disclosure requirements for asset-backed securities offerings. Please refer to https://www.sec.gov/corpfin/divisionscorpfinguidanceregulation-ab-interpshtm.

if they can be liquidated within a short period of time with no significant loss of value. In the context of our model, this translates into the ability of banks to always sell the security (irrespective of the realization of $Z$) and the floor on the price. By requiring $p \geq \rho\overline{f}^{\varphi}$, the regulator can guarantee that the maximal haircut on the true value of security is at most $1-\rho$ without knowing the bank's private information or having to trust bank's reporting. For instance, according to Basel III, banks should be able to liquidate level-2 assets over a 30-day period with maximal decline in price of 10%, which corresponds to $\rho = 90\%$.

The definition of liquidity requirements above makes the mapping into applications more apparent. The next lemma presents a more operational condition. It also shows that the liquidity requirements impose non-trivial joint restrictions on security and information design.

**Lemma 6.** *Security $\varphi$ satisfies liquidity requirements if and only if $u^{\varphi} \geq (\rho/\delta)\overline{f}^{\varphi}$.*

**Optimality of Debt.** Let us consider the simpler case of $\rho = \delta$. When $\rho = \delta$, Lemma 6 together with $u^{\varphi} \leq \overline{f}^{\varphi}$ implies that the security design program under liquidity requirements becomes

$$\max_{\varphi \in \Phi} \mathbf{V}(\varphi) \text{ subject to } u^{\varphi} = \overline{f}^{\varphi}. \tag{25}$$

**Theorem 5.** *Suppose that $H$ admits a continuous positive density on its support. There is a debt security $\varphi^*(X) \equiv \min\{X, D^*\}$ for some $D^*$ that solves the program (25).*

In practice, securities backed by mortgages or consumer/business loans are structured as tranches of the underlying asset pool. These securities are sold by originators to various investors, including banks, pension funds, and insurance companies, who face much higher regulatory scrutiny over liquidity of their holdings compared to investment funds or multidivisional firms. As predicted by Theorems 4 and 5, the former investors typically hold asset-backed securities structured as debt, while the latter often liquidate their assets to raise liquidity.

Theorem 5 provides a regulation-induced theory of debt that differs from classical papers on security design. Whereas the classical literature views debt as optimal, Theorem 4 suggests that debt arises from restrictions on information and security design imposed by the issuer, such as external liquidity requirements from regulators or shareholders. In practice, these restrictions may be in place to mitigate moral hazard on the issuer's side. However, our theory suggests that such restrictions come at a cost to the issuer, as they prevent optimal surplus extraction.

The optimality of debt under liquidity requirements can be intuitively understood as follows. According to Theorem 2, when the issuer has full freedom in information design, he will not receive overly positive signals about the security value. Specifically, the highest signal $u^{\varphi}$ is generally below the highest security payoff $\overline{f}^{\varphi}$. However, since the liquidity requirement mandates a price of $\delta\overline{f}^{\varphi}$ (when $\rho = \delta$), the signal $u^{\varphi}$ must be equal to $\overline{f}^{\varphi}$ to satisfy this requirement. Indeed, the liquidity supplier would find it suboptimal to offer prices above $\delta u^{\varphi}$. Thus, the liquidity requirement imposes non-trivial restrictions on the information design, which may disqualify certain securities, particularly pure equity. In such an environment, Theorem 5 establishes that the issuer finds it optimal to take advantage of the informational insensitivity of debt.

More formally, the proof (outlined below) explores the parallel between security design with exogenous information and our problem with the liquidity requirements. By Theorem 2, program (25) is equivalent to program (22) with $u = \overline{f}^{\varphi}$. We can strengthen the constraint in (22) to include $y = \overline{f}^{\varphi}$, as it trivially holds for $y = \overline{f}^{\varphi}$ (by Lemma 1). Thus, (25) is equivalent to

$$\max_{\varphi\in\Phi} \delta\left(\overline{f}^{\varphi} - \mu^{\varphi}\right) \text{ s.t. } \mu^{\varphi} - \delta\overline{f}^{\varphi} \geq \pi\left(M^{DD}_{y,\overline{f}^{\varphi}} \middle| H^{\varphi}\right), y \in \left[\underline{f}^{\varphi}, \overline{f}^{\varphi}\right]. \tag{26}$$

This program is similar to program (18), which states the security design problem with exogenous information, with the difference that the constraint is weaker in (26), because it contains suboptimal DD-mechanisms rather than optimal posted price mechanisms in the right-hand side. Biais and Mariotti (2005) uses optimal control tools to solve program (18). In our analysis, we adapt their approach to solve program (26), which however requires some care due to the more complicated constraint structure.

While debt securities solve both program (18) and (26), the face values $D^*$ and $D^{BM}$ and the program values $V^*$ and $V^{BM}$, respectively, are generally different.

**Proposition 1.** *It holds $D^* \geq D^{BM}$ and $V^* \geq V^{BM}$. Further, $V^* > V^{BM}$ if and only if $D^* > D^{BM}$.*

Proposition 1 reveals that, despite the restrictions imposed by the liquidity requirements, the issuer still gains from the possibility of choosing the signal distribution. The optimal signal distribution is generally more complex than simply learning cash flows.[12] In the latter case, the issuer's signal is distributed according to the prior $H^{\varphi}$, and for debt securities, this means that there is an atom at the face value $D$ and a continuous distribution below $D$. In the case of optimal signal distribution, constant-profit distributions $G^{CP}_{l,u}$ are without loss of optimality. Such distributions have an atom at the bottom $l$, but the distribution is smooth

[12] This result shows that not all insights obtained from the simple example in Section 3 are general. Under the signal distribution in Table 1b, the issuer chooses debt and perfectly learns the cash flows, which is not generally the case under the optimal information design.

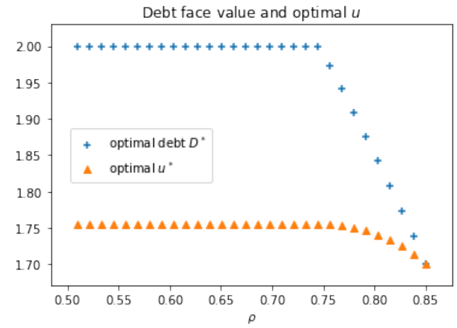



*Figure 5:* ***Optimal security for various*** $\rho$

***Note:*** *The distribution of cash flows is* $H(x) = x - 1, x \in [1,2]$ *and* $\delta = 0.85$. *The figure depicts the optimal security, which is debt with face value* $D^*$, *and the corresponding optimal* $u^*$ *as functions of parameter* $\rho$.

for $z \in (l, D)$. Thus, under the liquidity requirements, the issuer uses non-trivial design of both payoff and signal distributions when designing the security.

We analyze the general case $\rho \leq \delta$ in the Online Appendix, and here, summarize the main finding. We show that debt is optimal when the liquidity requirements are sufficiently stringent ($\rho$ is high), and pure equity is optimal when they are not binding. As an illustration, Figure 5 depicts the optimal security for different $\rho$'s in the uniform example. For high $\rho$'s, the constraint $u^{\varphi} \geq (\rho/\delta)\overline{f}^{\varphi}$ is binding and the optimal security is debt with face value $D_{\rho}$ that is weakly decreasing in $\rho$. For low $\rho$'s, the constraint is not binding, and the optimal security is pure equity (that is, $D_{\rho} = \overline{x}$).

**Proof of Theorem 5.** We next sketch the proof of Theorem 5 (see Appendix for the full proof). By Lemma 1, we can re-write program (26) more explicitly as

$$\max_{\varphi \in \Phi} \delta \left( \varphi(\overline{x}) - \int_{\underline{x}}^{\overline{x}} \varphi(x) \mathrm{d}H(x) \right) \text{ s.t. } C(\tilde{x}, \varphi) \geq 0, \tilde{x} \in [\underline{x}, \overline{x}], \text{ where}$$

$$C(\tilde{x}, \varphi) \equiv \int_{\underline{x}}^{\overline{x}} (\varphi(x) - \delta\varphi(\overline{x})) \mathrm{d}H(x) - (1-\delta)\, \varphi(\tilde{x})^{\frac{1}{1-\delta}} \varphi(\overline{x})^{-\frac{\delta}{1-\delta}} + \int_{\underline{x}}^{\tilde{x}} (\varphi(\tilde{x}) - \varphi(x)) \mathrm{d}H(x), \tag{27}$$

We first restrict (27) to debt securities $\varphi(X) = \min\{X, D\}, D \in [\underline{x}, \overline{x}]$, and solve

$$\max_{D \in [\underline{x}, \overline{x}]} \delta \int_{\underline{x}}^{D} (D - x)\, \mathrm{d}H(x) \text{ s.t. } \mathrm{L}(y, D) \geq 0, y \in [\underline{x}, D],$$

$$\text{where } \mathrm{L}(y, D) \equiv D(1-\delta) - \int_{\underline{x}}^{D} (D - x)\, \mathrm{d}H(x) - (1-\delta)\, y^{\frac{1}{1-\delta}} D^{-\frac{\delta}{1-\delta}} + \int_{\underline{x}}^{y} (y - x)\, \mathrm{d}H(x). \tag{28}$$

This allows us to (implicitly) identify binding constraints in (27) and gives us the candidate for the solution. The issuer's payoff in (28) is strictly increasing in $D$, hence, the solution is the highest $D^*$ that satisfies the constraints. Let $\varphi^*(X) \equiv \min\{X, D^*\}$ and $\hat{x}$ be the smallest $y$ at which the constraint in (28) binds for $D^*$ whenever $D^* < \overline{x}$ and let $\hat{x} = \overline{x}$ whenever $D^* = \overline{x}$.

To prove Theorem 5, we verify that debt security $\varphi^*$ is optimal among all $\varphi \in \Phi$. By the argument in Lemma 5 in Biais and Mariotti (2005), to prove that $\varphi^*$ solves (27), it is sufficient to find a distribution function $\Lambda$ (i.e., a non-decreasing and right-continuous function such that $\Lambda(\underline{x}) = 0$) that satisfies

$$\int_{\underline{x}}^{\overline{x}} C(\tilde{x}, \varphi^*) \mathrm{d}\Lambda(\tilde{x}) = 0, \tag{29}$$

$$\mathcal{L}(\varphi^*, \Lambda) \geq \mathcal{L}(\varphi, \Lambda), \varphi \in \Phi, \tag{30}$$

where the Lagrangian is $\mathcal{L}(\varphi, \Lambda) \equiv \delta \int_{\underline{x}}^{\overline{x}} (\varphi(\overline{x}) - \varphi(x)) \mathrm{d}H(x) + \int_{\underline{x}}^{\overline{x}} C(\tilde{x}, \varphi) \mathrm{d}\Lambda(\tilde{x})$. We choose the distribution $\Lambda_\lambda(\tilde{x}) = \lambda \mathbf{1}\{\tilde{x} \geq \hat{x}\}$ parametrized by $\lambda > 0$. By construction of $\hat{x}$, $\Lambda_\lambda$ and $\varphi^*$ satisfy (29). Using integration by parts, we re-write the Lagrangian as

$$\mathcal{L}(\varphi, \Lambda_\lambda) = \int_{\underline{x}}^{\overline{x}} L_\lambda(x, \dot{\varphi}(x)) \mathrm{d}x + \Phi_\lambda(\varphi(\overline{x}), \varphi(\hat{x})), \tag{31}$$

$$\text{where } L_\lambda(x, \dot{\varphi}) \equiv \dot{\varphi}\left(\delta H(x) \mathbf{1}\{x \leq \hat{x}\} + (\delta - \lambda) H(x) \mathbf{1}\{x > \hat{x}\}\right),$$
$$\Phi_\lambda(\varphi(\overline{x}), \varphi(\hat{x})) \equiv \lambda(1-\delta)\left\{\varphi(\overline{x}) - \varphi(\hat{x})^{1/(1-\delta)} \varphi(\overline{x})^{-\delta/(1-\delta)}\right\}.$$

Maximizing $\mathcal{L}(\varphi, \Lambda_\lambda)$ boils down to solving an optimal control problem with control $\dot{\varphi} \in [0, 1]$. In the Appendix, we use Pontryagin's Maximum principle to solve this problem for any fixed $\lambda$. We then construct $\lambda$ such that $\varphi^*$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ over $\varphi \in \Phi$, and so, it indeed solves the program (27).

# 7 Extensions

**Imperfectly Competitive Liquidity Suppliers.** We first relax the assumption that the liquidity supplier is monopolistic. Suppose there are two states of the world: high-liquidity state $\omega_H$ and low-liquidity state $\omega_L$. We suppose that both security and information design can be conditioned on $\omega$, that is, the issuer chooses at $t = 0$ two securities and two signal distributions, $(\varphi_H, G^{\varphi_H})$ and $(\varphi_L, G^{\varphi_L})$. In state $\omega_L$, the liquidity supply is scarce and there is a single monopolistic liquidity suppliers as in the baseline model. In state $\omega_H$, there are competitive liquidity suppliers and the issuer chooses a trading mechanism $M$ that maximizes his payoff subject to liquidity suppliers breaking even.[13] Formally, he offers the trading mechanism $M$ in state $\omega_H$ that solves

$$\max_{M\in\mathcal{M}} \int_{\underline{f}^{\varphi_H}}^{\overline{f}^{\varphi_H}} (zq(z) - \tau(z))\mathrm{d}G^{\varphi_H}(z)$$

$$\text{subject to conditions (2) and (3),}$$

$$\pi\left(M|G^{\varphi_H}\right) \geq 0. \tag{32}$$

The analysis is unchanged in the low-liquidity state. In state $\omega_H$, the maximal surplus from trading security $\varphi^H$ is $(1-\delta)\,\mu^{\varphi_H}$. If the issuer chooses to be uninformed about the security value and offers price $\mu^{\varphi_H}$, then the liquidity supplier gets payoff of 0, and so, it satisfies the constraint (32). This allows the issuer to extract the whole surplus from trade of any security. Therefore, in state $\omega_H$, it is optimal for the issuer to offer pure equity and trade it at price $\mu$, hence, fully extracting trade surplus.

**Proposition 2.** *The optimal security design is pure equity $\varphi_E(X) = X$ in both states $\omega_H$ and $\omega_L$. It is optimal for the issuer to choose the signal distribution described in Theorem 2 in state $\omega_L$ and to receive uninformative signal in state $\omega_H$. The security price is $\delta u^{\varphi_E}$ in state $\omega_L$ and $\mu^{\varphi_E}$ in state $\omega_H$.*

While the security design is pure equity in both high- and low-liquidity states, the signal distribution differs across states. In the high-liquidity state, the issuer chooses to be ignorant and has no informational advantage over liquidity suppliers. As a result, the liquidity suppliers bid the price up to the ex-ante value of the security, $\mu^{\varphi_E}$, and the issuer captures the whole surplus from trade. In the low-liquidity state, the issuer chooses a non-trivial signal structure, which allows him to capture part of the gains from trade even though the liquidity supplier has monopolistic power. This result is in contrast to Biais and Mariotti (2005) showing that, in the absence of information design, debt is optimal in both competitive and monopolistic setting and the face value of debt is sensitive to the degree of competition.

**Target Issuance Revenue.** One broad implication of our analysis is that optimal securities are shaped by external restrictions on information and security design. Pure equity arises when these restrictions do not contrain the issuer in a consequential way, while debt is a product of exogeneous liquidity requirements. Another security commonly used in practice is call-option (or equivalently, warrant or levered equity): $\varphi(X) = \max\{X - K, 0\}, K \geq 0$. We next show that call-options arise as constrained optima when the issuer needs to raise

[13] We can also allow cash flow distribution $H(\cdot|\omega)$ to vary across states, which would not change the results.

a fixed amount at $t = 1$. Specifically, we additionally require that, for some parameter $N \in (0, p^E]$, the price of the security satisfies $p^\varphi = N$, where $p^E$ is the price of pure equity under the optimal signal distribution.

Condition $p^\varphi = N$ can arise from a combination of known liquidity needs and agency frictions on the issuer side. For example, a multidivisional firm in anticipation of negative profitability shocks (e.g., Covid-19) needs to meet its current salary, lease, and debt obligations to continue operations. For a short horizon, it is reasonable to assume that there is not much uncertainty about those obligations and their size $N$ is known at $t = 0$. Suppose also that there is an agency problem: whatever money is raised at $t = 1$ in excess of $N$ (which must be spent to keep operations running) is diverted by the issuer's manager as private benefits. In this case, the shareholders would naturally demand that the issuer raises exactly $N$ at $t = 1$ through the security sale, which translates into $p^\varphi = N$.

