**FTG Working Paper Series**

# Monitor Reputation and Transparency

by

Martin Szydlowski
Ivan Marinovic

Working Paper No. 00045-01

Finance Theory Group

www.financetheory.com

# Monitor Reputation and Transparency*

Iván Marinovic† Martin Szydlowski‡

September 21, 2020

**Abstract**

We study the disclosure policy of a regulator overseeing a monitor with reputation concerns. The monitor faces a manager, who chooses how much to manipulate based on the monitor's reputation. Reputational incentives are strongest for intermediate reputations. Instead of providing transparency, the regulator's disclosure policy aims to keep the monitor' s reputation intermediate, even at the cost of diminished incentives. Beneficial schemes feature random delay. Commonly used ones, which feature immediate disclosure or fixed time delay, destroy reputational incentives. Surprisingly, the regulator discloses more aggressively when she has better enforcement tools.

***JEL Classification:*** G21, G28, D82, D83.

***Keywords:*** Monitoring, Regulatory Disclosure, Reputation, Bank Regulation, Auditing

*First version: December 2017. We thank Ehsan Azarmsa, Judson Caskey, Itay Goldstein, Dmitry Orlov, David Rahman, Larry Samuelson, Giulio Trigilia, and Ariel Zetlin-Jones for helpful comments as well as seminar participants at Dartmouth, Indiana, Iowa, Minnesota, Rochester, and Stanford and conference participants at the American Finance Association, the Finance Theory Group, the Financial Intermediation Research Society conference, the North American Summer Meeting of the Econometric Society, the Midwest Macroeconomics Meeting, the SFS Cavalcade, the Society for Economic Dynamics conference, and the Stony Brook Game Theory conference. Fangyuan Yu provided excellent research assistance, for which we are grateful.

†Stanford University, Graduate School of Business. Email: imvial@stanford.edu

‡University of Minnesota, Carlson School of Management. Email: szydl002@umn.edu

# 1 Introduction

Monitoring is a crucial task performed by intermediaries such as banks and auditors.[1] In practice, monitoring effort is unobservable which undermines a monitor's incentive to work diligently. The literature has observed that reputation *per se* can provide the monitor a strong incentive to work (Chemmanur and Fulghieri (1994a); Carter et al. (1998); Mathis et al. (2009)). When a monitor shirks and, as a result, fails to detect problems—the argument goes—these problems eventually become apparent, thereby damaging the monitor's reputation and, in some cases, even driving the monitor out of the market (see, e.g., "From Sunbeam to Enron, Andersen's Reputation Suffers", NYT, 2001).

Despite the compelling logic of this mechanism, recent events (notably, the financial crisis of 2007-2008) have led the public to believe that reputational incentives are insufficient, generating a demand for regulation and a call to "monitor the monitors." As a consequence, in many industries regulators collect information about monitor quality and eventually disclose it to the public. When the information is negative, a regulator faces a dilemma: should she disclose the information or conceal it from the public?

Scholars and market pundits often argue that regulators should disclose any information they learn, including negative information, or else the monitor—anticipating the regulator's opacity—would shirk.[2] On the other hand, disclosing negative information may damage the monitor's reputation potentially weakening his incentive to work. The regulator might thus prefer to conceal negative information which would damage the monitor's reputation.

For example, regulators like the SEC, FDIC, or OCC investigate banks and take enforcement actions. These actions, which range from cease and desist orders and written agreements to suspensions and fines, are publicly available. The bank's consumers and counterparties may e.g. learn that the bank lacks "lending and collection policies to provide effective guidance and control," or that its management does not have "qualifications and experience commensurate with the Bank's size and complexity."[3] Clearly, disclosure of such failings negatively affects the bank's reputation.

As another example, consider the Public Company Accounting Oversight Board (PCAOB). The PCAOB is a regulatory body, established by the Sarbannes Oxley act in 2002, to oversee the audit industry. The PCAOB conducts regular inspections to assess the quality of

[1]See e.g. Chemmanur and Fulghieri (1994b) for banks and Hansen and Torregrosa (1992) for underwriters.

[2]Indeed, many government programs implement this transparency principle. For example, the Los Angeles county restaurant hygiene program monitors restaurant hygiene randomly and requires the restaurants to display the outcome of the inspection immediately on their windows. See Jin and Leslie (2003).

[3]See `https://orders.fdic.gov/sfc/servlet.shepherd/document/download/069t0000002Z6IOAA0?operationContext=S1`.

auditors' control systems. The outcome of these inspections is not necessarily disclosed to the market, but remains private for at least a year, but potentially forever.[4] If disclosed, these reports often have significant consequences on an auditor's demand.

In this paper, we study why a regulator's commitment to delay or even conceal information about monitor quality can be desirable from an incentive purpose. We begin by studying a reputation game that features three players: a monitor, a (client) firm, and the client firm's manager. The monitor is a long-run player with reputation concerns whose quality is unknown. The manager of the client firm is a short-run player who may engage in "manipulation" but is subject to the monitor's oversight. Manipulation is unobservable, but the monitor can detect it. If the monitor does not detect the manipulation, the manager obtains a private benefit, but the firm may randomly suffer a negative shock (the shock could represent, for instance, a restatement of financial statements, or the default of a bank's creditor). This shock arrives at a random time, and its intensity is proportional to the magnitude of the manager's manipulation: larger manipulations, are more likely to cause negative shocks.

The monitor exerts hidden effort to detect manipulation and prevent the shock. There are two types of monitor: good and strategic. A good monitor always detects the manipulation. By contrast, the probability that a strategic monitor detects the manipulation depends on the effort he exerts. At each time, the firm hires the monitor and pays him a competitive fee based on the monitor's perceived ability and incentives to detect the manipulation.