Condition $p^\varphi = N$ implies $p(G^\varphi) = N/\delta$. Thus, the optimal design problem boils down to raising a fixed amont $p(G^\varphi) = N/\delta$ by pledging as little cash flows as possible. Formally, the issuer solves

$$\min_{\varphi, G^\varphi \in \mathcal{G}^\varphi} \mu^\varphi \text{ s.t. } p(G^\varphi) = N/\delta. \tag{33}$$

Consider any optimal $\varphi$ and corresponding optimal $G^\varphi$. Let $\tilde{\varphi}$ be the call-option with the same expected payoff $\mu^\varphi$. Theorem 3 implies that, since $\tilde{\varphi}$ is more informationally sensitive than $\varphi$, $\mathcal{G}^\varphi \subseteq \mathcal{G}^{\tilde{\varphi}}$, and so, $G^\varphi \in \mathcal{G}^{\tilde{\varphi}}$. This proves the (weak) optimality of call-options.

**Proposition 3.** *There is $K$ such that the call option $\varphi(X) = \max\{X - K, 0\}$ solves the program (33).*

# 8 Empirical Implications

Our focus is on the normative aspect of jointly designing securities and private information. In this section, we explore the positive implications of our theory.

Our theory predicts two most common ways of raising liquidity in practice – selling assets (as an unconstrained optimum) and debt (as a constrained optimum under external liquidity requirements). The underlying microfoundation for these securities differs from that in the classical literature which suggests that retaining assets' cash flows serves as a credible signal of quality in markets with significant information asymmetry. In models with exogenous information, debt is considered the optimal security, and selling the entire asset occurs only in rare cases when information asymmetry is relatively mild.

However, with optimal information design, retention is generally unnecessary, and selling the entire asset is strictly optimal. As a result, our novel empirical prediction is that, even in environments where information asymmetry is a major concern, investors can raise liquidity by selling assets rather than issuing more complex securities. This requires the seller to commit to having noisy private information about high security valuations and more detailed information about low valuations. It is worth noting that maintaining asset liquidity while maximizing information rents may not be the sole purpose of this private information structure. It can serve other purposes or simply reflect technological limitations on information acquisition (see examples below).

In turn, optimal information design can conflict with external liquidity requirements imposed by regulators or shareholders. In environments with high degree of regulatory or shareholder oversight, we predict that institutions will use debt securities instead of asset sales to raise liquidity. We now discuss various contexts in which our model is applicable, and these predictions align with common practices.

**Multidivisional Firms.** Multidivisional firms generally consist of core and periphery divisions. Under this organization structure, periphery divisions receive a great deal of autonomy in both daily operations and short and medium-term strategic planning. The firm's general management maintains a hands-off approach and only launches thorough investigations when a crisis occurs. This organization design serves as a commitment device for the management not to learn granular information about periphery divisions and be more aware of negative news.

Consistent with our theory, despite potentially a high degree of asymmetric information versus outsiders, liquidity-constrained multidivisional firms often divest entire divisions to raise funds (Lang, Poulsen and Stulz 1995, Officer 2007) rather than issue securities backed by division cash flows. Further, Kaplan and Weisbach (1992) and Maksimovic and Phillips (2001) find that parent units usually divest periphery, non-core divisions. A key insight of our analysis is that by not monitoring too closely periphery divisions, firms can maintain the liquidity of these assets. Consistent with this prediction, Schlingemann et al. (2002) find that multidivisional firms divest their divisions in highly-liquid markets, and that, perhaps surprisingly, firms are less likely to divest their worst-performing units but rather tend to divest their most liquid divisions.[14]

[14]Robot maker Boston Dynamics provides an interesting case study. It was bought by Google in 2013 who sold it to Softbank in 2017. In turn, Softbank sold it to Hyundai in 2021 partially in response to a liquidity shock caused by losses in its investment portfolio, such as Uber, WeWork, OneWeb. Throughout the years, Boston Dynamics maintained a high degree of autonomy by keeping the headquarters in Boston and maintaining its own research team. Our theory predicts that, because of this autonomy, the head companies were able to easily raise liquidity by selling it. Importantly, despite the complex nature of the business, the sale did not involve designing complex securities backed by Boston Dynamics' cash flows. *Heater, Brian. 2021. "Hyundai completes deal for controlling interest in Boston Dynamics." Tech Crunch, June 21. https://techcrunch.com/2021/06/21/hyundai-completes-deal-for-controlling-interest-in-boston-dynamics/.*

**Investment Funds.** Our model provides insights into the common practice of investment funds liquidating their assets, even though they may be susceptible to significant information asymmetry, instead of utilizing asset-backed securities as a means to raise liquidity.

In private equity funds, general partners (GPs) oversee investments and secure capital from limited partners (LPs). Despite LPs having access to internal performance reports, their ability to evaluate investment strategies is limited, leading them to delegate decisions to GPs. Our theory suggests that the passive role of LPs enables them to raise liquidity by selling their stakes, whereas GPs face more constraints in this regard. This aligns with the existence of an active secondary-market for LP stakes, where buyers, often funds of funds, provide liquidity to selling LPs impacted by unexpected liquidity needs (Nadauld et al. 2019). Interestingly, there is a segment of collateralized fund obligations issuing highly-rated bonds backed by pools of stakes in private equity funds, but its size remains relatively small. This indicates that the secondary markets for LP stakes are adequately liquid.[15]

Mutual and hedge funds face the possibility of meeting large redemptions, which can lead to the liquidation of less liquid assets like private equity or large blocks of public shares in decentralized markets. While these funds usually hold liquid securities as a safeguard, severe shocks during crises can disrupt this buffer. In such times, buyers in decentralized markets wield considerable market power due to limited liquidity and heightened demand. Except for activist funds with concentrated positions, fund managers oversee numerous firms and have limited knowledge and capabilities to provide effective governance for each company in its portfolio. Consequently, majority of investment funds tend to be passive, prioritizing liquidity needs in the face of shocks. Consistent with our theory, these funds do not issue securities backed by their holdings, opting instead to raise liquidity through portfolio liquidation.

VCs specialize in early-stage financing of startups with a finite life-span of around 12 years after which the fund must to return money to investors. Due to the growth of private equity markets and a recent cooling down of the IPO market, VC-backed startups often prefer to stay private for longer time. This shift makes conventional exit strategies of IPOs or mergers and acquisitions more challenging, leading VCs to liquidate their stakes in startups in an illiquid market for early-stage private equity (Nigro and Stahl 2021, Bian et al. 2022). Despite the significant information asymmetry between VCs and external investors, it is somewhat surprising, according to classical theory, that VCs simply sell their entire stakes without developing more intricate securities structures.

Nevertheless, this aligns with our theory, which stresses the role of information design.

[15] *Wiggins, Kaye. 2022. "Collateralised fund obligations: how private equity securitised itself." Financial Times, November 22. https://www.ft.com/content/e4c4fd61-341e-4f5b-9a46-796fc3bdcb03*

VCs often restrict themselves either contractually or through reputational mechanisms to take a hands-off approach to their investment, wherein they provide financial and operational support of the startup but refrain from interference unless the startup fails to meet predetermined milestones. This approach allows VCs to gain more detailed information when the firm performs poorly, prompting them to investigate the underlying causes. Conversely, as long as the startup remains on track, VCs have limited insight into its potential and day-to-day progress, ensuring that they do not set unrealistically high valuation expectations. This hands-off approach, which contrasts with the governance approach involving intensive monitoring of startups, has become prominent in recent years, with many leading VCs maintaining a founder-friendly reputation (Ewens et al. 2018, Lerner and Nanda 2020).

The qualitative properties of optimal information designs in Theorem 2 – specifically focus on downside risks rather than the upside potential – are in line with accounting principles and risk-management practices. Accounting standard-setters, such as GAAP, recommend the conservatism principle that is widely adopted by investment funds. According to the dictum, financial institutions should record losses as soon as they learn about them, whereas gains are not supposed to be recorded until they are realized (see Ruch and Taylor 2015). Standard risk management involves keeping track of market risk exposure of the investment portfolio and its different parts. While these principles and measures are in place for different purposes, our theory suggests that they also contribute to better liquidity management.

**Signaling with Retention.** A distinct prediction of our theory is that retention of cash flows is suboptimal if the issuer can properly curb his informational advantage. This prediction squares with recent evidence on the market for syndicated loans. Blickle et al. (2020) report that lead arrangers for syndicated loans, who are arguably the most informed investors in loans due to their prominent role in the underwriting process, often sell their entire loan stake to other investors, e.g., collateralized loan obligations, loan mutual funds, insurance companies, pension funds. They also show that reputational concerns seem to be important: lead arrangers of loans that turned sour tend to subsequently lose the market share. While this evidence contradicts the standard theory that highlights retention by the underwriter as a credible signaling device (e.g., Leland and Pyle 1977), it is consistent with our model. Maintaining reputation for focusing on the downside risk in their due diligence rather than the upside potential enables lead arrangers to offload completely their loan stake to institutional investors.

**Regulation-Induced Debt.** Many securities, such as MBSs and CLOs, are structured as debt securities. The traditional theory posits that debt is optimal under exogenous information. An alternative viewpoint is that debt arises from the "regulatory arbitrage:" institutional investors demand debt because their regulators view it as adequately safe and liquid. These two explanations are often presented as contradictory to each other.

Our theory of debt in Section 6 reconciles these view points. Similar to the traditional theory, debt is an optimal security, but only under additional external liquidity requirements. These requirements arise from regulatory or shareholder oversight over the securities holders and are similar in nature to the prudential regulation of banks, pension funds, and insurance companies. Importantly, similar to how they are formulated in practice, our liquidity requirements do not restrict the class of securities, but rather require that adequately liquid securities are sold in a short time without a significant loss of value. That the optimal security is debt comes as a result not an assumption. This result formalizes the regulatory-arbitrage viewpoint: debt allows financial institutions to optimally address their liquidity needs while complying with regulatory requirements. This theory aligns with the widespread presence of MBSs and CLOs marketed and held by heavily regulated entities such as banks, insurance companies, and pension funds. On the other hand, less regulated investment funds, as mentioned earlier, prefer to sell assets to generate liquidity.

Additionally, we diverge from the traditional theory regarding investors' private information about debt securities. In existing models, it is often assumed that investors perfectly learn cash flows before trading, leading to investors believing that debt is risk-free in most scenarios. In contrast, our optimal information design reveals to investors an expected debt value consistently lower than its face value, resulting in a generally positive credit spread recorded by investors. This prediction aligns with the industry's standard practice of marking securities to market value, rather than valuing them at face value on the balance sheet.

**Counter-Cyclicality of Private Information.** Our findings in Section 7 establish a connection between the competitive environment and the presence of private information. When liquidity is abundant and liquidity suppliers are competitive, issuers have no incentive to possess private information, as it would hinder liquidity without benefiting them. However, in periods of scarce liquidity when liquidity suppliers hold significant market power, issuers acquire private information about the downside of securities. This allows them to capture some information rents while preserving the liquidity of their securities. In essence, it is optimal to remain "ignorant" about asset quality during booms but gain sufficient private information during downturns to maintain market liquidity. As a result, our theory predicts a counter-cyclical pattern of private information among financial institutions.

Our prediction aligns with previous theoretical studies that highlight a negative relationship between economic activity and the extent of asymmetric information (e.g., Gorton and

Ordonez 2014, Fishman and Parker 2015). Nonetheless, our findings diverge by attributing the correlation to shifts in investors' bargaining power prompted by fluctuations in economic activity, rather than external shocks impacting asset quality.

# 9 Conclusion

We study the problem of joint information and security design. Optimal information design involves the issuer refraining from obtaining overly positive information about the security value while ensuring perfect liquidity of the security. Contrary to common intuition, when the issuer optimally selects his signal distribution, the optimal security is pure equity, the most sensitive to information. Additionally, we propose a theory linking regulatory liquidity requirements to the prevalence of debt securities.

One broad takeaway from our analysis is that the security design is shaped by external/institutional/technological restrictions on the joint information and security design. We focus on the classical question of what is the optimal shape of the security. In reality, there are many other features of securities that involve design of both payoffs and private information. For example, pooling and tranching commonly used in mortgage-backed securities and collateralized loan obligations; convertibility features often introduced in debt securities; downside protection common in startup equity contracts. We believe that our approach can be instrumental and fruitful in shedding light on these and other contractual features.

# References


Asriyan, V. and Vanasco, V.: Forthcoming, Security design in non-exclusive markets with asymmetric information, *The Review of Economic Studies* .

Axelson, U.: 2007, Security design with investor private information, *The Journal of Finance* **62**(6), 2587–2632.

Azarmsa, E. and Cong, L. W.: 2020, Persuasion in relationship finance, *Journal of Financial Economics* **138**(3), 818–837.

Barron, D., Georgiadis, G. and Swinkels, J.: 2020, Optimal contracts with a risk-taking agent, *Theoretical Economics* **15**(2), 715–761.

Bergemann, D., Brooks, B. and Morris, S.: 2015, The limits of price discrimination, *American Economic Review* **105**(3), 921–57.

Biais, B. and Mariotti, T.: 2005, Strategic liquidity supply and security design, *The Review of Economic Studies* **72**(3), 615–649.

Bian, B., Li, Y. and Nigro, C. A.: 2022, Conflicting fiduciary duties and fire sales of vc-backed start-ups, *Available at SSRN 4139724* .

Blickle, K., Fleckenstein, Q., Hillenbrand, S. and Saunders, A.: 2020, The myth of the lead arranger's share, *FRB of New York Staff Report* (922).

Boot, A. W. A. and Thakor, A. V.: 1993, Security design, *Journal of Finance* **48**(4), 1349–1378.

Daley, B., Green, B. and Vanasco, V.: 2020, Securitization, ratings, and credit supply, *The Journal of Finance* **75**(2), 1037–1082.

Daley, B., Green, B. and Vanasco, V.: 2023, Designing securities for scrutiny, *The Review of Financial Studies* .

Dang, T. V., Gorton, G. and Holmström, B.: 2013, The information sensitivity of a security, *Unpublished working paper, Yale University* .

Dellacherie, C. and Meyer, P.: 1982, *Probabilities and Potential B, Theory of Martingales*, Elsevier.

DeMarzo, P. and Duffie, D.: 1999, A liquidity-based model of security design, *Econometrica* **67**(1), 65–99.

DeMarzo, P. M.: 2005, The pooling and tranching of securities: A model of informed intermediation, *Review of Financial Studies* **18**(1), 1–35.

DeMarzo, P. M., Kremer, I. and Skrzypacz, A.: 2005, Bidding with securities: Auctions and security design, *American Economic Review* **95**(4), 936–959.

Ewens, M., Nanda, R. and Rhodes-Kropf, M.: 2018, Cost of experimentation and the evolution of venture capital, *Journal of Financial Economics* **128**(3), 422–442.

Fagiuoli, E., Pellerey, F. and Shaked, M.: 1999, A characterization of the dilation order and its applications, *Statistical papers* **40**(4), 393–406.

Fishman, M. J. and Parker, J. A.: 2015, Valuation, adverse selection, and market collapses, *The Review of Financial Studies* **28**(9), 2575–2607.

Fulghieri, P. and Lukin, D.: 2001, Information production, dilution costs, and optimal security design, *Journal of Financial Economics* **61**(1), 3–42.

Glode, V., Opp, C. C. and Zhang, X.: 2018, Voluntary disclosure in bilateral transactions, *Journal of Economic Theory* **175**, 652–688.

Gorton, G. and Ordonez, G.: 2014, Collateral crises, *American Economic Review* **104**(2), 343–378.

Inostroza, N. and Figueroa, N.: 2023, Screening with securities, *Working Paper* .

Kaplan, S. N. and Weisbach, M. S.: 1992, The success of acquisitions: Evidence from divestitures, *The Journal of Finance* **47**(1), 107–138.

Kartik, N. and Zhong, W.: 2023, Lemonade from lemons: Information design and adverse selection.

Lang, L., Poulsen, A. and Stulz, R.: 1995, Asset sales, firm performance, and the agency costs of managerial discretion, *Journal of financial economics* **37**(1), 3–37.

Leland, H. E. and Pyle, D. H.: 1977, Informational asymmetries, financial structure, and financial intermediation, *The Journal of Finance* **32**(2), 371–387.

Lerner, J. and Nanda, R.: 2020, Venture capital's role in financing innovation: What we know and how much we still need to learn, *Journal of Economic Perspectives* **34**(3), 237–61.

Li, Q.: 2022, Security design without verifiable retention, *Journal of Economic Theory* **200**, 105381.

Mahzoon, M., Shourideh, A. and Zetlin-Jones, A.: 2022, Indicator choice in pay-for-performance, *Working Paper* .

Mailath, G. J. and Von Thadden, E.-L.: 2013, Incentive compatibility and differentiability: New results and classic applications, *Journal of Economic Theory* **148**(5), 1841–1861.

Maksimovic, V. and Phillips, G.: 2001, The market for corporate assets: Who engages in mergers and asset sales and are there efficiency gains?, *The Journal of Finance* **56**(6), 2019–2065.

Mas-Colell, A., Whinston, M. D. and Green, J. R.: 1995, *Microeconomic theory*, Vol. 1, Oxford university press New York.

Myers, S. C.: 1984, The capital structure puzzle, *The Journal of Finance* **39**(3), 574–592.

Myers, S. C. and Majluf, N. S.: 1984, Corporate financing and investment decisions when firms have information that investors do not have, *Journal of Financial Economics* **13**(2), 187–221.

Nachman, D. C. and Noe, T. H.: 1994, Optimal design of securities under asymmetric information, *Review of Financial Studies* **7**(1), 1–44.

Nadauld, T. D., Sensoy, B. A., Vorkink, K. and Weisbach, M. S.: 2019, The liquidity cost of private equity investments: Evidence from secondary market transactions, *Journal of Financial Economics* **132**(3), 158–181.

Nigro, C. A. and Stahl, J. R.: 2021, Venture capital-backed firms, unavoidable value-destroying trade sales, and fair value protections, *European Business Organization Law Review* **22**(1), 39–86.

Officer, M. S.: 2007, The price of corporate liquidity: Acquisition discounts for unlisted targets, *Journal of Financial Economics* **83**(3), 571–598.

Plantin, G.: 2009, Learning by holding and liquidity, *The Review of Economic Studies* **76**(1), 395–412.

Roesler, A.-K. and Szentes, B.: 2017, Buyer-optimal learning and monopoly pricing, *American Economic Review* **107**(7), 2072–80.

Ruch, G. W. and Taylor, G.: 2015, Accounting conservatism: A review of the literature, *Journal of Accounting Literature* **34**, 17–38.

Schlingemann, F. P., Stulz, R. M. and Walkling, R. A.: 2002, Divestitures and the liquidity of the market for corporate assets, *Journal of financial Economics* **64**(1), 117–144.

Szydlowski, M.: 2021, Optimal financing and disclosure, *Management Science* **67**(1), 436–454.

Yang, M.: 2020, Optimality of debt under flexible information acquisition, *The Review of Economic Studies* **87**(1), 487–536.

Yang, M. and Zeng, Y.: 2018, Financing entrepreneurial production: security design with flexible information acquisition, *The Review of Financial Studies* **32**(3), 819–863.

Yang, M. and Zeng, Y.: 2019, Financing entrepreneurial production: security design with flexible information acquisition, *The Review of Financial Studies* **32**(3), 819–863.

# Appendix: Omitted Proofs

The proofs of Lemmas 1, 2, 4, 7, and Proposition 1 are either standard or purely technical and are relegated to Online Appendix.

***Proof of Theorem 1.*** Since distributions $\tilde{G}$ and $G^{\varphi}$ have the same mean, we will show that $\tilde{G}$ second-order stochastically dominates $G^{\varphi}$ if and only if $[\tilde{l}, \tilde{u}] \subseteq [l, u]$ and (10) holds.

*Direction* $\Longrightarrow$ *:* Suppose that

$$\int_{-\infty}^{y} G^{\varphi}(z)\,\mathrm{d}z \geq \int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z, \text{ for all } y. \tag{34}$$

Let us first show that $[\tilde{l}, \tilde{u}] \subseteq [l, u]$. If it were that $l > \tilde{l}$, then the inequality (34) would fail for $y \in (\tilde{l}, l)$. If it were that $u < \tilde{u}$, then by the integration by parts, $\int_{-\infty}^{\tilde{u}} G^{\varphi}(z)\,\mathrm{d}z = \int_{-\infty}^{\tilde{u}} \tilde{G}(z)\,\mathrm{d}z = \tilde{u} - \mu$. Since $\tilde{G}(z) < G^{\varphi}(z) = 1$ for $z \in [u, \tilde{u})$, $\int_{-\infty}^{y} G^{\varphi}(z)\,\mathrm{d}z < \int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z,$ for $y \in [u, \tilde{u})$, which contradicts (34). Thus, $[\tilde{l}, \tilde{u}] \subseteq [l, u]$.

For $y \in [\tilde{l}, \tilde{u})$,

$$\begin{aligned}\pi\left(M_{y,\tilde{u}}^{DD}\middle|\tilde{G}\right) &= (1-\delta)y^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)} + \left(\mathbb{E}_{\tilde{G}}\left[Z|Z \leq y\right] - y\right)\tilde{G}(y) \\ &\geq (1-\delta)y^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)} + \left(\mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right] - y\right)G^{\varphi}(y) = \pi\left(M_{y,\tilde{u}}^{DD}\middle|G_{\tilde{u}}^{\varphi}\right),\end{aligned}$$

where the first equality is by Lemma 1 and $\tilde{G}(\tilde{u}) = 1$; the inequality is by integrating by parts (34); the second equality is by the equation (11) and Lemma 1. Thus, we obtain the condition (10) for $y \in [\tilde{l}, \tilde{u})$, which completes the first part of the proof.

*Direction* $\Longleftarrow$ *:* Suppose $[\tilde{l}, \tilde{u}] \subseteq [l, u]$ and inequalities (10) hold. Let us establish inequalities (34). Since $u \geq \tilde{u}$, $\int_{-\infty}^{y} G^{\varphi}(z)\,\mathrm{d}z = \int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z = y - \mu$, for $y \geq u$. Since $\int_{-\infty}^{u} G^{\varphi}(z)\,\mathrm{d}z = \int_{-\infty}^{u} \tilde{G}(z)\,\mathrm{d}z$ and $G^{\varphi}(z) < \tilde{G}(z) = 1$ for $z \in [\tilde{u}, u)$, $\int_{-\infty}^{y} G^{\varphi}(z)\,\mathrm{d}z > \int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z$, for $y \in [\tilde{u}, u)$. Since $l \leq \tilde{l}$, $\int_{-\infty}^{y} G^{\varphi}(z)\,\mathrm{d}z \geq \int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z = 0$, for $y < \tilde{l}$. Finally, for $y \in [\tilde{l}, \tilde{u})$,

$$\underbrace{(1-\delta)y^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)} - \left(y - \mathbb{E}_{\tilde{G}}\left[Z|Z \leq y\right]\right)\tilde{G}(y)}_{=\pi\left(M_{y,\tilde{u}}^{DD}|\tilde{G}\right)} \geq \underbrace{(1-\delta)y^{1/(1-\delta)}\tilde{u}^{-\delta/(1-\delta)} - \left(y - \mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]\right)G^{\varphi}(y)}_{=\pi\left(M_{y,\tilde{u}}^{DD}|G_{\tilde{u}}^{\varphi}\right)}$$

where the inequality is by (10). Thus, we get $\left(y - \mathbb{E}_{G^{\varphi}}\left[Z|Z \leq y\right]\right)G^{\varphi}(y) \geq \left(y - \mathbb{E}_{\tilde{G}}\left[Z|Z \leq y\right]\right)\tilde{G}(y)$. Integrating by parts, this expression is equivalent to inequalities (34), which completes the proof. □

***Proof of Lemma 3***. Let $\tilde{u}$ and $\tilde{l}$ be given by (15). By construction, $G_{\tilde{l},\tilde{u}}^{CP} = \tilde{G}_{\mu,\pi}^{CP}$ and $\mu - \delta\tilde{u} = \pi$. To simplify notations and facilitate application of Theorem 1, we write $\tilde{G}^{\varphi}$ in place of $\tilde{G}_{\mu,\pi}^{CP}$, and denote $l = l(G^{\varphi})$ and $u = u(G^{\varphi})$.

Consider an optimal posted price $p^* \in p^*(G^{\varphi})$ and the corresponding cutoff type $z^* = p^*/\delta \in [l, u]$ such that all types $z \leq z^*$ accept $p^*$. Then, $\pi = \Pi(G^{\varphi}) = (\mu^* - \delta z^*)\,G^{\varphi}(z^*)$, where $\mu^* \equiv \mathbb{E}_{G^{\varphi}}[Z|Z \leq z^*]$. Note that $\mu \geq \mathbb{E}_{G^{\varphi}}[Z|Z \leq z^*] = \mu^*$.

First, we show that $\tilde{u} \geq z^*$. Suppose to contradiction that $\tilde{u} < z^*$. Then, $\pi = (\mu^* - \delta z^*)\,G^{\varphi}(z^*) \leq \mu^* - \delta z^* < \mu^* - \delta\tilde{u} \leq \mu - \delta\tilde{u} = \pi$, where the first inequality is by $\pi \geq 0$ and $G^{\varphi}(z^*) \in [0, 1]$; the second inequality is by $\tilde{u} < z^*$; the third inequality is by $\mu \geq \mu^*$. Thus, we obtain a contradiction. Hence, $\tilde{u} \geq z^*$.

To show the first statement, note that under distribution $G^{\varphi}$, $u$ is the highest issuer type, and so, the offer $p = \delta u$ is accepted by all issuer types. Since $\pi$ is the maximal expected profit of the liquidity supplier under $G^{\varphi}$, it is weakly greater than the profit from offering $p = \delta u$. Hence, $\pi \geq \mu - \delta u$. By construction of $\tilde{u}$, $\mu - \delta\tilde{u} = \pi$. Thus, $u \geq \tilde{u}$.

Define $G_{\tilde{u}}$ as in the equation (11). That $\tilde{u} \geq z^*$ then implies that $\pi = \max_p \pi\left(p|G^\varphi\right) = \max_{p \leq \delta\tilde{u}} \pi\left(p|G^\varphi\right)$. Thus, the optimal posted price under $G^\varphi_{\tilde{u}}$ either coincides with that under $G^\varphi$ or equals $\delta\tilde{u}$. The expected profit of the liquidity supplier equals $\Pi\left(G^\varphi\right) = \pi$ in the former case and $\mu\left(G^\varphi_{\tilde{u}}\right) - \delta\tilde{u}$ in the latter. Hence, $\Pi\left(G^\varphi_{\tilde{u}}\right) = \max\left\{\pi, \mu\left(G^\varphi_{\tilde{u}}\right) - \delta\tilde{u}\right\}$. By construction, $G^\varphi$ first-order stochastically dominates $G^\varphi_{\tilde{u}}$, and so, $\mu\left(G^\varphi_{\tilde{u}}\right) \leq \mu$. Hence, $\mu\left(G^\varphi_{\tilde{u}}\right) - \delta\tilde{u} \leq \mu - \delta\tilde{u} = \pi$, and so, $\Pi\left(G^\varphi_{\tilde{u}}\right) = \pi$.

Since $\pi$ is the maximal expected profit of the liquidity supplier under $G^\varphi_{\tilde{u}}$, $\pi \geq \pi\left(M^{DD}_{y,\tilde{u}}\middle|G^\varphi_{\tilde{u}}\right)$, $y \in [l, \tilde{u})$.By Lemma 1, for $y = l$, this expression implies $\pi \geq (1-\delta)\, l^{1/(1-\delta)} \tilde{u}^{-\delta/(1-\delta)}$. This inequality combined with the equation (15) implies $\tilde{l} = \left(\pi/(1-\delta)\right)^{1-\delta} \tilde{u}^\delta \geq l$. By Lemma 2, $\pi = \pi\left(M^{DD}_{y,\tilde{u}}\middle|\tilde{G}^\varphi\right), y \in [\tilde{l}, \tilde{u})$, and so, $\pi\left(M^{DD}_{y,\tilde{u}}\middle|\tilde{G}^\varphi\right) \geq \pi\left(M^{DD}_{y,\tilde{u}}\middle|G^\varphi_{\tilde{u}}\right)$, for $y \in \left[\tilde{l}, \tilde{u}\right)$. Therefore, by Theorem 1 $\tilde{G}^\varphi$ is a mean-preserving contraction of $G^\varphi$, which proves the first statement. Since $G^\varphi \in \mathcal{G}^\varphi$, we have $\tilde{G}^\varphi \in \mathcal{G}^\varphi$, and so, the second statement is established.

To prove the third statement, under $\tilde{G}^\varphi$, the liquidity supplier posts price $\delta\tilde{u}$, which all types of the issuer accept. Thus, $V\left(\tilde{G}^\varphi\right) = \delta\left(\tilde{u} - \mu\right) = (1-\delta)\mu - \pi$. At the same time, under $G^\varphi$, the surplus is at most $(1-\delta)\mu$ and the liquidity supplier's expected profit is $\pi$. Hence, $V\left(G^\varphi\right) \leq (1-\delta)\mu - \pi$. Thus, $V\left(\tilde{G}^\varphi\right) \geq V\left(G^\varphi\right)$, which proves the last statement. □

***Proof of Theorem 2.*** We first prove that $G^{CP}_{l^\varphi,u^\varphi}$ is an optimal signal distribution for $\varphi$. As we argued in the main text, it is sufficient to show that it solves the program (16). By Lemma 2, under $G^{CP}_{l,u}$, it is optimal for the liquidity supplier to offer $p = \delta u$ and buy the whole issue of the security, which gives the expected payoff of $\delta\left(u - \mu^\varphi\right)$ to the issuer and $\mu^\varphi - \delta u$ to the liquidity supplier. For any admissible $G^{CP}_{l,u}$, $\mathbb{E}_{G^{CP}_{l,u}}[Z] = \mu^\varphi$, and so, by equations (15), $l = \left(\frac{\mu^\varphi - \delta u}{1-\delta}\right)^{1-\delta} u^\delta$. By Theorem 1, constraint $G^{CP}_{l,u} \in \mathcal{G}^\varphi$ is equivalent to the requirement that $[l, u] \subseteq \left[\underline{f}^\varphi, \overline{f}^\varphi\right]$ and that the profit of the liquidity supplier under $G^{CP}_{l,u}$, $\mu^\varphi - \delta u$, is greater than the profit from DD-mechanisms $M^{DD}_{y,u}, y \in [l, u)$, under $H^\varphi_u$.Given $l = \left(\frac{\mu^\varphi - \delta u}{1-\delta}\right)^{1-\delta} u^\delta$, the constraint $l \geq \underline{f}^\varphi$ is equivalent to $\mu^\varphi - \delta u \geq (1-\delta)\left(\underline{f}^\varphi\right)^{1/(1-\delta)} u^{-\delta/(1-\delta)} = \pi\left(M^{DD}_{\underline{f}^\varphi,u}\middle|H^\varphi_u\right)$. Further, for $y \in \left(\underline{f}^\varphi, l\right)$,

$$\begin{aligned}\pi\left(M^{DD}_{y,u}\middle|H^\varphi_u\right) &= (1-\delta)\underbrace{y^{1/(1-\delta)}}_{<l^{1/(1-\delta)}} u^{-\delta/(1-\delta)} - \underbrace{\left(y - \mathbb{E}_{H^\varphi_u}[Z|Z \leq y]\right) H^\varphi_u(y)}_{\geq 0} \\ &< (1-\delta) l^{1/(1-\delta)} u^{-\delta/(1-\delta)} = \mu^\varphi - \delta u,\end{aligned}$$

where the first equality is by Lemma 1; the first inequality is by $y < l$; the second inequality is by $y \geq \mathbb{E}_{H^\varphi_u}[Z|Z \leq y]$; the second equality is by $l = \left(\frac{\mu^\varphi - \delta u}{1-\delta}\right)^{1-\delta} u^\delta$. Thus, inequalities in (17) for $y \in \left(\underline{f}^\varphi, l\right)$ trivially hold, which means that adding the latter does not change the value of program (17). Thus, $G^{CP}_{l^\varphi,u^\varphi}$ that solves program (17) also solves program (16), and so, is optimal for $\varphi$.

We next show that $G^\varphi \in \mathcal{G}^\varphi$ is optimal for $\varphi$ if and only if (i) trade occurs with probability one under $G^\varphi$; (ii) $u\left(G^\varphi\right) = u^\varphi$. To prove sufficiency, suppose that $G^\varphi$ satisfies these two properties. Then, $V\left(G^\varphi\right) = \delta\left(u^\varphi - \mu^\varphi\right) = V\left(G^{CP}_{l^\varphi,u^\varphi}\right)$, and so, by the optimality of $G^{CP}_{l^\varphi,u^\varphi}$, $G^\varphi$ is also optimal for $\varphi$.