This game features a unique Markov perfect equilibrium in which the monitor's reputation and his behavior evolve over time based on the history of shocks. In equilibrium the monitor shirks when his reputation is below a threshold—because prospects are low—but also shirks when his reputation is above a threshold—because manipulation is less prevalent. Thus, extreme reputations, whether high or low, weaken the monitor's incentive to exert effort. The equilibrium thus features a "Shirk-Work-Shirk" structure. If the manager's manipulation were independent of the monitor's reputation, the equilibrium would feature the Shirk-Work structure that is typical in reputation settings with bad news (see e.g. Board and Meyer-ter Vehn (2013)). However, because manipulation decreases endogenously as monitor reputation improves, the monitor's incentive to exert effort also decreases, thus explaining why top reputation monitors shirk.

As our main contribution, we study whether and how a regulator should disclose information about monitor quality to maximize the monitor's effort incentives, and thus mitigate

[4] Sarbanes-Oxley Act Section 104 prescribes "no portions of the inspection report that deal with defects in the quality control systems of the firm under inspection shall be made public if those defects are addressed by the firm not later than 12 months after the date of the inspection report."

the manager's manipulation.

We show that disclosures can indeed strengthen reputational incentives and motivate the monitor to work harder. Intuitively, by disclosing information, the regulator can keep the monitor's reputation in the intermediate region in which the monitor exerts effort. This, however, comes at a cost. The strategic monitor anticipates future disclosures, and, knowing his own type, expects that any future disclosure reduces the reputation on average.[5] This reduces the monitor's value from building reputation, and he exerts less monitoring effort. In this sense, the regulator's ability to affect reputational incentives via disclosure is limited.

Specifically, the regulator commits to a disclosure policy ex-ante, which consists of verifiable signals about the monitor's type with a Poisson delay chosen by the regulator. We allow for both "good news" disclosure, in which the regulator reveals whether the monitor is good with a delay, and "bad news" disclosure, in which the regulator reveals whether the monitor is strategic. We find that revealing good news is optimal whenever reputation is high (at the upper end of the work region), while revealing bad news is optimal whenever reputation is low (at the lower end of the work region). Intuitively, revealing good news ends the game in case the monitor is good, but lowers the monitor's reputation if no news arrives. This is beneficial at the upper end of the work region, since keeps the monitor's reputation in the work region longer. Thus, it leads the monitor to exert more effort. Bad news similarly either ends the game by revealing that the monitor is bad or increases reputation if no news arrives. This is beneficial for low reputations, since the absence of news drives reputation towards the work region faster. In other words, bad news induces the reputation level to leave the lower shirking region more quickly.

The optimal disclosure policy prescribes no disclosure for top reputations. Effectively, this no-disclosure-at-the-top is a prize that rewards the monitor with opacity for his past performance (i.e., a history free of negative shocks). This result illustrates, as a general principle, that the regulator is more willing to interfere and disclose information for lower reputations. Part of the incentives that a low reputation monitor derives, come from the promised opacity (i.e., lack of scrutiny) that he will face when his reputation grows sufficiently.

In sum, by strategically choosing which kind of information to reveal, the regulator can strengthen reputational incentives provided by the market. Importantly, this sometimes involves deliberately destroying or lowering the monitor's reputation.

To be optimal, the arrival of information has to be uncertain. Indeed, no deterministic disclosure policy, in which the regulator commits to reveal verifiable information at a fixed

[5] In technical terms, since the strategic monitor knows that he is strategic, any information disclosure induces his reputation to follow a super-martingale.

time, can be optimal. The logic is simple. When faced with such a policy, the strategic monitor anticipates that his type will be revealed and that his reputation will be driven down to zero. Thus, the strategic monitor's reputation will be short-lived, which reduces his willingness to build a reputation in the first place. As a result, the monitor exerts less effort and the manager manipulates more. From an ex-ante perspective, deterministic disclosures are dominated by not disclosing any information at all.

Our results have concrete implications for regulatory disclosure. First, commonly used schemes which feature immediate disclosure or a fixed time delay are suboptimal.[6] While it is difficult for regulators to disclose information with a random time delay once it is collected, they can choose *how* to investigate monitors and which evidence to collect. Thus, our results imply that regulators should aim to uncover evidence that the monitor is "bad" whenever reputation is low and evidence that the monitor is "good" whenever it is high. In Section 6, we provide comparative statics about the range of beliefs for which regulators should collect evidence of the monitor being bad. Perhaps counterintuitively, we show that regulators should be less aggressive when the manager's manipulation, which the monitor is supposed to prevent, has more severe consequences, and that they should be more aggressive when they have more effective enforcement tools which directly punish the manager. These results arise because reputational incentives and regulatory intervention are substitutes in our model.

**Literature** Our paper contributes to the literature on monitoring and reputational incentives for banks (e.g. Diamond (1991), Hansen and Torregrosa (1992), Chemmanur and Fulghieri (1994a), Carter et al. (1998)), auditors (e.g. Beatty (1989), Firth (1990), Dye (1991), and Dye (1993)), and other intermediaries (e.g. Biglaiser (1993), Bar-Isaac (2003), and Mathis et al. (2009)). We expand on this literature by introducing a regulator who has learned the monitor's type and chooses whether to disclose it. Generally, while the literature focuses on the role of reputation, it does not consider how regulatory disclosure affects reputational incentives. However, how to disclose information is a key concern for regulators across many contexts (e.g. Jovanovic (1982), Alvarez and Barlevy (2014), Acito et al. (2017), and Goldstein and Leitner (2018)).

[6] For example, the Dodd-Frank Act requires banks to publish company-run stress test results within a certain time window. See `https://www.fdic.gov/regulations/reform/dfast/index.html`, esp. "all covered institutions [...] are required to submit the results of their company-run stress tests to the FDIC by July 31 and publish those results between October 15 and October 31." The PCAOB withholds inspection reports of non-compliant firms for one year. Specifically, Section 104(g)(2) of the SOX states that "no portions of the inspection report that deal with [...] defects in the quality control systems of the firm under inspection shall be made public if those [...] defects are addressed by the firm, to the satisfaction of the Board, not later than 12 months after the date of the inspection report."