Conversely, suppose that $G^\varphi \in \mathcal{G}^\varphi$ is optimal for $\varphi$. Let $\pi = \Pi(G^\varphi)$. Then, $V(G^\varphi) \leq (1-\delta)\mu^\varphi - \pi = V\left(\tilde{G}^{CP}_{\mu^\varphi,\pi}\right)$ and the inequality is strict if and only if trade occurs with probability less than one under $G^\varphi$. By Lemma 3, $\tilde{G}^{CP}_{\mu^\varphi,\pi}$ is admissible, and so, $V(G^\varphi) = V\left(\tilde{G}^{CP}_{\mu^\varphi,\pi}\right)$ by optimality of $G^\varphi$. Thus, trade occurs with probability one under $G^\varphi$ and $V(G^\varphi) = \delta(u(G^\varphi) - \mu^\varphi)$. By Lemma 2, $V\left(\tilde{G}^{CP}_{\mu^\varphi,\pi}\right) = \delta(u^\varphi - \mu^\varphi)$. Therefore, $u(G^\varphi) = u^\varphi$, which completes the proof of necessity. □

***Proof of Theorem 3.*** We first show $\tilde{\varphi} \succeq_{\text{dil}} \varphi$. Note that $\tilde{\varphi} \succeq_{\text{dil}} \varphi$ is equivalent to $\tilde{\varphi}_0 \succeq_{\text{cvx}} \varphi_0$, where $\tilde{\varphi}_0 \equiv \tilde{\varphi} - \mu^{\tilde{\varphi}}$ and $\varphi_0 \equiv \varphi - \mu^\varphi$. Thus, it is without loss of generality to suppose that $\mu^{\tilde{\varphi}} = \mu^\varphi$ and we only need to prove that $\tilde{\varphi} \succeq_{\text{cvx}} \varphi$. For any weakly increasing right-continuous function $\phi(t)$, let us define its inverse $\phi^{-1}(q) \equiv \sup\{t : \phi(t) \leq q\}$ to be the (right-continuous) inverse of $\phi$. We first prove that

$$\int_0^p (H^\varphi)^{-1}(q)\mathrm{d}q \geq \int_0^p \left(H^{\tilde{\varphi}}\right)^{-1}(q)\mathrm{d}q, \text{ for } p \in [0,1], \tag{35}$$

with equality for $p=1$. Since $\tilde{\varphi}$ is more informationally sensitive than $\varphi$, there exists $x^* \in [\underline{x}, \bar{x}]$ such that $\tilde{\varphi}(x) \leq \varphi(x)$ for $x < x^*$ and $\tilde{\varphi}(x) \geq \varphi(x)$ for $x > x^*$. Let $f^* \equiv \inf\{\varphi(x) | x : \tilde{\varphi}(x) \geq \varphi(x)\}$. By monotonicity of securities, we can choose $x^*$ such that $f^* = \varphi(x^*) \leq \tilde{\varphi}(x^*)$. Recalling that $H^\varphi = H \circ \varphi^{-1}$, we have

$$H^{\tilde{\varphi}}(f) = H(\sup\{x : \tilde{\varphi}(x) \leq f\}) \geq H(\sup\{x : \varphi(x) \leq f\}) = H^\varphi(f), \text{ for } f < f^*; \tag{36}$$

$$H^{\tilde{\varphi}}(f) = H(\sup\{x : \tilde{\varphi}(x) \leq f\}) \leq H(\sup\{x : \varphi(x) \leq f\}) = H^\varphi(f), \text{ for } f > f^*. \tag{37}$$

Let $p^* \equiv H^{\tilde{\varphi}}(f^*)$, and $p_* \equiv \lim_{f \uparrow f^*} H^{\tilde{\varphi}}(f)$. Inequalities (36) and (37) imply that

$$(H^\varphi)^{-1}(q) \geq \left(H^{\tilde{\varphi}}\right)^{-1}(q), \text{ for } q \in [0, p_*], \tag{38}$$

$$(H^\varphi)^{-1}(q) = \left(H^{\tilde{\varphi}}\right)^{-1}(q), \text{ for } q \in (p_*, p^*), \tag{39}$$

$$(H^\varphi)^{-1}(q) \leq \left(H^{\tilde{\varphi}}\right)^{-1}(q), \text{ for } q \in [p^*, 1]. \tag{40}$$

Inequalities (38) imply (35) for $p \in [0, p_*]$. Note that, since $\mu^{\tilde{\varphi}} = \mu^\varphi$, we have that for $p = 1$,

$$\int_0^1 (H^\varphi)^{-1}(q)\mathrm{d}q = \int_{\underline{f}^\varphi}^{\overline{f}^\varphi} f\mathrm{d}H^\varphi(f) = \int_{\underline{f}^{\tilde{\varphi}}}^{\overline{f}^{\tilde{\varphi}}} f\mathrm{d}H^{\tilde{\varphi}}(f) = \int_0^1 \left(H^{\tilde{\varphi}}\right)^{-1}(q)\mathrm{d}q,$$

which proves that (35) holds as equality for $p = 1$. Further, inequalities (40) and $\mu^{\tilde{\varphi}} = \mu^\varphi$ imply that for $p \in [p^*, 1)$.

$$\int_0^p (H^\varphi)^{-1}(q)\mathrm{d}q = \underbrace{\mu^\varphi}_{=\mu^{\tilde{\varphi}}} - \int_p^1 \underbrace{(H^\varphi)^{-1}(q)}_{\leq (H^{\tilde{\varphi}})^{-1}(q)}\mathrm{d}q \geq \mu^{\tilde{\varphi}} - \int_p^1 \left(H^{\tilde{\varphi}}\right)^{-1}(q)\mathrm{d}q = \int_0^p \left(H^{\tilde{\varphi}}\right)^{-1}(q)\mathrm{d}q,$$

which is the desired inequality (35). For $p \in [p_*, p^*]$,

$$\int_0^p (H^\varphi)^{-1}(q)\mathrm{d}q = \int_0^{p_*} \underbrace{(H^\varphi)^{-1}(q)}_{\geq (H^{\tilde{\varphi}})^{-1}(q)} \mathrm{d}q + \int_{p_*}^p \underbrace{(H^\varphi)^{-1}(q)}_{=(H^{\tilde{\varphi}})^{-1}(q)} \mathrm{d}q \geq \int_0^p \left(H^{\tilde{\varphi}}\right)^{-1}(q)\mathrm{d}q$$

which is the desired inequality (35). Therefore, we have proven inequalities (35). By Lemma 2.1 in Fagiuoli et al. (1999), inequalities (35) imply that $\tilde{\varphi}(X) \succeq_{\text{cvx}} \varphi(X)$, which is the desired conclusion.

To show (19), $\tilde{\varphi} \succeq_{\text{dil}} \varphi$ implies that $\int_{-\infty}^{+\infty} \phi(f)\,\mathrm{d}H^{\tilde{\varphi}-\mu^{\tilde{\varphi}}}(f) \geq \int_{-\infty}^{+\infty} \phi(f)\,\mathrm{d}H^{\varphi-\mu^\varphi}(f)$, for any convex $\phi$. Using $\phi(z) = \max\{y - z, 0\}$ for some $y$,

$$\int_{-\infty}^{y} H^{\tilde{\varphi}-\mu^{\tilde{\varphi}}}(f)\,\mathrm{d}f = \int_{-\infty}^{+\infty} \phi(f)\,\mathrm{d}H^{\tilde{\varphi}-\mu^{\tilde{\varphi}}}(f) \geq \int_{-\infty}^{+\infty} \phi(f)\,\mathrm{d}H^{\varphi-\mu^\varphi}(f) = \int_{-\infty}^{y} H^{\varphi-\mu^\varphi}(f)\,\mathrm{d}f.$$

Hence, $\int_{-\infty}^{y+\mu^{\tilde{\varphi}}} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y+\mu^\varphi} H^\varphi(f)\,\mathrm{d}f$, or equivalently, $\int_{-\infty}^{y+\Delta_\mu} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \geq \int_{-\infty}^{y} H^\varphi(f)\,\mathrm{d}f$, which implies inequalities (19).

Finally, if $\mu^{\tilde{\varphi}} = \mu^\varphi$, then inequalities (19) become (20), and so, any $\mathcal{G}^\varphi \subseteq \mathcal{G}^{\tilde{\varphi}}$. Since $\mu^{\tilde{\varphi}} = \mu^\varphi$, $\mathbf{V}(\tilde{\varphi}) \geq \mathbf{V}(\varphi)$. □

***Omitted Details of Proof of Lemma 5***. Here, we consider two remaining cases omitted in the main text. First, suppose $\tilde{y} = \tilde{u}$. Then,

$$L(\tilde{y}|\tilde{\varphi}, \tilde{u}) = \mu^{\tilde{\varphi}} - \tilde{u} + \int_d^{\tilde{u}} H^{\tilde{\varphi}}(f)\,\mathrm{d}f = \overline{f}^{\tilde{\varphi}} - \int_d^{\overline{f}^{\tilde{\varphi}}} H^{\tilde{\varphi}}(f)\,\mathrm{d}f - \tilde{u} + \int_d^{\tilde{u}} H^{\tilde{\varphi}}(f)\,\mathrm{d}f = \int_{\tilde{u}}^{\overline{f}^{\tilde{\varphi}}} \left(1 - H^{\tilde{\varphi}}(f)\right)\mathrm{d}f \geq 0.$$

Second, suppose $\tilde{y} \in [d, \underline{x})$. Then,

$$\begin{aligned} L(\tilde{y}|\tilde{\varphi}, \tilde{u}) &= \underbrace{\mu^{\tilde{\varphi}} - \delta\tilde{u}}_{>\mu^\varphi - \delta u} - (1-\delta)\underbrace{\tilde{y}^{1/(1-\delta)}}_{<\underline{x}^{1/(1-\delta)}} \underbrace{\tilde{u}^{-\delta/(1-\delta)}}_{\leq u^{-\delta/(1-\delta)}} + \underbrace{(\tilde{y} - \mathbb{E}_{H^{\tilde{\varphi}}}[Z|Z \leq \tilde{y}])\,H^{\tilde{\varphi}}(\tilde{y})}_{\geq 0} \\ &> \mu^\varphi - \delta u - (1-\delta)\,\underline{x}^{1/(1-\delta)} u^{-\delta/(1-\delta)} = L(\underline{x}|\varphi, u) \geq 0, \end{aligned}$$

where we used $\mu^{\tilde{\varphi}} - \mu^\varphi > \delta(\tilde{u} - u)$ and $\tilde{u} > u$. □

***Proof of Theorem 4***. See the argument in the main text before Lemma 5. □

***Proof of Lemma 6***. The "if" direction is trivial. To prove the "only if" statement, suppose to contradiction that $u^\varphi < (\rho/\delta)\overline{f}^\varphi$ but security $\varphi$ satisfies the liquidity requirements, that is, it is always sold at price $p \geq \rho\overline{f}^\varphi$. The latter implies that the issuer's expected payoff equals $p - \delta\mu^\varphi \geq \rho\overline{f}^\varphi - \delta\mu^\varphi > \delta(u^\varphi - \mu^\varphi)$, which contradicts Theorem 2. □

The proof of Theorem 5 uses the following technical lemma proven in the Online Appendix.

**Lemma 7.** *If* $D^* < \overline{x}$, *then* $1 - \delta - H(D^*) + \delta(\hat{x}/D^*)^{1/(1-\delta)} < 0$.

***Proof of Theorem 5***. The proof is outlined in the main text. Here, we provide omitted details. We argue in the main text that showing that debt $\varphi^*$ solves program (27) boils down to finding $\lambda > 0$ such that $\varphi^*$ maximizes the Lagrangian $\mathcal{L}(\varphi, \Lambda_\lambda)$. Equation (31) for $\mathcal{L}(\varphi, \Lambda_\lambda)$ in the main text is obtained as follows:

$$
\begin{aligned}
\mathcal{L}(\varphi, \Lambda_\lambda) = {} & \delta \int_{\underline{x}}^{\overline{x}} (\varphi(\overline{x}) - \varphi(x))\,\mathrm{d}H(x) \\
& + \lambda \left\{ \int_{\underline{x}}^{\overline{x}} (\varphi(x) - \delta\varphi(\overline{x}))\,\mathrm{d}H(x) - (1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\,\varphi(\overline{x})^{-\delta/(1-\delta)} + \int_{\underline{x}}^{\hat{x}} (\varphi(\hat{x}) - \varphi(x))\,\mathrm{d}H(x) \right\} \\
& = (1-\lambda)\,\delta \int_{\underline{x}}^{\overline{x}} (\varphi(\overline{x}) - \varphi(x))\,\mathrm{d}H(x) + \lambda \int_{\underline{x}}^{\hat{x}} (\varphi(\hat{x}) - \varphi(x))\,\mathrm{d}H(x) \\
& + \lambda \left\{ \int_{\underline{x}}^{\overline{x}} (1-\delta)\,\varphi(x)\,\mathrm{d}H(x) - (1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\,\varphi(\overline{x})^{-\delta/(1-\delta)} \right\} \\
& = (1-\lambda)\,\delta \int_{\underline{x}}^{\overline{x}} \dot{\varphi}(x)\,H(x)\,\mathrm{d}x + \lambda \int_{\underline{x}}^{\hat{x}} \dot{\varphi}(x)\,H(x)\,\mathrm{d}x \\
& + \lambda \left\{ (1-\delta)\,\varphi(\overline{x}) - (1-\delta) \int_{\underline{x}}^{\overline{x}} \dot{\varphi}(x)\,H(x)\,\mathrm{d}x - (1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\,\varphi(\overline{x})^{-\delta/(1-\delta)} \right\} \\
& = \delta \int_{\underline{x}}^{\hat{x}} \dot{\varphi}(x)\,H(x)\,\mathrm{d}x + (\delta - \lambda) \int_{\hat{x}}^{\overline{x}} \dot{\varphi}(x)\,H(x)\,\mathrm{d}x + \lambda(1-\delta) \left\{ \varphi(\overline{x}) - \varphi(\hat{x})^{1/(1-\delta)}\,\varphi(\overline{x})^{-\delta/(1-\delta)} \right\}.
\end{aligned}
$$

For any $\lambda > 0$, we first maximize $\mathcal{L}(\varphi, \Lambda_\lambda)$ over $\varphi \in \Phi$ that are piecewise continuously differentiable. Then, maximizing $\mathcal{L}(\varphi, \Lambda_\lambda)$ boils down to solving an optimal control problem. By double monotonicity of $\varphi$, the control is $\dot{\varphi}(x) \in [0, 1]$. We introduce two state variables: $\varphi(x)$ and an auxiliary state variable $\psi(x)$ satisfying $\dot{\psi}(x) = \dot{\varphi}(x)\mathbf{1}\{x \leq \hat{x}\}$. Thus, $\psi(\overline{x}) = \varphi(\hat{x})$ and we can re-write the terminal value function in (31) in the canonical form $\Phi_\lambda(\varphi(\overline{x}), \varphi(\hat{x})) = \Phi_\lambda(\varphi(\overline{x}), \psi(\overline{x}))$. We have the boundary condition $\psi(\underline{x}) = \varphi(\underline{x}) = \underline{x}$ and free boundary conditions on $\psi$ and $\varphi$ at $\overline{x}$.

The Hamiltonian of this optimal control problem is given by:

$$
\mathcal{H}_\lambda(x, \varphi, \psi, \dot{\varphi}, p, q) = \dot{\varphi}\left\{\delta H(x)\,\mathbf{1}\{x \leq \hat{x}\} + (\delta - \lambda)\,H(x)\,\mathbf{1}\{x > \hat{x}\}\right\} + \dot{\varphi}p + \dot{\varphi}q\mathbf{1}\{x \leq \hat{x}\},
$$

where $p$ and $q$ are costate variables that cannot be equal to zero simultaneously. By Pontryagin's Maximum principle, a necessary condition for $(\varphi_\lambda, \psi_\lambda, \dot{\varphi_\lambda})$ to be optimal is that, for all $x$, $\dot{\varphi}_\lambda(x)$ maximizes $\mathcal{H}_\lambda(x, \varphi_\lambda, \psi_\lambda, \cdot, p_\lambda, q_\lambda)$ for some piecewise continuously differentiable functions $p_\lambda$ and $q_\lambda$ that satisfy

$$
\begin{aligned}
\dot{p}_\lambda(x) &= -\partial \mathcal{H}_\lambda(x, \varphi_\lambda(x), \psi_\lambda(x), \dot{\varphi}_\lambda(x), p_\lambda(x), q_\lambda(x))/\partial\varphi = 0, \\
\dot{q}_\lambda(x) &= -\partial \mathcal{H}_\lambda(x, \varphi_\lambda(x), \psi_\lambda(x), \dot{\varphi}_\lambda(x), p_\lambda(x), q_\lambda(x))/\partial\psi = 0,
\end{aligned}
$$

with boundary conditions

$$p_\lambda(\overline{x}) = \partial\Phi\left(\varphi_\lambda\left(\overline{x}\right), \psi_\lambda\left(\overline{x}\right)\right)/\partial\varphi\left(\overline{x}\right) = \lambda\left(1-\delta\right)\left\{1+\frac{\delta}{1-\delta}\left(\psi_\lambda(\overline{x})/\varphi_\lambda(\overline{x})\right)^{1/(1-\delta)}\right\},$$
$$q_\lambda(\overline{x}) = \partial\Phi\left(\varphi_\lambda\left(\overline{x}\right), \psi_\lambda\left(\overline{x}\right)\right)/\partial\psi\left(\overline{x}\right) = -\lambda\left(\psi_\lambda(\overline{x})/\varphi_\lambda(\overline{x})\right)^{\delta/(1-\delta)}.$$

Thus, using $\psi_\lambda\left(\overline{x}\right) = \varphi_\lambda\left(\hat{x}\right)$, we can solve these differential equations to get that, for all $x \in [\underline{x}, \overline{x}]$,

$$p_\lambda(x) = p_\lambda \equiv \lambda\left(1-\delta\right)\left\{1+\frac{\delta}{1-\delta}\left(\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x})\right)^{1/(1-\delta)}\right\},$$
$$q_\lambda(x) = q_\lambda \equiv -\lambda\left(\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x})\right)^{\delta/(1-\delta)}.$$

Given the linearity of $\mathcal{H}_\lambda$ in $\dot{\varphi}$,

$$\dot{\varphi}_\lambda(x) = \begin{cases} \mathbf{1}\left\{\delta H(x) + p_\lambda + q_\lambda > 0\right\}, & x \leq \hat{x}, \\ \mathbf{1}\left\{\left(\delta-\lambda\right) H(x) + p_\lambda > 0\right\}, & x > \hat{x}. \end{cases} \tag{41}$$

maximizes $\mathcal{H}_\lambda$. Since $\mathcal{H}_\lambda$ is linear in $(\psi, \varphi, \dot{\varphi})$, Mangasarian's sufficiency condition is satisfied, and so, $\dot{\varphi}_\lambda$ is an optimal control and $\varphi_\lambda$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ over all piecewise continuously differentiable functions in $\Phi$. By the Stone-Weierstrass Theorem, this space is dense in $\Phi$ endowed with the sup norm and $\mathcal{L}(\varphi, \Lambda_\lambda)$ is continuous in this topology. Hence, $\varphi_\lambda$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ on $\Phi$.

To complete the proof, we construct $\lambda$ such that $\varphi_\lambda$ coincides with the optimal debt security $\varphi^*$. Observe that $p_\lambda + q_\lambda = \lambda\left(1-\delta+\delta\left(\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x})\right)^{1/(1-\delta)} - \left(\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x})\right)^{\delta/(1-\delta)}\right)$. The right-hand side is decreasing in $\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x})$ and equals to 0 when $\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x}) = 1$.[16] Since $\varphi_\lambda(\hat{x})/\varphi_\lambda(\overline{x}) \leq 1$ (by $\dot{\varphi}_\lambda(x) \geq 0$ for all $x$), $p_\lambda + q_\lambda \geq 0$. Hence, (41) implies that $\dot{\varphi}_\lambda(x) = \mathbf{1}\left\{\delta H(x) + p_\lambda + q_\lambda > 0\right\} = 1$ for $x \leq \hat{x}$. In particular, $\varphi_\lambda(x) = x$ for $x \leq \hat{x}$.

For $x > \hat{x}$, $\dot{\varphi}_\lambda\left(x\right) = \mathbf{1}\left\{\left(\delta-\lambda\right) H\left(x\right) + p_\lambda > 0\right\}$. If $\delta - \lambda \geq 0$, then $\dot{\varphi}_\lambda\left(x\right) = 1$ and $\varphi_\lambda\left(x\right) = x$ for all $x > \hat{x}$. If $\delta - \lambda < 0$, then since $H$ is strictly increasing, the optimal policy takes the form: for some $x^*_\lambda \in [\hat{x}, \overline{x}]$ such that $\left(\lambda-\delta\right) H\left(x^*_\lambda\right) = p_\lambda$,

$$\dot{\varphi}_\lambda\left(x\right) = \mathbf{1}\left\{x < x^*_\lambda\right\} \text{ and } \varphi_\lambda\left(x\right) = \min\left\{x, x^*_\lambda\right\}. \tag{42}$$

To construct $\lambda$, we consider separately two cases depending on whether $D^* < \overline{x}$ or $D^* = \overline{x}$.

*Case 1: $D^* < \overline{x}$.* Let us set

$$\lambda = \frac{\delta H(D^*)}{H(D^*) - (1-\delta) - \delta\left(\hat{x}/D^*\right)^{1/(1-\delta)}}. \tag{43}$$

By Lemma 7, the denominator is positive. Further, since $\delta\left(\hat{x}/D^*\right)^{1/(1-\delta)} > -(1-\delta)\left(1-H\left(D^*\right)\right)$, $\lambda > 1$, and so, $\delta - \lambda < 0$. We need to show that $x^*_\lambda = D^*$. We have that $x^*_\lambda$ solves $\left(\delta-\lambda\right) H\left(x^*_\lambda\right) +$

[16] Indeed, $\frac{\mathrm{d}}{\mathrm{d}\chi}\left(1-\delta+\delta\chi^{1/(1-\delta)} - \chi^{\delta/(1-\delta)}\right) = \frac{\delta}{1-\delta}\chi^{\delta/(1-\delta)}\left(1-\chi^{-1}\right) \leq 0$ for $\chi \leq 1$.

$p_\lambda = 0$, which given $x^*_\lambda \in [\hat{x}, \overline{x}]$, (42), and (43), is equavalent to

$$H\left(x^*_\lambda\right)/H\left(D^*\right) = \left(1 - \delta + \delta\left(\hat{x}/x^*_\lambda\right)^{1/(1-\delta)}\right) / \left(1 - \delta + \delta\left(\hat{x}/D^*\right)^{1/(1-\delta)}\right).$$

The left-hand side is strictly increasing in $x^*_\lambda$ and the right-hand side is strictly decreasing in $x^*_\lambda$. Further, the equality obtains at $x^*_\lambda = D^*$. Thus, $x^*_\lambda = D^*$ is the only possible $x^*_\lambda$, and so, solution (42) indeed coincides with $\varphi^*$ (i.e., debt with face value $D^*$).

*Case 2:* $D^* = \overline{x}$. Then, $\hat{x} = \overline{x}$, and so, $\dot{\varphi}_\lambda(x) = 1$ is optimal for all $x \leq \overline{x} = \hat{x}$. Thus, we can set $\lambda$ to arbitrary positive number, and $\varphi^*$ (i.e., pure equity) is the solution to the optimal control problem. □

# Online Appendix (Not for Publication)

## Omitted Proofs

***Proof of Lemma 1.*** The incentive compatibility for types $z \in [y, w]$ follows from the argument in Mailath and Von Thadden (2013). To check individual rationality for $z \in [y, w]$,

$$\begin{aligned}\tau(z) - \delta z q(z) =& y^{1/(1-\delta)} z^{-\delta/(1-\delta)} - (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} - \delta z (y/z)^{1/(1-\delta)} \\ =& (1-\delta) y^{1/(1-\delta)} \left( z^{-\delta/(1-\delta)} - w^{-\delta/(1-\delta)} \right) \geq 0.\end{aligned}$$

Consider $z > w$. The incentive compatibility for type $w$ implies that $\tau(w) - \delta w q(w) \geq \tau(\tilde{z}) - \delta w q(\tilde{z})$ for all $\tilde{z} \in [y, w]$, and so, given that $q(w) \leq q(\tilde{z})$,$\tau(w) - \tau(\tilde{z}) \geq \delta w (q(w) - q(\tilde{z})) \geq \delta z (q(w) - q(\tilde{z}))$. Thus, type $z$ prefers $(q(w), \tau(w))$ to any other $(q(\tilde{z}), \tau(\tilde{z}))$. Since $\tau(w) - \delta z q(w) < \tau(w) - \delta w q(w) = 0$, $(q(w), \tau(w))$ is not individually rational for type $z > w$, hence, he optimally chooses $(0, 0)$ from the menu.

Consider $z < y$. The incentive compatibility for type $y$ implies that $\tau(y) - \delta y q(y) \geq \tau(\tilde{z}) - \delta y q(\tilde{z})$ for $\tilde{z} \in [y, w]$, and so, given that $q(y) \geq q(\tilde{z})$,$\tau(y) - \tau(\tilde{z}) \geq \delta y (q(y) - q(\tilde{z})) \geq \delta z (q(y) - q(\tilde{z}))$. Thus, type $z$ prefers $(q(z), \tau(z)) = (q(y), \tau(y))$ to any other $(q(\tilde{z}), \tau(\tilde{z}))$, which establishes incentive compatibility for such $z$. Since

$$\begin{aligned}\tau(y) - \delta z q(y) &= y - (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} - \delta z \\ &\geq y - (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} - \delta y \\ &= (1-\delta) y^{1/(1-\delta)} \left( y^{-\delta/(1-\delta)} - w^{-\delta/(1-\delta)} \right) \geq 0,\end{aligned}$$

individual rationality is satisfied for $z < y$.

Given equations (8),

$$\begin{aligned}\pi\left(M^{DD}_{y,w} | G\right) =& \int_l^u (z q(z) - \tau(z)) \, \mathrm{d}G(z) \\ =& \int_l^y \left( z - y + (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} \right) \mathrm{d}G(z) + \int_y^w (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} \mathrm{d}G(z) \\ =& \int_l^y (z - y) \, \mathrm{d}G(z) + (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} G(w) \\ =& \left( \mathbb{E}_G [Z | Z \leq y] - y \right) G(y) + (1-\delta) y^{1/(1-\delta)} w^{-\delta/(1-\delta)} G(w),\end{aligned}$$

which proves (9). Since posted price mechanisms are optimal in the liquidity supplier's screening problem, the profit from the optimal posted price is weakly greater than the profit from any alternative mechanism, including DD-mechanisms, which proves the last statement. □

***Proof of Lemma 2.*** By Lemma 1 in Biais and Mariotti (2005), the information rent of type $z$ equals $r(z) = \delta \int_z^u q(c) \, \mathrm{d}c + r(u)$. Using $r(z) = \tau(z) - \delta q(z) z$ and $r(u) = 0$ (by $\tau(u) = \delta q(u) u$),

$\tau(z) = \delta \int_z^u q(c)\, \mathrm{d}c + \delta q(z)\, z$. Hence,

$$
\begin{aligned}
\pi\left(M \middle| G_{l,u}^{CP}\right) &= \int_l^u \left(zq(z) - \tau(z)\right) \mathrm{d}G_{l,u}^{CP}(z) \\
&= \int_l^u \left((1-\delta)\, zq(z) - \delta \int_z^u q(c)\, \mathrm{d}c\right) \mathrm{d}G_{l,u}^{CP}(z) \\
&= \int_l^u (1-\delta)\, zq(z)\, \mathrm{d}G_{l,u}^{CP}(z) - \int_l^u \delta G_{l,u}^{CP}(z)\, q(z)\, \mathrm{d}z \\
&= (1-\delta)\, lG_{l,u}^{CP}(l) + \int_l^u q(z)\, \delta \left\{(z/u)^{\delta/(1-\delta)} - G_{l,u}^{CP}(z)\right\} \mathrm{d}z \\
&= (1-\delta)\, l^{1/(1-\delta)} u^{-\delta/(1-\delta)},
\end{aligned}
$$

where the third equality is by the integration by parts; the last two equalities are by plugging in $G_{l,u}^{CP}$ from the equation (13). □

***Proof of Lemma 4***. Let $\tilde{x} \equiv \sup\{x : \varphi(x) < \underline{x}\}$. Since $\varphi$ is weakly increasing in $x$, $\varphi(x) < \underline{x}$ for $x < \tilde{x}$. Let $\tilde{\varphi}(x) \equiv \varphi(x) + \varepsilon, \varepsilon > 0$. Since $\varphi \in \Phi$ and $\varphi(x) < \underline{x}$ for $x < \tilde{x}$, we have $\tilde{\varphi} \in \Phi$ for all $\varepsilon$ sufficiently small. By Theorem 2, there is optimal signal distribution for security $\varphi$, $G^{\varphi}$, such that under $G^{\varphi}$, the liquidity supplier allocates to all issuer types by offering $\delta u(G^{\varphi})$. Let $G^{\tilde{\varphi}}(z) \equiv G^{\varphi}(z - \varepsilon)$ for all $z$. Since $G^{\varphi} \in \mathcal{G}^{\varphi}$, $G^{\tilde{\varphi}} \in \mathcal{G}^{\tilde{\varphi}}$. By Lemma 4 in Biais and Mariotti (2005), under $G^{\tilde{\varphi}}$, the liquidity supplier optimally chooses a screening cutoff type that is weakly greater than $u(G^{\varphi}) + \varepsilon = u\left(G^{\tilde{\varphi}}\right)$. Hence, the liquidity supplier finds it optimal under $G^{\tilde{\varphi}}$ to buy from all types and offers $\delta u\left(G^{\tilde{\varphi}}\right) = \delta\left(u(G^{\varphi}) + \varepsilon\right)$. Thus, the issuer's expected payoff under $G^{\tilde{\varphi}}$ equals $\delta\left(u\left(G^{\tilde{\varphi}}\right) - \mu^{\tilde{\varphi}}\right) = \delta\left(u(G^{\varphi}) - \mu^{\varphi}\right)$, which equals the issuer's maximal expected payoff from security $\varphi$. Since the issuer's expected payoff from security $\tilde{\varphi}$ is greater than or equal to his expected payoff under the (not necessarily optimal) signal distribution $G^{\tilde{\varphi}}$, security $\tilde{\varphi}$ brings a weakly higher expected payoff than $\varphi$, which is the desired conclusion. □

***Proof of Lemma*** 7. Re-write the constraints in (28) as

$$
\Psi(D) \equiv \min_{y \in [\underline{x}, D]} \mathrm{L}(y, D) \geq 0. \tag{44}
$$

Since $\mathrm{L}(D, D) = 0$, $\Psi(D) \leq 0$ for all $D$. By Berge's maximum theorem, the solution of (44), $Y(D)$, is compact-valued and upper-hemicontinuous in $D$. Then, $y(D) \equiv \min Y(D)$ is well-defined for any $D$. Note that by the definition of $\hat{x}$, $y(D^*) = \hat{x}$.

Since $\mathrm{L}_y(\underline{x}, D) = -(\underline{x}/D)^{\delta/(1-\delta)} < 0$ (by $\underline{x} > 0$), $y(D) > \underline{x}$. When $y(D) = D$, $\Psi(D) = \mathrm{L}(D, D) = 0$. When $y(D) < D$, it satisfies the first-order condition

$$
\mathrm{L}_y(y, D) = -(y/D)^{\delta/(1-\delta)} + H(y) = 0. \tag{45}
$$

If $y(D) = D$ for some $D \in (D^*, \overline{x})$ (this interval in non-empty by $D^* < \overline{x}$), then $\Psi(D) = 0$ and constraint (44) is satisfied for $D$, which contradicts optimality of $D^*$.[17] Thus, for all $D \in (D^*, \overline{x})$, $y(D) < D$ and $y(D)$ satisfies (45). Consider a limit point $y^*$ of the sequence $\{y(D), D \downarrow D^*\}$. By the upper-hemicontinuity of $Y(D)$, $y^* \in Y(D^*)$ and it satisfies the first-order condition (45), which implies $y^* < D^*$ (by $(y^*/D^*)^{\delta/(1-\delta)} = H(y^*) \leq H(D^*) < 1$). Since $y(D^*) \leq y^*$, $y(D^*) < D^*$, and so, $y(D^*)$ also satisfies the first-order condition (45).

Fix some $D_0 \in [y(D^*), D^*)$ and consider an auxiliary program

$$\Theta(D|D_0) \equiv \min_{y \in [\underline{x}, D_0]} \mathrm{L}\,(y, D)\,. \tag{46}$$

Since $D_0 \geq y(D^*)$, $\Theta(D^*|D_0) = \Psi(D^*) \leq 0$. Let $\gamma(D|D_0)$ be the smallest solution to (46). Note that $\gamma(D^*|D_0) = y(D^*) = \hat{x}$. Since $\mathrm{L}_y(y, D) = -(y/D)^{\delta/(1-\delta)} + H(y)$ is increasing in $D$, $\mathrm{L}(y, D)$ has increasing differences in $(y, D)$, and so, $\gamma(D|D_0)$ is weakly decreasing in $D$. Then, there is $\underline{D} \in [y(D^*), D^*)$ such that $\gamma(\underline{D}|\underline{D}) = \underline{D}$. Indeed, otherwise, for any $D_0 \in [y(D^*), D^*)$, $\gamma(D_0|D_0) < D_0$ and $\gamma(D_0|D_0) \geq y(D^*)$ (by monotonicity of $\gamma(D|D_0)$), which is not possible. Let us denote $\Theta(D) \equiv \Theta(D|\underline{D})$ and $\gamma(D) \equiv \gamma(D|\underline{D})$. By the choice of $\underline{D}$, $\Theta(\underline{D}) = \mathrm{L}\,(\underline{D}, \underline{D}) = 0$.