Our model builds on the reputation literature and the literature on exponential bandits (e.g. Keller et al. (2005)). Our model features perfect bad news as in Board and Meyer-ter Vehn (2013), which leads to a shirk-work equilibrium in their paper.[7] Our equilibrium differs because we explicitly model the manager's behavior, which changes the dynamics of reputation.[8] By contrast, in Board and Meyer-ter Vehn (2013), there is only a single decision maker and an anonymous market. We also consider disclosures about the monitor, which are absent in Board and Meyer-ter Vehn (2013).

Varas et al. (2020) study the optimal monitoring mechanism when the agent has reputation concerns and the principal derives utility from learning the agent's type, but inspections are costly. Monitoring plays a dual role: learning and incentive provision. In our setting, disclosure provides incentives to two agents at the same time: the monitor and the monitored. Also, there is no role for information acquisition: the principal learns the monitor's type at the outset and can implement disclosure policies that depend on the monitor's type.

Disclosure in reputation games has been studied by Ekmekci (2011), Horner and Lambert (2016), and Di Pei (2016). These papers focus on disclosure about *actions* rather than disclosure about the *type*. Thus, our setting better fits a regulator who e.g. during the course of investigation has learned information about the monitor's type as opposed to the monitor's past behavior.

Finally, our paper is related to the recent literature on disclosure and monitoring in finance. Frenkel et al. (2020) study voluntary disclosure and its effect on analyst coverage and Banerjee et al. (2018) consider the effect of transparency on market prices. Gryglewicz and Mayer (2019) study monitoring in a dynamic contracting environment. Recent papers using Poisson learning include Martel et al. (2018), Hu and Varas (2019), and Geelen (2019).

# 2 Model

In this section, we study a continuous-time game between a long-run monitor and a sequence of firm managers who may opportunistically engage in manipulation. The analysis of the regulator's disclosure policy is deferred until Section 5.

[7]Also, Dilmé and Garrett (2015) features an inspector with switching costs and the reputation is about the inspector's state.

[8]Shirk-work-shirk equilibria have been obtained in other works in reputation, which do not feature perfect bad news, e.g. Board and Meyer-ter Vehn (2014) and Bonatti and Hörner (2017a). Both use perfect good news and the role of effort in these papers is different. In Board and Meyer-ter Vehn (2013), effort means investment in an imperfectly persistent type. In Bonatti and Hörner (2017a), effort increases the arrival rate of a breakthrough, but not affect an agent's incentives as it does in our paper. None of these papers feature disclosure.

**Players and Actions** There is a single monitor and a competitive market of identical firms. In each firm, there is a single manager. At any time $t \geq 0$, the monitor is matched randomly with a firm. The monitor and the firm's manager simultaneously choose monitoring effort $a_t \in [0,1]$ and manipulation $m_t \in [0,1]$. Following Holmström (1999), we assume that the firm pays a fee $p_t$ for the monitor equal to its expected value from hiring the monitor (see Equation (4)). The monitor is either *good* ($\theta = G$) or *strategic* ($\theta = S$). The good monitor exerts maximal effort at all times, i.e. $a_t = 1$ for all $t$, while the strategic monitor chooses effort optimally given a flow cost $ca_t$.[9] We henceforth refer to the strategic monitor simply as "monitor." Whenever no confusion can arise, we do not distinguish between the firms and managers that the monitor is matched with at each time and instead call them the "firm" and the "manager."

**Information** The monitor's type and effort and the manager's manipulation are all private. A public signal arrives with a Poisson rate $\lambda m_t(1 - a_t)$, which we interpret as a "loss" to the firm. This can represent e.g. loan defaults or an accounting scandal. Intuitively, a loss occurs with Poisson rate $\lambda m_t$ without the monitor and more manipulation leads to a higher arrival rate. By exerting effort, the monitor can "detect" manipulation and prevent the loss from occurring, so that the overall arrival rate is $\lambda m_t(1-a_t)$. Since the good monitor always chooses $a_t = 1$, a loss can only arrive when monitor is strategic.

Let $h^t$ be the public history of loss arrivals up to (and including) time $t$. The monitor's strategy $\{a_t\}_{t\geq 0}$ is predictable with respect to the filtration generated by $h^t$ and $\theta$. Let $\{m_t\}_{t\geq 0}$ denote the process induced by each time-$t$ manager's manipulation. Let $\{\hat{a}_t\}_{t\geq 0}$, with $\hat{a}_t \in [0,1]$, denote the conjecture about the monitor's effort by the firm and manager. Let $\{\hat{m}_t\}_{t\geq 0}$, with $\hat{m}_t \in [0,1]$ denote the conjectures about the manager's manipulation by the firm and monitor. The processes $\{\hat{a}_t\}_{t\geq 0}$, $\{m_t\}_{t\geq 0}$, and $\{\hat{m}_t\}_{t\geq 0}$ are predictable with respect to the filtration generated by $h^t$. Intuitively, the monitor conditions his effort on his type and on the public history $h^{t-}$ up to (but not including) time $t$, while the manager and the firm condition only on the public history.

The initial belief that the entrepreneur is good is $x_0 \in (0,1)$. Given conjectures $\{\hat{a}_t\}_{t\geq 0}$ and $\{\hat{m}_t\}_{t\geq 0}$, we denote the public belief at history $h^t$ as $x_t = \Pr(\theta = G|h^t)$. We define $x_{t-} = \lim_{s\uparrow t} x_s$.[10] This is the monitor's *reputation*. Since losses never occur for the good monitor, the firm must take the reputation into account when forming beliefs about loss arrival. From the firm's perspective, the arrival rate is $\lambda(1 - x_{t-})\hat{m}_t(1 - \hat{a}_t)$. Since the strategic monitor knows his own type, the arrival rate is $\lambda\hat{m}_t(1 - a_t)$ from his perspective.