By the envelope theorem, for almost all $D$,

$$\Theta'(D) = \mathrm{L}_D\,(\gamma(D), D) = 1 - \delta - H(D) + \delta(\gamma(D)/D)^{1/(1-\delta)}. \tag{47}$$

To complete the proof, suppose in contradiction to the lemma statement that $\lim_{D \uparrow D^*} \Theta'(D) = 1 - \delta - H\,(D^*) + \delta\,(\hat{x}/D^*)^{1/(1-\delta)} \geq 0$. By (47) and monotonicity of $\gamma(D)$, $\Theta'(D)$ is strictly decreasing in $D$. Hence, for $D \in [\underline{D}, D^*]$, $\Theta'(D) > \lim_{D \uparrow D^*} \Theta'(D) \geq 0$, and so, $0 = \Theta(\underline{D}) < \Theta(D^*) = \Psi(D^*) \leq 0$, which is a contradiction. Therefore, the conclusion of the lemma obtains. □

***Proof of Proposition 1***. Since $\varphi^{BM}$ satisfies the constraints in program (18) and DD-mechanisms are dominated by posted price mechanisms, $\varphi^{BM}$ also satisfies the constraints in program (27). Thus, $V^* \geq V^{BM}$. Note $V^{BM} = V\left(D^{BM}\right)$ and $V^* = V\,(D^*)$, where $V\,(D) \equiv \delta\,(D - \mu\,(D))$. Since $V'\,(D) = \delta H\,(D) > 0$, $D^* \geq D^{BM}$, and the inequality is strict if and only if $V^* > V^{BM}$. □

## Optimality of Debt under Liquidity Requirements: General Case

In this online appendix, we consider liquidity requirements of general form. Specifically, we require that the security is sold with probability one at a price $p \geq \rho\overline{f}^{\varphi}$, $\rho \in [0, \delta]$. Denoting $\alpha \equiv \rho/\delta \in [0, 1]$, by the same argument as in Section 6, the optimal security under liquidity requirements solves the program

$$\max_{\varphi \in \Phi} \mathbf{V}\,(\varphi) \text{ s.t. } u^{\varphi} \geq \alpha\overline{f}^{\varphi}. \tag{48}$$

[17] Indeed, the issuer's payoff $\delta\left(D - \mu^{\varphi^D}\right) = \delta\left(D - \int_{\underline{x}}^{\overline{x}} \min\{x, D\}\,\mathrm{d}H\,(x)\right) = \delta\int_{\underline{x}}^{D}(D - x)\,\mathrm{d}H\,(x)$ is strictly increasing in $D$.

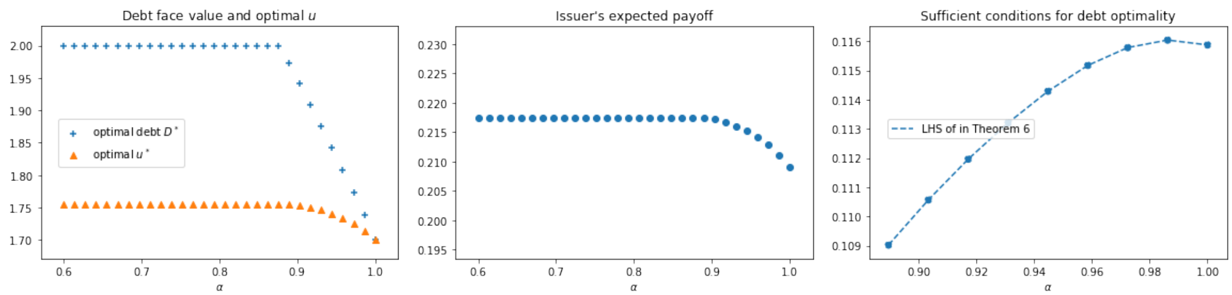



*Figure 6: Optimal security and issuer's maximal expected payoff for various $\alpha$*

***Note:*** *The distribution of cash flows is given by $H(x) = x - 1, x \in [1,2]$ and $\delta = 0.85$. The left panel depicts the optimal security, which is debt with face value $D^*$ and the corresponding optimal $u^*$ that solves the program (49) for a range of $\alpha \in [0.6, 1]$. The middle panel depicts the issuer's expected payoff under the optimal security and information design that solves the program (49). The right panel verifies that regularity condition (55) for optimality of debt are indeed satisfied in this example.*

We use Theorem 2 to re-write this program more explicitly as follows

$$\max_{\varphi \in \Phi, u \in \left[\alpha \overline{f}^{\varphi}, \overline{f}^{\varphi}\right]} \delta\left(u - \mu^{\varphi}\right) \text{ s.t. } \mu^{\varphi} - \delta u \geq \pi\left(M_{y,u}^{DD} \middle| H_u^{\varphi}\right), y \in \left[\underline{f}^{\varphi}, u\right). \tag{49}$$

This program differs from the program (22) in that we additionally require $u \geq \alpha \overline{f}^{\varphi}$. As we showed in Section 6, debt solves this program for $\alpha = 1$. In this online appendix, we show that under certain conditions, debt is still optimal for $\alpha < 1$ sufficiently close to 1.

**Example** Before proceeding with the formal results, we provide an example that illustrates the changes in the optimal security as we vary the parameter $\alpha$. This example also motivates the conditions in the formal results in the next subsection.

Consider a uniform example, $H(x) = x - 1$ for $x \in [1, 2]$. Lemma 9 below implies that either the solution to the program (49) is pure equity $\varphi(X) = X$ or it satisfies $u^{\varphi} = \alpha \overline{f}^{\varphi}$. By Proposition **??** below, under certain conditions, the optimal security under the constraint $u^{\varphi} = \alpha \overline{f}^{\varphi}$ is debt, which we denote by $D^{\alpha}$. Thus, the optimal security under the general liquidity requirement is either debt $D^{\alpha}$ or pure equity, which corresponds to debt with face value $D^{\alpha} = \overline{x}$. Figure 6 depicts the optimal security for different $\alpha$'s in the uniform example. For low $\alpha$'s, the constraint $u^{\varphi} \geq \alpha \overline{f}^{\varphi}$ is not binding and the optimal security is pure equity as in the program without the liquidity requirements. For high $\alpha$'s, the constrain $u^{\varphi} \geq \alpha \overline{f}^{\varphi}$ becomes binding and the optimal security is debt with face value $D^{\alpha}$. The face value $D^{\alpha}$ is weakly decreasing in $\alpha$. The case $\alpha = 1$ corresponds to the liquidity requirement $u^{\varphi} = \overline{f}^{\varphi}$ analyzed in Section 6.

**Main Result** The analysis proceeds as follows. We first show that Lemma 4 still holds when we additionally impose the general liquidity requirements.

**Lemma 8.** *For any security $\varphi \in \Phi$ such that $\varphi(X) < \underline{x}$ with positive probability and that satisfies $u^{\varphi} \geq \alpha \overline{f}^{\varphi}$, there is $\varepsilon > 0$ such that $\tilde{\varphi}(X) = \varphi(X) + \varepsilon$ belongs to $\Phi$, satisfies $u^{\tilde{\varphi}} \geq \alpha \overline{f}^{\tilde{\varphi}}$, and is weakly preferred by the issuer. Further, $\varphi(X) > \underline{x}$ with positive probability for any optimal security $\varphi$.*

*Proof.* The proof proceeds identically to that of Lemma 4. We only need to verify that for $\tilde{\varphi}$ and $G^{\tilde{\varphi}}$ constructed there, we have $u\left(G^{\tilde{\varphi}}\right) \geq \alpha \overline{f}^{\tilde{\varphi}}$, which follows from $u\left(G^{\tilde{\varphi}}\right) = u\left(G^{\varphi}\right) + \varepsilon \geq \alpha \overline{f}^{\varphi} + \varepsilon > \alpha\left(\overline{f}^{\varphi} + \varepsilon\right) = \alpha \overline{f}^{\tilde{\varphi}}$. □

By Lemma 8, it is without loss of optimality to focus on securities satisfying $\varphi(\underline{x}) \geq \underline{x}$. The following lemma is an analogue of Theorem 3 that states that the issuer can increase his expected payoff by issuing a more informationally sensitive security, unless liquidity requirements or limited liability constraints do not allow for this.

**Lemma 9.** *Suppose security $\varphi \in \Phi$ satisfies $\varphi(\underline{x}) \geq \underline{x}$. Suppose additionally that $u^{\varphi} > \alpha \overline{f}^{\varphi}$ and $\varphi(X) < X$ with positive probability. Then, there is another security $\tilde{\varphi} \in \Phi$ that is more informationally sensitive than $\varphi$, satisfies $u^{\tilde{\varphi}} > \alpha \overline{f}^{\tilde{\varphi}}$, and brings a weakly higher expected payoff to the issuer.*

*Proof.* Consider $\varphi'(x) = \underline{x} + \max\{0, x - K\}$, where $K$ is such that $\mu^{\varphi'} = \mu^{\varphi}$. Fix some $\varepsilon \in (0,1)$ and let $\tilde{\varphi}(x) \equiv \varepsilon \varphi'(x) + (1-\varepsilon)\varphi(x)$. By construction, $\tilde{\varphi} \in \Phi$, $\mu^{\tilde{\varphi}} = \mu^{\varphi}$, and $\tilde{\varphi}$ is more information sensitive than $\varphi$. By Theorem 3, any $G \in \mathcal{G}^{\varphi}$ also belongs to $\mathcal{G}^{\tilde{\varphi}}$. Further, $u^{\varphi} > \alpha \overline{f}^{\varphi}$ implies that $u^{\varphi} > \alpha \overline{f}^{\tilde{\varphi}}$ for sufficiently small $\varepsilon$. The issuer can thus guarantee a payoff for himself at least equal to $\delta\left(u^{\varphi} - \mu^{\varphi}\right)$ with $\tilde{\varphi}$ and still satisfy the liquidity requirement $u^{\varphi} > \alpha \overline{f}^{\tilde{\varphi}}$. Thus, $\tilde{\varphi}$ is the desired security. □

The next lemma is a generalization of Lemma 5 that incorporates the liquidity requirements.

**Lemma 10.** *Consider $\varphi(x) = \underline{x} + \max\{0, x - K\}, K \in (\underline{x}, \overline{x})$ suchthat $u^{\varphi} > \alpha \overline{f}^{\varphi}$. There exists a more informationally sensitive security $\tilde{\varphi} \in \Phi$ and $\tilde{u} \geq \alpha \overline{f}^{\varphi}$ such that $\mu^{\tilde{\varphi}} > \mu^{\varphi}$, $\delta\left(\tilde{u} - \mu^{\tilde{\varphi}}\right) > \delta\left(u^{\varphi} - \mu^{\varphi}\right)$, and conditions (23) hold for $\tilde{\varphi}$ and $\tilde{u}$. Furthermore, $\mathbf{V}(\tilde{\varphi}) > \mathbf{V}(\varphi)$.*

*Proof.* The proof of Lemma 10 is analogous to Lemma 5 in the main text. We show that increasing both the security's average payoff and its information sensitivity relaxes the constraint in (17) sufficiently so that we can choose a signal distribution that is strictly preferred by the issuer. Fix $\Delta \in \left(0, \min\left\{K - \underline{x}, (u^{\varphi} - \alpha \overline{f}^{\varphi})/\alpha\right\}\right)$. Note that $\min\{K - \underline{x}, \gamma/\alpha\} > 0$, by $u^{\varphi} > \alpha \overline{f}^{\varphi}$ and $K \in (\underline{x}, \overline{x})$. Consider the security $\tilde{\varphi}(x) \equiv d + \Delta_{\mu} + \max\{0, x - k\}$, where $k = K - \Delta$ and $d = \underline{x} - \int_k^K (1 - H(x))\, dx$ and $\Delta_{\mu}$ is chosen below. Security $\tilde{\varphi}$ is a combination of the safe debt $d + \Delta_{\mu}$ and the call option with a lower strike $k \in [\underline{x}, K)$. By construction, $\tilde{\varphi}$ satisfies $\mu^{\tilde{\varphi}} = \mu^{\varphi} + \Delta_{\mu}$. Fix $\varepsilon > 0$, and let $\Delta_u = \varepsilon$ and $\Delta_{\mu} = \varepsilon - \varepsilon^2$. Provided that $\varepsilon$ is small, $\tilde{\varphi} \in \Phi$. Further, let $\tilde{u} \equiv u^{\varphi} + \Delta_u$. The same arguments as in the proof of Lemma 5 imply that inequalities (23) hold for $\tilde{\varphi}$ and $\tilde{u}$, provided that $\varepsilon$ is sufficiently small. Importantly, under the new security, the issuer's payoff increases by $\delta\left(\Delta_u-\Delta_\mu\right)=\delta\varepsilon^2>0$. It remains to verify that the liquidity requirements are satisfied. Note that

$$\begin{aligned}
u^{\tilde{\varphi}}=u^{\varphi}+\Delta_u &= \alpha\varphi\left(\overline{x}\right)+u^{\varphi}-\alpha\overline{f}^{\varphi}+\Delta_u\\
&= \alpha\left(\underline{x}+\overline{x}-K\right)+u^{\varphi}-\alpha\overline{f}^{\varphi}+\Delta_u\\
&\geq \alpha\left(\underline{x}+\overline{x}-K\right)+\alpha\Delta+\Delta_u\\
&= \alpha\left(\underline{x}+\overline{x}-k\right)+\Delta_u\\
&> \alpha\underbrace{\left(d+\Delta_\mu+\overline{x}-k\right)}_{=\tilde{\varphi}(\overline{x})}+\Delta_u>\alpha\tilde{\varphi}\left(\overline{x}\right),
\end{aligned}$$

which completes the proof. □

Before stating and proving the main result, we consider the following auxiliary program, which restricts the search for optimal securities in program (49) to debt securities.

$$\begin{aligned}
&\max_{u\in[\alpha D,D],D\in[\underline{x},\overline{x}]}\delta\left(u-D\left(1-H\left(D\right)\right)-\int_{\underline{x}}^{D}x\mathrm{d}H\left(x\right)\right)\\
&\text{s.t. }\underbrace{D\left(1-H\left(D\right)\right)+\int_{\underline{x}}^{D}x\mathrm{d}H\left(x\right)-\delta u-\left(1-\delta\right)y^{1/(1-\delta)}u^{-\delta/(1-\delta)}+\int_{\underline{x}}^{y}\left(y-x\right)\mathrm{d}H\left(x\right)}_{\equiv L(y|D,u)}\geq 0,\ y\in[\underline{x},u)\,.
\end{aligned}\tag{50}$$

The next lemma establishes that, in this program, either the constraint $D\leq\overline{x}$ or $\alpha D\leq u$ must bind in the solution.

**Lemma 11.** *In the solution $(D^*,u^*)$ to (50), either $D^*=\overline{x}$ or $u^*=\alpha D^*$.*

*Proof.* Suppose to contradiction that $D^*<\overline{x}$ and $\alpha D^*<u^*$. Let $\tilde{u}=u^*+\varepsilon$ and $\tilde{D}=D^*+\frac{\max\{2-H(D^*),1+\delta\}}{2(1-H(D^*))}\varepsilon$, for some $\varepsilon>0$. Note that

$$\tilde{D}-\tilde{u}\geq D^*-u^*+\frac{H(D^*)}{2\left(1-H(D^*)\right)}\varepsilon\geq\frac{H(D^*)}{2\left(1-H(D^*)\right)}\varepsilon.\tag{51}$$

Choosing $(\tilde{D}, \tilde{u})$ increases the issuer's expected payoff:

$$\begin{aligned}
&\delta\left(\tilde{u}-\tilde{D}\left(1-H\left(\tilde{D}\right)\right)-\int_{\underline{x}}^{\tilde{D}} x \mathrm{d} H(x)\right) \\
=&\delta\left(u^*+\varepsilon-D^*\left(1-H\left(D^*\right)\right)-\int_{\underline{x}}^{D^*} x \mathrm{d} H(x)-\int_{D^*}^{\tilde{D}}(1-H(x)) \mathrm{d} x\right) \\
\geq&\delta\left(u^*-D^*\left(1-H\left(D^*\right)\right)-\int_{\underline{x}}^{D^*} x \mathrm{d} H(x)\right)+\delta\left(\varepsilon-\left(\tilde{D}-D^*\right)\left(1-H\left(D^*\right)\right)\right) \\
=&\delta\left(u^*-D^*\left(1-H\left(D^*\right)\right)-\int_{\underline{x}}^{D^*} x \mathrm{d} H(x)\right)+\frac{\delta \varepsilon}{2} \max \left\{H(D^*), 1-\delta\right\} \\
>&\delta\left(u^*-D^*\left(1-H\left(D^*\right)\right)-\int_{\underline{x}}^{D^*} x \mathrm{d} H(x)\right).
\end{aligned}$$

Next, for all $y \in [\underline{x}, u^*)$,

$$\begin{aligned}
&L(y|\tilde{D}, \tilde{u})-L(y|D^*, u^*) \\
=&\tilde{D}\left(1-H\left(\tilde{D}\right)\right)+\int_{\underline{x}}^{\tilde{D}} x \mathrm{d} H(x)-\delta \tilde{u}-(1-\delta) y^{1/(1-\delta)} \tilde{u}^{-\delta/(1-\delta)} \\
&-\left(D^*\left(1-H\left(D^*\right)\right)+\int_{\underline{x}}^{D^*} x \mathrm{d} H(x)-\delta u^*-(1-\delta) y^{1/(1-\delta)}\left(u^*\right)^{-\delta/(1-\delta)}\right) \\
=&\int_{D^*}^{\tilde{D}}(1-H(x)) \mathrm{d} x-\delta \int_{u^*}^{\tilde{u}}\left(1-(y/u)^{1/(1-\delta)}\right) \mathrm{d} u \\
\geq&\left(\tilde{D}-D^*\right)\left(1-H\left(\tilde{D}\right)\right)-\delta\left(\tilde{u}-u^*\right) \\
=&\left\{\frac{\max \left\{2-H(D^*), 1+\delta\right\}}{2\left(1-H(D^*)\right)}\left(1-H\left(\tilde{D}\right)\right)-\delta\right\} \varepsilon \\
>&\left\{\frac{1+\delta}{2}\left(\frac{1}{2}+\frac{\delta}{1+\delta}\right)-\delta\right\} \varepsilon \\
=&\left(\frac{1-\delta}{4}\right) \varepsilon>0,
\end{aligned}$$

where in the second inequality, we choose $\varepsilon$ sufficiently small so that $\left(1-H(\tilde{D})\right)/(1-H(D^*)) > 1/2+\delta/(1+\delta)$. Since $L(y|D^*, u^*) \geq 0$ for all $y \in [\underline{x}, u^*)$, we have $L(y|\tilde{D}, \tilde{u}) > 0$ for all $y \in [\underline{x}, u^*)$.