[9]Thus, the good monitor is either a behavioral type or his cost of monitoring is zero.

[10]Thus, we have $x_{t-} = \Pr(\theta = G|h^{t-})$.

**Payoffs** At each time $t$, the (strategic) monitor receives fees from the firm he is matched with and his payoff is given by

$$V_0 = E^M \left[ \int_0^\infty e^{-rt}(p_t - ca_t)dt \right], \tag{1}$$

where $E^M[\cdot]$ denotes the expectation given the monitor's information and his conjecture about manipulation.

The (flow) payoff of the time-$t$ manager given manipulation $m_t$ and conjectured monitoring $\hat{a}_t$ is

$$(1 - x_{t-})(1 - \hat{a}_t)m_t - \frac{m_t^2}{2}. \tag{2}$$

This captures the following (heuristic) intuition. Manipulation yields a private benefit $m_t$ whenever it is not detected by the monitor, but to manipulate, the manager faces a quadratic effort cost, e.g. because of existing control mechanisms within the firm. Manipulation goes undetected if he faces the strategic monitor (with probability $1 - x_{t-}$) and if the strategic monitor fails to detect manipulation (with probability $1 - \hat{a}_t$). Conversely, if the monitor detects manipulation, the manager receives no private benefit.[11]

Similarly, the time-$t$ firm's flow loss from manipulation when hiring the monitor, given conjectures $\hat{a}_t$ and $\hat{m}_t$, is

$$\alpha(1 - x_{t-})(1 - \hat{a}_t)\hat{m}_t. \tag{3}$$

Here, $\alpha$ parameterizes the severity of losses for the firm.[12] Following Holmström (1999), we assume that the firm at time $t$ pays a fee

$$p_t = \alpha(1 - (1 - x_{t-})(1 - \hat{a}_t)\hat{m}_t) \tag{4}$$

to the monitor. Intuitively, if the firm does not hire a monitor, the manager optimally chooses $m_t = 1$, and the firm's flow loss is $\alpha$. Thus, the fee $p_t$ equals the benefit of hiring the monitor relative to not hiring one. Equivalently, the monitor is a monopolist who captures all rents from his monitoring ability.

We assume that

$$\alpha > c,$$

otherwise, we have $p_t < c$ for all $t$ and exerting effort is never optimal for the monitor.

[11] Including explicit penalties for manipulation being detected does not qualitatively change our results.

[12] We use a slight abuse of notation here. From the firm's perspective, losses arrive at rate $\lambda(1 - x_{t-})(1 - \hat{a}_t)\hat{m}_t$, so the expected loss is $\lambda\hat{\alpha}(1 - x_{t-})(1 - \hat{a}_t)\hat{m}_t$, where $\hat{\alpha}$ is the severity. To save notation, we combine the two parameters, i.e. $\alpha = \lambda\hat{\alpha}$.

**Admissibility** We restrict attention to Markovian strategies $m_t = m(x_{t-})$ and $a_t = a(x_{t-})$, and Markovian conjectures $\hat{a}_t = \hat{a}(x_{t-})$ and $\hat{m}_t = \hat{m}(x_{t-})$. As is standard in models with Poisson learning, we require that the functions $a(x)$, $m(x)$, $\hat{a}(x)$, and $\hat{m}(x)$ are piecewise Lipschitz continuous in $x$.[13] Additionally, we require that $\hat{m}(x), m(x) > 0$ and $\hat{a}(x) < 1$ for all $x \in (0, 1)$. We call such strategies *admissible*.

**Reputation Dynamics** Given admissibility, standard results guarantee that the law of motion for reputation is well-defined. Specifically, Bayes' rule implies that reputation follows

$$dx_t = \lambda x_{t-}(1 - x_{t-})(1 - \hat{a}(x_{t-}))\hat{m}(x_{t-})dt - x_{t-}dN_t, \tag{5}$$

where $N_t$ is the Poisson process marking the arrival of a loss. Intuitively, since the good monitor always exerts effort, observing a loss reveals that the monitor is strategic. Conversely, the absence of a loss is "good news" about the monitor's type and his reputation drifts upwards.

Let $\tau$ be the arrival time of the first loss. For $t < \tau$, reputation satisfies the ordinary differential equation (ODE)

$$\dot{x} = \lambda x(1 - x)(1 - \hat{a}(x))\hat{m}(x), \tag{6}$$

with initial condition $x_0$. We prove existence and uniqueness of a solution to this ODE under our admissibility assumption in Appendix A.

**Continuation Value** Given admissible strategies and conjectures, the firm's fee is Markovian and given by $p_t = p(x_{t-})$, where

$$p(x) = \alpha(1 - (1 - x)(1 - \hat{a}(x))\hat{m}(x)), \tag{7}$$

and the monitor's continuation value at time $t$ depends only on his current reputation, i.e.

$$V(x_t) = \sup_{\{a_s\}_{s\geq t}} E_t^M \left[ \int_t^\infty e^{-r(s-t)}(p(x_s) - ca_s)ds \right]. \tag{8}$$

**Markov Perfect Equilibrium** A Markov Perfect Equilibrium (MPE) is a collection of admissible strategies $(a(x), m(x))$ and conjectures $(\hat{a}(x), \hat{m}(x))$, and a price $p(x)$ such that (i) monitoring effort $a(x)$ maximizes the monitor's continuation value (8) at each $x$, (ii)

[13] That is, there exists a finite partition $x_1, ..., x_N$ of $[0, 1]$, such that $m$ and $a$ are Lipschitz continuous on any $[x_n, x_{n+1}]$ for $n = 1, .., N - 1$. See e.g. Board and Meyer-ter Vehn (2013).

manipulation $m(x)$ maximizes the manager's payoff (2), (iii) the price is given by (7), (iv) reputation follows (5), and (v) all conjectures are correct, i.e. $m(x) = \hat{m}(x)$ and $a(x) = \hat{a}(x)$ for all $x$. We provide a formal definition in Appendix A. We focus on MPE throughout the paper.