Note that

$$\begin{aligned}
L(\tilde{u}|\tilde{D},\tilde{u}) &= \tilde{D}\left(1-H\left(\tilde{D}\right)\right)+\int_{\underline{x}}^{\tilde{D}} x\mathrm{d}H\left(x\right)-\tilde{u}+\int_{\underline{x}}^{\tilde{u}}\left(\tilde{u}-x\right)\mathrm{d}H\left(x\right)\\
&= \tilde{D}-\tilde{u}-\int_{\underline{x}}^{\tilde{D}}\left(\tilde{D}-x\right)\mathrm{d}H\left(x\right)+\int_{\underline{x}}^{\tilde{u}}\left(\tilde{u}-x\right)\mathrm{d}H\left(x\right)\\
&= \tilde{D}-\tilde{u}-\int_{\underline{x}}^{\tilde{D}} H\left(x\right)\mathrm{d}x+\int_{\underline{x}}^{\tilde{u}} H\left(x\right)\mathrm{d}x\\
&= \tilde{D}-\tilde{u}-\int_{\tilde{u}}^{\tilde{D}} H\left(x\right)\mathrm{d}x\\
&= \int_{\tilde{u}}^{\tilde{D}}\left(1-H\left(x\right)\right)\mathrm{d}x>0,
\end{aligned}$$

where the inequality follows from (51). For $y\in[u^*,\tilde{u})$,

$$L_y(y|\tilde{D},\tilde{u})=-(y/\tilde{u})^{1/(1-\delta)}+H\left(y\right)\leq-(u^*/\tilde{u})^{1/(1-\delta)}+H\left(\tilde{u}\right)<0,$$

where the last inequality holds for sufficiently small $\varepsilon$ and follows from the fact that $\lim_{\varepsilon\to0}u^*/\tilde{u}=1$ and $\lim_{\varepsilon\to0}H\left(\tilde{u}\right)=H\left(u^*\right)\leq H\left(D^*\right)<1$. Hence, for $y\in[u^*,\tilde{u})$,

$$L(y|\tilde{D},\tilde{u})=L(\tilde{u}|\tilde{D},\tilde{u})-\int_{y}^{\tilde{u}}L_y(y|\tilde{D},\tilde{u})\mathrm{d}y>L(\tilde{u}|\tilde{D},\tilde{u})>0.$$

Thus, we showed that constraints in (50) hold for $(\tilde{D},\tilde{u})$ and $(\tilde{D},\tilde{u})$ brings a strictly higher expected payoff to the issuer, which contradicts the optimality of $(D^*,u^*)$. Therefore, either $D^*\leq\overline{x}$ and $\alpha D^*\leq u^*$ must bind. □

**Theorem 6.** *Suppose regularity condition (55) below holds. Then, the debt security of the form $\varphi^*(X)=\min\{X,D^*\}$ solves program (49).*

*Proof.* Lemmas 8, 9, and 10 jointly imply that either the solution $(\hat{\varphi},\hat{u})$ to program (49) is pure equity or it satisfies $\hat{u}=\alpha\hat{\varphi}(\overline{x})$. The former case implies the conclusion of the theorem so in the rest of the proof we suppose that the solution to (49) is not pure equity and the constraint $u=\alpha\varphi(\overline{x})$ is binding. We will show that a debt security solves (49). We can re-write this program with binding constraint $u=\alpha\varphi(\overline{x})$ more explicitly as:

$$\begin{aligned}
&\max_{\varphi\in\Phi}\delta\left(\alpha\varphi\left(\overline{x}\right)-\int_{\underline{x}}^{\overline{x}}\varphi\left(x\right)\mathrm{d}H\left(x\right)\right)\\
&\text{s.t.}\int_{\underline{x}}^{\overline{x}}\varphi\left(x\right)\mathrm{d}H\left(x\right)-\alpha\delta\varphi\left(\overline{x}\right)-\left(1-\delta\right)\varphi\left(\tilde{x}\right)^{\frac{1}{1-\delta}}\left(\alpha\varphi\left(\overline{x}\right)\right)^{-\frac{\delta}{1-\delta}}+\int_{\underline{x}}^{\tilde{x}}\left(\varphi\left(\tilde{x}\right)-\varphi\left(x\right)\right)\mathrm{d}H\left(x\right)\geq0,\\
&\quad\text{for all }\tilde{x}\in[\underline{x},\overline{x}]\text{ such that }\varphi\left(\tilde{x}\right)<\alpha\varphi\left(\overline{x}\right).
\end{aligned}\tag{52}$$

We want to prove that a debt security solves this program.

Consider first the auxiliary program (50). By Lemma 11, in the solution $(D^*, u^*)$ to (50), either $D^* = \overline{x}$ or $u^* = \alpha D^*$. If $D^* = \overline{x}$ and $u^* \geq \alpha D^*$ is slack, then by Lemma 6 in the main text, pure equity satisfies the liquidity requirements and therefore solves the program (49), which as we supposed above, is not the case. Thus, we focus on the case $u^* = \alpha D^*$, with $D^* < \overline{x}$. Then, $D^*$ solves program (52) restricted to debt securities of the form $\varphi(X) = \min\{X, D\}, D \in [\underline{x}, \overline{x}]$:

$$\max_{D \in [\underline{x}, \overline{x}]} \delta \left( \alpha D - D(1 - H(D)) - \int_{\underline{x}}^{D} x \mathrm{d}H(x) \right)$$
$$\text{s.t. } \mathrm{L}(y, D) \geq 0, \text{ for all } y \in [\underline{x}, \alpha D),$$
$$\text{where } \mathrm{L}(y, D) \equiv D(1 - H(D)) + \int_{\underline{x}}^{D} x \mathrm{d}H(x) - \delta \alpha D - (1 - \delta) y^{1/(1-\delta)} (\alpha D)^{-\delta/(1-\delta)} + \int_{\underline{x}}^{y} (y - x) \mathrm{d}H(x). \tag{53}$$

The derivative of the objective function equals $\alpha - 1 + H(D)$, which is strictly increasing in $D$. This implies that the interior critical point satisfying $H(D) = 1 - \alpha$ is a global minimum and the unconstrained maximum is attained at either $D = \underline{x}$ or $D = \overline{x}$. In the former case, the issuer's expected payoff from $D = \underline{x}$ and $u = \alpha \underline{x}$ is $-(1 - \alpha)\underline{x} < 0$. This cannot be the optimal security for the issuer, because the issuer is guaranteed zero by choosing $D = \underline{x}$ and $u = \underline{x}$ in program (50). In the latter case, the solution to (53) is the highest $D^*$ that satisfies the constraint. Let $\hat{x}$ be the smallest $y$ at which the constraint in (53) binds for $D^*$ whenever $D^* < \overline{x}$.[18] Observe that $D^* = \overline{x}$ is not possible. In this case, it must be the case that $u^{\varphi^*} = u^*$, and so, the solution to (49) is pure equity, which as we supposed above, is not the case.

By the same argument as in Lemma 7 in the Appendix, we can show that if $D^* < \overline{x}$, then

$$1 - \alpha\delta - H(D^*) + \alpha\delta \left( \frac{\hat{x}}{\alpha D^*} \right)^{1/(1-\delta)} < 0, \tag{54}$$

and in particular, $\hat{x} < \alpha D^*$. We further assume the regularity condition:

$$\alpha \left( 1 - \delta + \delta \left( \frac{\hat{x}}{\alpha D^*} \right)^{1/(1-\delta)} \right) - \left( \frac{\hat{x}}{\alpha D^*} \right)^{\delta/(1-\delta)} > 0. \tag{55}$$

[18]Note that

$$\begin{aligned} L(\alpha D, D) &= D(1 - H(D)) + \int_{\underline{x}}^{D} x \mathrm{d}H(x) - \alpha D + \int_{\underline{x}}^{\alpha D} (\alpha D - x) \mathrm{d}H(x) \\ &= D(1 - H(D)) + DH(D) - \int_{\underline{x}}^{D} H(x) \mathrm{d}x - \alpha D + \int_{\underline{x}}^{\alpha D} H(x) \mathrm{d}x \\ &= \int_{\alpha D}^{D} (1 - H(x)) \mathrm{d}x > 0, \end{aligned}$$

and therefore $L(\cdot, D)$ never binds at $y = \alpha D$, thereby implying that $\hat{x}$ is well-defined.

The role of this condition will become clearer further in the argument. This condition is easy to verify for a particular distribution $H$: one only needs to solve for the optimal debt security (program (53)) and find $\hat{x}$ at which the constraint in (53) is binding. For example, the right panel of Figure 6 verifies condition (55) in the uniform example for all $\alpha$'s considered.

We verify next that the debt security $\varphi^*(X) = \min\{X, D^*\}$ solves program (52). Denote by

$$C(\tilde{x}, \varphi) \equiv \int_{\underline{x}}^{\overline{x}} (\varphi(x) - \delta\alpha\varphi(\overline{x}))\,\mathrm{d}H(x) - (1-\delta)\,\varphi(\tilde{x})^{1/(1-\delta)}(\alpha\varphi(\overline{x}))^{-\delta/(1-\delta)} + \int_{\underline{x}}^{\tilde{x}} (\varphi(\tilde{x}) - \varphi(x))\,\mathrm{d}H(x)$$

the left-hand side of the constraints in the program (52). By the same argument as in Lemma 5 in Biais and Mariotti (2005), to solve the program (52), it is sufficient to find a distribution function $\Lambda$ (that is, a non-decreasing and right-continuous function such that $\Lambda(\underline{x}) = 0$) that satisfies

$$\int_{\underline{x}}^{\overline{x}} C(\tilde{x}, \varphi^*)\,\mathbf{1}\{\varphi^*(\tilde{x}) < \alpha\varphi^*(\overline{x})\}\,\mathrm{d}\Lambda(\tilde{x}) = 0 \tag{56}$$

and

$$\mathcal{L}(\varphi^*, \Lambda) \geq \mathcal{L}(\varphi, \Lambda), \text{ for all } \varphi \in \Phi, \tag{57}$$

where $\mathcal{L}(\varphi, \Lambda)$ is the Lagrangian given by

$$\mathcal{L}(\varphi, \Lambda) \equiv \delta \int_{\underline{x}}^{\overline{x}} (\alpha\varphi(\overline{x}) - \varphi(x))\,\mathrm{d}H(x) + \int_{\underline{x}}^{\overline{x}} C(\tilde{x}, \varphi)\,\mathbf{1}\{\varphi(\tilde{x}) < \alpha\varphi(\overline{x})\}\,\mathrm{d}\Lambda(\tilde{x}) \tag{58}$$

We choose the distribution $\Lambda_\lambda = \lambda 1_{[\hat{x}, \overline{x}]}$ parametrized by $\lambda > 0$. By construction of $\hat{x}$ above, $\Lambda_\lambda$ and $\varphi^*$ satisfy (56). Let

$$\lambda = \frac{\delta\,(H(D^*) - 1 + \alpha)}{H(D^*) - 1 + \alpha\delta\left(1 - \left(\frac{\hat{x}}{\alpha D^*}\right)^{1/(1-\delta)}\right)}, \tag{59}$$

By inequality (54), the denominator and numerator are positive, and so $\lambda > 0$. We show that $\varphi^*$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ over $\varphi \in \Phi$, and so, it indeed solves the program (52).

*Claim* 1. *$\mathcal{L}(\varphi^*, \Lambda_\lambda) \geq \mathcal{L}(\varphi, \Lambda_\lambda)$ holds for all securities $\varphi \in \Phi$ satisfying $\varphi(\hat{x}) < \alpha\varphi(\overline{x})$.*

*Proof:* Consider any security $\varphi$ satisfying $\varphi(\hat{x}) < \alpha\varphi(\overline{x})$. In this case, we can re-write the Lagrangian as follows:

$$
\begin{aligned}
\mathcal{L}(\varphi,\Lambda) &= \delta\int_{\underline{x}}^{\overline{x}}\left(\alpha\varphi(\overline{x})-\varphi(x)\right)\mathrm{d}H(x)\\
&+\lambda\left\{\int_{\underline{x}}^{\overline{x}}\left(\varphi(x)-\delta\alpha\varphi(\overline{x})\right)\mathrm{d}H(x)-(1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}+\int_{\underline{x}}^{\hat{x}}\left(\varphi(\hat{x})-\varphi(x)\right)\mathrm{d}H(x)\right\}\\
&=\delta\int_{\underline{x}}^{\overline{x}}\left(\alpha\varphi(\overline{x})-\varphi(x)\right)\mathrm{d}H(x)\\
&+\lambda\left\{\int_{\underline{x}}^{\overline{x}}(1-\delta)\,\varphi(x)\,\mathrm{d}H(x)+\delta\int_{\underline{x}}^{\overline{x}}\left(\varphi(x)-\alpha\varphi(\overline{x})\right)\mathrm{d}H(x)-(1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}\right\}\\
&+\lambda\int_{\underline{x}}^{\hat{x}}\left(\varphi(\hat{x})-\varphi(x)\right)\mathrm{d}H(x)\\
&=(1-\lambda)\,\delta\int_{\underline{x}}^{\overline{x}}\left(\alpha\varphi(\overline{x})-\varphi(x)\right)\mathrm{d}H(x)\\
&+\lambda\left\{\int_{\underline{x}}^{\overline{x}}(1-\delta)\,\varphi(x)\,\mathrm{d}H(x)-(1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}\right\}\\
&+\lambda\int_{\underline{x}}^{\hat{x}}\left(\varphi(\hat{x})-\varphi(x)\right)\mathrm{d}H(x)\\
&=-(1-\alpha)(1-\lambda)\,\delta\varphi(\overline{x})+(1-\lambda)\,\delta\int_{\underline{x}}^{\overline{x}}\dot{\varphi}(x)\,H(x)\,\mathrm{d}x\\
&+\lambda\left\{(1-\delta)\,\varphi(\overline{x})-(1-\delta)\int_{\underline{x}}^{\overline{x}}\dot{\varphi}(x)\,H(x)\,\mathrm{d}x-(1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}\right\}\\
&+\lambda\int_{\underline{x}}^{\hat{x}}\dot{\varphi}(x)\,H(x)\,\mathrm{d}x\\
&=-(1-\alpha)(1-\lambda)\,\delta\varphi(\overline{x})+(1-\lambda)\,\delta\int_{\underline{x}}^{\overline{x}}\dot{\varphi}(x)\,H(x)\mathrm{d}x\\
&+\lambda\left\{(1-\delta)\varphi(\overline{x})+\delta\int_{\underline{x}}^{\hat{x}}\dot{\varphi}(x)H(x)\mathrm{d}x-(1-\delta)\int_{\hat{x}}^{\overline{x}}\dot{\varphi}(x)H(x)\mathrm{d}x-(1-\delta)\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}\right\}\\
&=\delta\int_{\underline{x}}^{\hat{x}}\dot{\varphi}(x)\,H(x)\,\mathrm{d}x+(\delta-\lambda)\int_{\hat{x}}^{\overline{x}}\dot{\varphi}(x)\,H(x)\,\mathrm{d}x\\
&+\lambda(1-\delta)\left\{\varphi(\overline{x})-\frac{(1-\alpha)(1-\lambda)\,\delta}{\lambda(1-\delta)}\varphi(\overline{x})-\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{L}(\varphi,\Lambda_\lambda) &= \int_{\underline{x}}^{\overline{x}} L_\lambda\left(x,\dot{\varphi}(x)\right)\mathrm{d}x+\Psi_\lambda\left(\varphi(\overline{x}),\varphi(\hat{x})\right),\\
&\text{where } L_\lambda(x,\dot{\varphi})\equiv\dot{\varphi}(x)\left(\delta H(x)\,\mathbf{1}\{x\le\hat{x}\}+(\delta-\lambda)\,H(x)\,\mathbf{1}\{x>\hat{x}\}\right),\\
&\qquad\Psi_\lambda\left(\varphi(\overline{x}),\varphi(\hat{x})\right)\equiv\left(\lambda-\delta+\alpha(1-\lambda)\,\delta\right)\varphi(\overline{x})-\lambda(1-\delta)\,\varphi(\hat{x})^{1/(1-\delta)}\left(\alpha\varphi(\overline{x})\right)^{-\delta/(1-\delta)}.
\end{aligned}
$$

We solve a relaxed problem of maximizing $\mathcal{L}(\varphi,\Lambda_\lambda)$ over all $\varphi\in\Phi$, and then verify that in the optimum, $\varphi(\hat{x}) < \alpha\varphi(\overline{x})$ is satisfied.