# 3 Applications and Assumptions

In this section we discuss applications of our model, and explain our modeling choices.

**Bank Regulation** When banks screen borrowers or monitor their loans, they are driven by reputation concerns.[14] In our model, reputation is about the quality of the bank's internal control mechanisms, which affect its cost of monitoring. Specifically, banks with strong control mechanisms have a lower cost of monitoring loans, for example because moral hazard problems of loan officers are reduced. We can interpret the Poisson shock as the arrival of a loan default. The borrower can choose whether to put in costly effort, which makes loan default less likely. Thus, the borrower's value is equivalent to Equation (2). In particular, we can describe the borrower's effort as $1 - m_t$, where $m_t$ is defined in Equation (9). Thus, exerting more effort is equivalent to "manipulating less" in our main model. The fee in Equation (4) is the price at which the bank can sell off its loans, e.g. to be securitized, which depends on its reputation. It represents the premium the bank can charge for having a reputation.

Regulators routinely investigate banks and publicize failures of banks' quality control systems, which we interpret as the bank's type. For example, the SEC publishes the outcomes of its administrative proceedings online[15] and frequently finds that banks' "supervisory policies and procedures were not reasonably designed and implemented to provide sufficient oversight," that banks "did not have adequate processes to verify that the information it was reporting was accurate," or that banks "failed to reasonably design and implement [...] policies."[16] Similarly, the FDIC publishes correction notices[17], where it routinely finds shortcomings in banks' internal controls. Thus, we can interpret such disclosures as being about the bank's type. How these disclosures affect the bank's reputational incentives is our key object of interest.

[14] See e.g. Chemmanur and Fulghieri (1994b).

[15] See https://www.sec.gov/litigation/admin.shtml. Often, these are cease and desist orders which are accompanied by either no fines or very small ones.

[16] See https://www.sec.gov/litigation/admin/2019/34-85395.pdf, https://www.sec.gov/litigation/admin/2018/34-84759.pdf, and https://www.sec.gov/litigation/admin/2018/ia-5061.pdf, respectively.

[17] See https://orders.fdic.gov/s/.

**Auditors** The audit industry is a good application of our model. First, auditors' incentives are strongly driven by reputation concerns[18] and auditors' demands are sensitive to their reputation.[19] Second, the regulator (PCAOB) collects information and discloses auditor quality to the market to strengthen auditors' reputational incentives. Third, the regulator commits to a disclosure policy, as described by the law (See SOX, section 404).

Specifically, we can think of the monitor as the auditor hired by a firm to prevent its manager's accounting manipulation. The firm's manager may engage in manipulation for a private gain, which he realizes only if the auditor does not prevent it. Whether he is caught depends on the auditor's effort and his type —which represents the quality of the auditor's control system. Auditors with weak quality controls have a higher cost for exerting auditing effort. They hence correspond to the strategic type in our model. Finally, we can interpret the Poisson shock as the discovery of discrepancies by investors or by the firm itself, in which case the firm is forced to issue a restatement.

Auditors are regulated and the regulator discloses auditor quality, as a means to provide incentives. Indeed, the Public Company Accounting Oversight Board (PCAOB) oversees audit companies, by conducting inspections and uncovering audit discrepancies.[20] Importantly, the PCAOB's inspections are designed to assess the effectiveness of the auditor's quality control policies,[21] which corresponds to the monitor's type in our model.[22] Releasing this information to the public, which is one of the PCAOB's main enforcement tools, may serve as a deterrent.[23] Indeed, when the inspection finds deficiencies, auditors are more likely to lose clients, have lower growth in audit fees, and lose market share.[24] However, the release of information also affects the auditor's incentive to build a reputation in the first place. This channel has, to our knowledge, been absent in the debate about the PCAOB's effectiveness.

[18]For example, the Economist points asserts "KPMG's troubles tarnish its main asset -its reputation. A big enough blow could knock it over, disrupting capital markets in turn. According to Audit Analytics, a research firm, KPMG audited 19% of the S&P 500 in 2017 and a quarter of companies in the FTSE 350. If clients fled, other firms would have to absorb that work."

[19]For example Skinner and Srinivasan 2012 document the Japanese affiliate of PwC lost one quarter of its clients after it failed to detect a clients' large accounting fraud.

[20] See e.g. PCAOB (2010) for a description of the PCAOB's activities.

[21]See PCAOB (2010), p. 6.

[22]The PCAOB inspections have two parts. Part I evaluates the auditor's past performance, by examining a sample of the auditor's past audit engagements. Part II evaluates the quality of the auditor's control system, which conceptually is closer to our notion of auditor's type. The Part II findings are disclosed online (see https://pcaobus.org/Inspections/Reports/Pages/FirmsFailedToAddressQCSatisfactorily.aspx), when a firm fails to address the PCAOB criticism within a year. The actual delay with which these findings are disclosed varies, since there is a negotiation process after the twelve months deadline.

[23]See PCAOB (2016) for a breakdown of enforcement actions.

[24]See Acito et al. (2017), Boone et al. (2014), and Aobdia and Shroff (2017), respectively. See also Firth (1990) for an early study.

**Brokers and Financial Advisers** Financial advisers have an incentive to recommend unsuitable products to investors, thus effectively manipulating investors' perceptions.[25] Financial advisory (or brokerage) *firms*, however, are supposed to have supervisory systems in place to prevent individual advisers from recommending unsuitable investments. In the context of our model, the brokerage firm is the monitor and an individual adviser is the manager. By recommending unsuitable products, the adviser increases his payoff, via commissions or kickbacks, but is punished by the firm if his behavior is discovered. The arrival of public news can be interpreted as lawsuits by customers, after their investments realize losses. The monitor's type is again the quality of the firm's internal control systems. A high reputation is valuable because it allows the firm to attract more customers[26] or to charge higher fees.

FINRA oversees brokerage firms and publishes enforcement notices online, which frequently cite failures of firms' quality control systems.[27] These disclosures, in turn, affect the broker's reputation among customers.

**Modeling Assumptions** To make our model tractable, we use perfect bad news, i.e. a realization of the Poisson shock reveals the strategic type perfectly. This is a common tool to ensure tractability in many different settings involving dynamic learning.[28] In our model, perfect bad news requires that the good monitor always detects manipulation. While this assumption is stylized, it allows us to avoid significant technical difficulties.[29]

Because the monitor's type is fixed, his reputation is transitory and, eventually, his type will be revealed. This outcome is expected in models of imperfect monitoring with fixed types.[30] A model with changing types, as in Board and Meyer-ter Vehn (2013), would prevent reputations from becoming degenerate in the long run, at the cost of significantly increased complexity. This generalization would weaken the negative effect of a loss arrival, but we don't have a reason to think this would qualitatively change our results.

To abstract from repeated-games like interactions between the monitor and manager, we assume that the manager is a short-term player, or, equivalently, the monitor is matched with a sequence of firms. This assumption is reasonable in the context of our applications.

[25]See e.g. the 2015 White House report: `https://obamawhitehouse.archives.gov/sites/default/files/docs/cea_coi_report_final.pdf`.

[26]Who, in equilibrium, know that they are less likely to be defrauded.

[27]See `http://www.finra.org/industry/enforcement`.

[28]See Keller and Rady (2015), Bonatti and Hörner (2017b), and Halac and Kremer (2018) for experimentation models, Board and Meyer-ter Vehn (2013) and Dilmé (2014) for reputation models, and Marinovic and Varas (2016) for a model of dynamic disclosure.

[29]Specifically, our equilibrium can be characterized by a nonlinear ODE, in Equation (13). With imperfect news, this would become a nonlinear delay-differential equation.

[30]See Cripps et al. (2004).

First, a long literature argues that managers behave myopically,[31]. Second, in our banking application, banks often sell loans to a third party, so there is little room for long-term interactions between the borrower and the bank.

# 4 Analysis

In any MPE, the manager's optimal manipulation is given by

$$m(x, \hat{a}(x)) = (1-x)(1-\hat{a}(x)), \tag{9}$$

which follows from plugging the conjecture $\hat{a}(x)$ into Equation (2) and maximizing. This result is intuitive: if the manager expects the monitor to exert more effort, he manipulates less; if he believes that the monitor is good, he manipulates less as well.

Given the monitor's optimal strategy, the ODE for reputation (Equation 6) becomes

$$\dot{x} = \lambda x m(x, \hat{a}(x))^2. \tag{10}$$

In equilibrium, the monitor's value function solves the following HJB equation:

$$rV(x) = \max_{a\in[0,1]} p(x) - ca + \dot{x}V'(x) + \lambda m(x, \hat{a}(x))(1-a)(V(0) - V(x)), \tag{11}$$

subject to the ODE for reputation (10), the price (7), and an appropriate conjecture $\hat{a}(x)$. We prove this formally as part of establishing Propositions 1 and 2 below.

The right-hand side of the above equation captures the return to the monitor. The monitor's flow benefit consists of the fee $p$ net of monitoring cost $ca$. He also receives the capital gains associated with changes in his reputation. The latter come in "two flavors": the positive drift in reputation arising when there is no loss, and the decrease in reputation caused by a loss, which is equal to $V(x) - V(0)$.

The relevant boundary conditions are $V(0) = 0$ and $V(1) = \frac{\alpha}{r}$. That is, when the strategic monitor is discovered ($x = 0$), he has no incentive to exert effort and the fee is 0, leading to a continuation value of zero. When the firm and manager believe that the monitor is good ($x = 1$), the manager does not manipulate for fear of being detected, i.e. $m(1) = 0$, and the fee becomes $p(1) = \alpha$. The monitor's continuation value is then $\alpha/r$.

Since the HJB Equation (11) is linear in monitoring effort, the monitor exerts effort at $x \in (0, 1)$ whenever

$$\lambda m(x, \hat{a}(x))V(x) \geq c.$$

[31] See e.g. Stein (1988).

Intuitively, higher effort reduces the expected loss of reputation by $\lambda m(x) V(x)$, but it costs the monitor $c$. Thus, the monitor's incentive to work comes from the threat of losing his reputation. The monitor prefers to shirk whenever his continuation value $V(x)$ is low. In this case, losing his reputation when the loss arrives is not very costly and hence he faces little incentive to exert effort. The monitor also prefers to shirk whenever the manager manipulates little. In this case, a loss is unlikely to arrive even if the monitor shirks, and the monitor is unlikely to lose his reputation.

The last point implies that monitoring can never be perfect in equilibrium. Intuitively, if the monitor exerts full effort at any interior reputation level, then the manager does not manipulate since he expects to be detected (see Equation (9)). But then, if the monitor deviates to shirking, no loss can arrive, so exerting effort is not optimal. In equilibrium, the monitor never chooses full effort and there is always a positive likelihood of observing a loss ex-post. Thus, reputational incentives are never strong enough to prevent manipulation in our model.

More starkly, whenever the monitoring cost $c$ is sufficiently high, the monitor always shirks, regardless of his reputation level. In that case, reputational incentives are too weak at any level of reputation.