For any $\lambda > 0$, we first maximize $\mathcal{L}(\varphi, \Lambda_\lambda)$ over $\varphi \in \Phi$ that are piecewise continuously differentiable. Then, maximizing $\mathcal{L}(\varphi, \Lambda_\lambda)$ boils down to solving an optimal control problem. By double monotonicity of $\varphi$, the control is $\dot{\varphi}(x) \in [0, 1]$. We introduce two state variables: $\varphi(x)$ and an auxiliary state variable $\psi(x)$ satisfying $\dot{\psi}(x) = \dot{\varphi}(x)\mathbf{1}\{x \leq \hat{x}\}$. Thus, $\psi(\overline{x}) = \varphi(\hat{x})$ and we can re-write the terminal value function in (31) in the canonical form $\Psi_\lambda(\varphi(\overline{x}), \varphi(\hat{x})) = \Psi_\lambda(\varphi(\overline{x}), \psi(\overline{x}))$. We have the boundary condition $\psi(\underline{x}) = \varphi(\underline{x}) = \underline{x}$ and free boundary conditions on $\psi$ and $\varphi$ at $\overline{x}$.

The Hamiltonian of this optimal control problem is given by:

$$\mathcal{H}_\lambda(x, \varphi, \psi, \dot{\varphi}, p, q) = \dot{\varphi}\{\delta H(x)\mathbf{1}\{x \leq \hat{x}\} + (\delta - \lambda) H(x)\mathbf{1}\{x > \hat{x}\}\} + \dot{\varphi}p + \dot{\varphi}q\mathbf{1}\{x \leq \hat{x}\},$$

where $p$ and $q$ are costate variables that cannot be equal to zero simultaneously. By Pontryagin's Maximum principle, a necessary condition for $(\varphi_\lambda, \psi_\lambda, \dot{\varphi}_\lambda)$ to be optimal is that, for all $x$, $\dot{\varphi}_\lambda(x)$ maximizes $\mathcal{H}_\lambda(x, \varphi_\lambda, \psi_\lambda, \cdot, p_\lambda, q_\lambda)$ for some piecewise continuously differentiable functions $p_\lambda$ and $q_\lambda$ that satisfy

$$\begin{aligned}
\dot{p}_\lambda(x) &= -\frac{\partial \mathcal{H}_\lambda(x, \varphi_\lambda(x), \psi_\lambda(x), \dot{\varphi}_\lambda(x), p_\lambda(x), q_\lambda(x))}{\partial \varphi} = 0, \\
\dot{q}_\lambda(x) &= -\frac{\partial \mathcal{H}_\lambda(x, \varphi_\lambda(x), \psi_\lambda(x), \dot{\varphi}_\lambda(x), p_\lambda(x), q_\lambda(x))}{\partial \psi} = 0,
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
p_\lambda(\overline{x}) &= \frac{\partial \Psi_\lambda(\varphi_\lambda(\overline{x}), \psi_\lambda(\overline{x}))}{\partial \varphi(\overline{x})} = -\delta(1 - \alpha) + \lambda - \lambda\alpha\delta + \lambda\alpha\delta\left(\frac{\psi_\lambda(\overline{x})}{\alpha\varphi_\lambda(\overline{x})}\right)^{1/(1-\delta)}, \\
q_\lambda(\overline{x}) &= \frac{\partial \Psi_\lambda(\varphi_\lambda(\overline{x}), \psi_\lambda(\overline{x}))}{\partial \psi(\overline{x})} = -\lambda\left(\frac{\psi_\lambda(\overline{x})}{\alpha\varphi_\lambda(\overline{x})}\right)^{\delta/(1-\delta)}.
\end{aligned}$$

Thus, using $\psi_\lambda(\overline{x}) = \varphi_\lambda(\hat{x})$, we can solve these differential equations to get that for all $x \in [\underline{x}, \overline{x}]$,

$$\begin{aligned}
p_\lambda(x) = p_\lambda &\equiv -\delta(1 - \alpha) + \lambda - \lambda\alpha\delta + \lambda\alpha\delta\left(\frac{\varphi_\lambda(\hat{x})}{\alpha\varphi_\lambda(\overline{x})}\right)^{1/(1-\delta)}, \\
q_\lambda(x) = q_\lambda &\equiv -\lambda\left(\frac{\varphi_\lambda(\hat{x})}{\alpha\varphi_\lambda(\overline{x})}\right)^{\delta/(1-\delta)}.
\end{aligned}$$

Given the linearity of $\mathcal{H}_\lambda$ in $\dot{\varphi}$,

$$\dot{\varphi}_\lambda(x) = \begin{cases} \mathbf{1}\{\delta H(x) + p_\lambda + q_\lambda > 0\}, & x \leq \hat{x}, \\ \mathbf{1}\{(\delta - \lambda) H(x) + p_\lambda > 0\}, & x > \hat{x}. \end{cases} \tag{60}$$

Since $\mathcal{H}_\lambda$ is linear in $(\psi, \varphi, \dot{\varphi})$, Mangasarian's sufficiency condition is satisfied, and so, $\dot{\varphi}_\lambda$ is an optimal control and $\varphi_\lambda$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ over all piecewise continuously differentiable functions in $\Phi$. This space is dense in $\Phi$ endowed with the sup norm and $\mathcal{L}(\varphi, \Lambda_\lambda)$ is continuous in this topology. Hence, $\varphi_\lambda$ maximizes $\mathcal{L}(\varphi, \Lambda_\lambda)$ on $\Phi$.

Consider $\lambda$ defined in equation (59). Note that $\lambda > \delta$ (follows directly from (59)). Using $\varphi^*(\hat{x}) = \hat{x}$ and $\varphi^*(\overline{x}) = D^*$, we get that for all $x \in [\underline{x}, \overline{x}]$,

$$p_\lambda = -\delta(1-\alpha) + \lambda - \lambda\alpha\delta + \lambda\alpha\delta\left(\frac{\hat{x}}{\alpha D^*}\right)^{1/(1-\delta)},$$
$$q_\lambda = -\lambda\left(\frac{\hat{x}}{\alpha D^*}\right)^{\delta/(1-\delta)}.$$

Then, condition (55) implies

$$p_\lambda + q_\lambda = -\delta(1-\alpha) + \lambda\left\{1 - \alpha\delta - \left(\frac{\hat{x}}{\alpha D^*}\right)^{\delta/(1-\delta)} + \alpha\delta\left(\frac{\hat{x}}{\alpha D^*}\right)^{1/(1-\delta)}\right\} > (\lambda - \delta)(1-\alpha) > 0.$$

Thus, for $x \leq \hat{x}$, $\dot{\varphi}_\lambda(x) = \mathbf{1}\{\delta H(x) + p_\lambda + q_\lambda > 0\} = 1$, and so, $\varphi_\lambda(x) = x$. For $x > \hat{x}$, $\dot{\varphi}_\lambda(x) = 1\{(\delta - \lambda) H(x) + p_\lambda > 0\}$. Since $\lambda > \delta$ and $H$ is strictly increasing, the optimal policy takes the form: for some $x^*_\lambda \in [\hat{x}, \overline{x}]$, $\dot{\varphi}_\lambda(x) = \mathbf{1}\{x < x^*_\lambda\}$ and $\varphi_\lambda(x) = \min\{x, x^*_\lambda\}$. We need to show that $x^*_\lambda = D^*$. We have that $x^*_\lambda$ solves $(\delta - \lambda) H(x^*_\lambda) + p_\lambda = 0$, which given our definition of $\lambda$, is equavalent to

$$\frac{H(x^*_\lambda) - (1-\alpha)}{H(x^*_\lambda) - 1 + \alpha\delta\left(1 - \left(\frac{\hat{x}}{\alpha D^*}\right)^{1/(1-\delta)}\right)} = \frac{\delta(H(D^*) - (1-\alpha))}{H(D^*) - 1 + \alpha\delta\left(1 - \left(\frac{\hat{x}}{\alpha D^*}\right)^{1/(1-\delta)}\right)}.$$

The left-hand side is strictly decreasing in $x^*_\lambda$, hence, this equation has a unique solution $x^*_\lambda = D^*$, which is the desired conclusion. Further, since $\hat{x} < \alpha D^*$, the constraint $\varphi^*(\hat{x}) < \alpha\varphi^*(\overline{x})$ is indeed satisfied. *q.e.d.*

*Claim* 2. *$\mathcal{L}(\varphi^*, \Lambda) \geq \mathcal{L}(\varphi, \Lambda)$ holds for all securities $\varphi \in \Phi$ satisfying $\varphi(\hat{x}) \geq \alpha\varphi(\overline{x})$.*

*Proof:* Consider any security $\varphi$ satisfying

$$\varphi(\hat{x}) \geq \alpha\varphi(\overline{x}). \tag{61}$$

For any such security, $\mathcal{L}(\varphi, \Lambda) = \delta\int_{\underline{x}}^{\overline{x}}(\alpha\varphi(\overline{x}) - \varphi(x))\,\mathrm{d}H(x)$. Consider two state variables: $\varphi(x)$ and an auxiliary state variable $\psi(x)$ satisfying $\dot{\psi}(x) = \dot{\varphi}(x)(\mathbf{1}\{x \leq \hat{x}\} - \alpha)$. Thus, $\psi(\overline{x}) = \varphi(\hat{x}) - \alpha\varphi(\overline{x})$ and we can re-write the restriction as $\psi(\overline{x}) \geq 0$. The problem of maximizing $\mathcal{L}(\varphi, \Lambda)$ across all securities satisfying $\varphi(\hat{x}) \geq \alpha\varphi(\overline{x})$ is thus equivalent to maximizing,

$$\tilde{\mathcal{L}}(\varphi, \gamma) \equiv \delta\int_{\underline{x}}^{\overline{x}}(\alpha\varphi(\overline{x}) - \varphi(x))\,\mathrm{d}H(x) + \gamma\psi(\overline{x}) = -\delta(1-\alpha)\varphi(\overline{x}) + \gamma\psi(\overline{x}) + \int_{\underline{x}}^{\overline{x}}\underbrace{\delta\dot{\varphi}(x)H(x)}_{\equiv L_\lambda(x,\dot{\varphi})}\mathrm{d}x,$$

for some Lagrange multiplier $\gamma \geq 0$ that satisfies the complementary slackness condition $\gamma\psi(\overline{x}) = 0$.

Consider first the case $\gamma = 0$. Then,

$$\tilde{\mathcal{L}}(\varphi, 0) = \underbrace{-\delta(1-\alpha)\varphi(\overline{x})}_{\equiv\Psi_\lambda(\varphi(\overline{x}))} + \int_{\underline{x}}^{\overline{x}} \delta\dot{\varphi}(x) H(x)\, \mathrm{d}x,$$

The Hamiltonian corresponding to the program maximizing $\tilde{\mathcal{L}}(\varphi, 0)$ is given by $\mathcal{H}(x, \varphi, \psi, \dot{\varphi}, p) = \dot{\varphi}(\delta H(x) + p)$, where the costate variable $p(x)$ satisfies $\dot{p}(x) = -\partial\mathcal{H}(x, \varphi(x), \psi(x), \dot{\varphi}(x), p(x))/\partial\varphi = 0$, with the boundary condition $p(\overline{x}) = \partial\Psi_\lambda(\varphi(\overline{x}))/\partial\varphi(\overline{x}) = -\delta(1-\alpha)$. Hence, $p(x) = -\delta(1-\alpha)$ for all $x \in [\underline{x}, \overline{x}]$. Thus, $\dot{\varphi}^\dagger(x) = \mathbf{1}\{H(x) > 1-\alpha\}$ maximizes $\mathcal{H}$. Let $x_\alpha$ be such that $H(x_\alpha) = 1-\alpha$. Then,

$$\varphi^\dagger(x) = \begin{cases} 0, & x < x_\alpha, \\ x - x_\alpha, & x \geq x_\alpha. \end{cases}$$

Note that the condition $\varphi^\dagger(\hat{x}) \geq \alpha\varphi^\dagger(\overline{x})$ is equivalent to requiring $\frac{\hat{x}-x_\alpha}{\alpha(\overline{x}-x_\alpha)} \geq 1$. At the same time, for $x_\alpha < \hat{x}$,[19] $\frac{\hat{x}-x_\alpha}{\alpha(\overline{x}-x_\alpha)} < \frac{\hat{x}}{\alpha\overline{x}} < \frac{\hat{x}}{\alpha D^*} < 1$, which implies $\varphi^\dagger(\hat{x}) < \alpha\varphi^\dagger(\overline{x})$. Thus, $\mathcal{L}(\varphi, \Lambda)$ is maximized with the binding constraint (61).

Consider now the case when the constraint (61) is binding at the solution $\varphi^\#$, and so, $\varphi^\#(\hat{x}) = \alpha\varphi^\#(\overline{x})$. We then have that

$$\begin{aligned} C(\hat{x}, \varphi^\#) &= \int_{\underline{x}}^{\overline{x}} \left(\varphi^\#(x) - \delta\alpha\varphi^\#(\overline{x})\right) \mathrm{d}H(x) - (1-\delta)\alpha\varphi^\#(\overline{x}) + \int_{\underline{x}}^{\hat{x}} \left(\alpha\varphi^\#(\overline{x}) - \varphi^\#(x)\right) \mathrm{d}H(x) \\ &= \int_{\underline{x}}^{\overline{x}} \left(\varphi^\#(x) - \alpha\varphi^\#(\overline{x})\right) \mathrm{d}H(x) - \int_{\underline{x}}^{\hat{x}} \left(\varphi^\#(x) - \alpha\varphi^\#(\overline{x})\right) \mathrm{d}H(x) \\ &= \int_{\hat{x}}^{\overline{x}} \left(\varphi^\#(x) - \alpha\varphi^\#(\overline{x})\right) \mathrm{d}H(x) \\ &= \int_{\hat{x}}^{\overline{x}} \left(\varphi^\#(x) - \varphi^\#(\hat{x})\right) \mathrm{d}H(x) \geq 0. \end{aligned}$$

Thus,

$$\mathcal{L}\left(\varphi^\#, \Lambda\right) \leq \delta \int_{\underline{x}}^{\overline{x}} \left(\alpha\varphi^\#(\overline{x}) - \varphi^\#(x)\right) \mathrm{d}H(x) + \lambda C\left(\hat{x}, \varphi^\#\right) = \lim_{n\to\infty} \mathcal{L}(\varphi_n, \Lambda),$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence of piecewise continuously differentiable functions in $\Phi$ such that $\varphi_n(\hat{x}) < \alpha\varphi_n(\overline{x})$ for all $n \in \mathbb{N}$ that converges to $\varphi^\#$ in the sup norm. The limit then follows from the continuity of $\mathcal{L}(\varphi, \Lambda)$ derived explicitly in the proof of Claim 1. Since $\mathcal{L}(\varphi^*, \Lambda) \geq \mathcal{L}(\varphi_n, \Lambda)$ by Claim 1, $\mathcal{L}(\varphi^*, \Lambda) \geq \mathcal{L}\left(\varphi^\#, \Lambda\right)$. *q.e.d.* □

[19]Indeed, $\frac{\mathrm{d}}{\mathrm{d}x_\alpha}(\ln(\hat{x}-x_\alpha) - \ln(\overline{x}-x_\alpha)) = -\frac{1}{\hat{x}-x_\alpha} + \frac{1}{\overline{x}-x_\alpha} = \frac{\hat{x}-\overline{x}}{(\hat{x}-x_\alpha)(\overline{x}-x_\alpha)} < 0$.