**Proposition 1.** *If $\frac{c}{\lambda} > \max_x (1-x) V_s(x)$, there is a unique equilibrium, in which the monitor always shirks.*[32] *The monitor value $V_s$ is the unique solution to the HJB equation*

$$rV_s(x) = \alpha\left(1-(1-x)^2\right) + V_s'(x)\lambda(1-x)^2 x - \lambda(1-x)V_s(x), \tag{12}$$

*with boundary condition*

$$V_s(1) = \frac{\alpha}{r}.$$

*The manager's manipulation is*

$$m(x) = 1-x,$$

*and the fee is*

$$p(x) = \alpha\left(1-(1-x)^2\right).$$

In the shirking equilibrium, the firm does not expect the (strategic) monitor to exert effort and thus the equilibrium fees and the monitor's continuation value are both low. Since the monitor's value is low, his incentives to exert effort are low as well and the monitor always shirks. Despite this apparent circularity, the shirking equilibrium is unique. Intuitively, since the monitor is forward-looking and the arrival rate of losses is strictly positive, the monitor's value function is continuous in reputation. Any equilibrium in which the monitor

[32] See Proposition 10 for sharp sufficient conditions not involving the function $V_s(x)$.

works, however, requires that $V(x) \geq c/\lambda(1-x)$, which under the parametric assumption of Proposition 1 implies a jump in the monitor's value, which is impossible.

We next characterize the unique equilibrium for moderate effort costs. This equilibrium has a shirk-work-shirk structure, i.e. the monitor shirks for both high and lower reputations, and exerts effort at intermediate ones. Thus, in our monitoring game, both high and low reputations are detrimental to monitor incentives.

**Proposition 2.** *If $\frac{c}{\lambda} < \max_x (1-x) V_s(x)$ and $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}} \frac{2}{3\sqrt{3}} < rc$, there is a unique equilibrium which takes the form of a shirk-work-shirk equilibrium. There are thresholds $0 < x_l < x_h < 1$ such that $a(x) \in (0,1)$ for $x \in (x_l, x_h)$ and $a(x) = 0$ for $x \leq x_l$ and $x \geq x_h$.*

*Manipulation $m(x)$ is strictly decreasing in reputation. On $(x_l, x_h)$, $m(x)$ solves the ODE*

$$rc = \lambda(\alpha - c)\, m(x) - \alpha\lambda m(x)^3 - \lambda c x m'(x)\, m(x) \tag{13}$$

*with boundary conditions $m(x_h) = 1 - x_h$ and $m(x_l) = 1 - x_l$. For $x \leq x_l$ and $x \geq x_h$ we have $m(x) = 1 - x$.*

*The monitor value $V(x)$ is strictly increasing and satisfies the shirking ODE (12) on $[x_h, 1]$, with boundary condition $V(1) = \frac{\alpha}{r}$, and on $[0, x_l]$, with boundary condition $V(x_l) = \frac{c}{\lambda(1-x_l)}$. On $(x_l, x_h)$, the monitor's value satisfies the indifference condition $V(x) = \frac{c}{\lambda m(x)}$.*

In equilibrium, a monitor with low reputation shirks because of low fees and poor prospects, and therefore low continuation values. Intuitively, once the monitor's reputation is low, losing it is not particularly costly, and the monitor shirks. When the monitor's reputation is high, the manager manipulates little, since he believes that he faces the good monitor with high probability. Then, the monitor also shirks, because he is unlikely to lose his reputation due to a loss arrival. For intermediate reputations, incentives are strong enough to induce the monitor to exert effort. As we outlined above, the monitor never exerts full effort, so that his effort on the work region is interior. Then, monitor is indifferent between working and shirking, which yields the indifference condition $\lambda m(x)V(x) = c$. The monitor anticipates future effort, fees, and manipulation and, given these expectations, forms a value. On any interval where the monitor works, the anticipated manipulation must be such that the monitor remains indifferent. This determines the evolution of manipulation in the work region, via the ODE (13).

Figure 1 illustrates the equilibrium levels of monitor effort and manipulation. Whenever the monitor shirks, manipulation decreases linearly with reputation, because the manager is more likely to face the good monitor. However, in the work region, manipulation is lower because the manager expects the strategic monitor to exert effort.

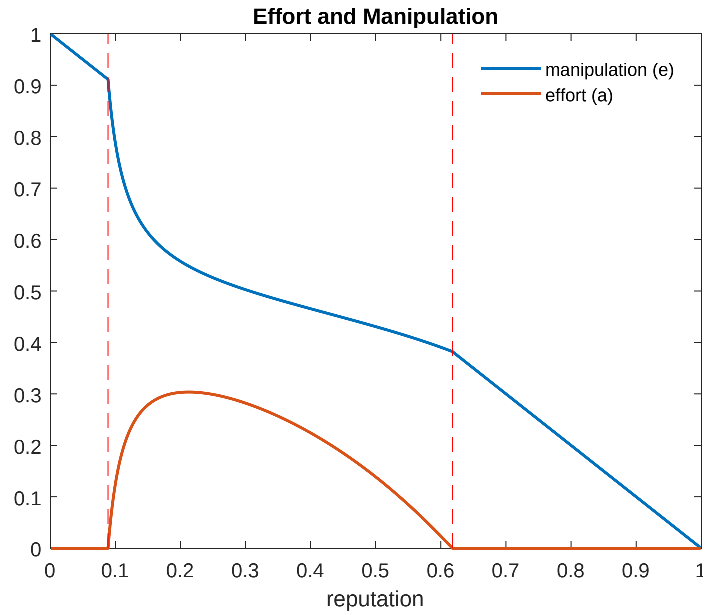



**Figure 1:** Parameters: $\lambda = 1, r = 0.5, \alpha = 1.75, c = 0.75$. The blue line represents the manager's manipulation strategy. Intuitively, manipulation decreases in monitor reputation, $x$. The red line represents the monitor effort strategy. The monitor shirks in both tails of the support of reputations and exerts effort over an intermediate range.

Overall, our analysis suggests that reputational incentives for monitors are strongest for intermediate reputations. Both high and low reputations are detrimental to monitor incentives, which encourages manipulation.[33]

# 5 Disclosing Monitor Quality

In the shirk-work-shirk equilibrium, the monitor only exerts effort for moderate reputations. In this section, we show that this has an important implication: uncertainty about the monitor's type is valuable, because then reputational incentives are the strongest. A regulator, who has collected information about the monitor's type and chooses how to disclose it, may prefer to induce this uncertainty. Such disclosures frequently occur in reality, in the context of banking or brokerage regulation, as well as auditing.

Specifically, we consider a regulator who learns the monitor's type at time 0 and can commit to verifiable disclosures. For tractability, we restrict attention to two classes of disclosure policies: (1) deterministic disclosures, i.e. the regulator reveals the monitor's type at some fixed time $T \geq 0$, and (2) Poisson disclosures, which either take the form of *good news* or *bad news*. Under a good news policy, the regulator discloses that the monitor is

[33]For completeness, we consider the case when effort cost is sufficiently low, so that the equilibrium is work-shirk, in Appendix B.3.

good with Poisson rate $\gamma_{Gt}$, while under a bad news policy, the regulator discloses that the monitor is strategic with Poisson rate $\gamma_{Bt}$.

The regulator's goal is to minimize the expected manipulation, for example because it carries an unmodeled social cost. Her value is given by

$$W_0 = E^R \left[ \int_0^\infty e^{-rt} \alpha \left(1 - m\left(x_t\right)\right) dt \right]. \tag{14}$$

Here, $E^R[\cdot]$ denotes the expectation given the regulator's information, which includes the monitor's type and any public information.[34]

To focus on the regulator's disclosure, we do not consider other tools such as fines, lawsuits, bans, or other enforcement actions. Disclosure remains important, and frequently used, in practice, even though regulators have access to other enforcement tools.

**Remark 3.** *In our model, the value of disclosure is purely driven by its impact on the monitor's incentives. If we are in the shirking equilibrium (see Proposition 1), these reputational incentives are absent and disclosing information cannot be beneficial. In that case, we have $m(x) = 1 - x$, so that the regulator's value is linear in reputation, i.e.,*

$$W_s\left(x_0\right) = E^R \left[ \int_0^\infty e^{-rs} \alpha x_t dt \right] = \frac{\alpha x_0}{r}.$$

*The regulator is indifferent about disclosing information, since under any disclosure policy, reputation must follow a martingale (see e.g. Kamenica and Gentzkow (2011)). In particular, the regulator's value is the same under any disclosure policy, including a policy of no disclosure.*

## 5.1 Deterministic Disclosure

Regulators often adopt disclosure policies which are deterministic in time. For example, the Dodd-Frank Act requires banks to publish company-run stress test results within a certain time window[35] and the PCAOB withholds inspection reports of non-compliant firms for one year.[36] Such policies cannot be optimal in our model, because they diminish the monitor's

[34] In particular, the regulator does not observe monitoring effort or manipulation.

[35] See `https://www.fdic.gov/regulations/reform/dfast/index.html`, esp. "all covered institutions [...] are required to submit the results of their company-run stress tests to the FDIC by July 31 and publish those results between October 15 and October 31."

[36] Section 104(g)(2) of the SOX states that "no portions of the inspection report that deal with [...] defects in the quality control systems of the firm under inspection shall be made public if those [...] defects are addressed by the firm, to the satisfaction of the Board, not later than 12 months after the date of the inspection report." The PCAOB thus withholds this information for one year (see also PCAOB (2006)).

reputational incentives.

**Proposition 4.** *Consider two policies: (1) disclose the monitor's type after a fixed time delay $T \geq 0$ and (2) disclose the monitor's type when the reputation reaches a certain level $x_1$. Both policies yield a lower value to the regulator than not disclosing any information.*

Intuitively, the regulator's value in a shirking equilibrium is a lower bound for her payoff. Disclosing the monitor's type, at any reputation level $x_1$, leads to shirking by the strategic monitor thereafter, generating an expected value of $x_1 W(1) + (1 - x_1)W(0) = x_1 \frac{\alpha}{r}$, which is exactly the regulator's value in a shirking equilibrium. Such a deterministic disclosure policy also leads to zero continuation value for the strategic monitor at $x = x_1$. Relative to no disclosure, revealing the monitor type at $x_1$ thus weakens the monitor's incentives to exert effort before $x_1$, which in turn increases the manager's manipulation. Therefore, it is never optimal to deterministically disclose the monitor's type.[37]

## 5.2 Random Delay

Since deterministic disclosure policies cannot be optimal in our model, how should regulators disclose information? We now show that disclosure policies with random delay are valuable. Specifically, the regulator may choose to reveal information if the monitor is strategic with a random time delay, but reveal nothing if the regulator is good, and vice versa. We refer to these policies as bad and good news respectively.

### 5.2.1 Disclosure of Bad News

Suppose that, at each point in time, the regulator discloses the monitor type with intensity $\gamma_t \in [0, \bar{\gamma}]$ when the monitor is strategic, but never discloses information when the monitor is good. Formally, the arrival rate of the Poisson process $N_t$ is now given by $\lambda m_t(1 - a_t) + \gamma_t$ if the monitor is strategic, and, as before, by zero if the monitor is good. The regulator chooses the process $\{\gamma_t\}_{t \geq 0}$ ex-ante, before she learns the monitor's type. Here, we focus on Markovian disclosure policies, i.e. $\gamma_t = \gamma(x_{t-})$ under the same admissibility requirements as in Section 2.[38] We consider more general policies in Section 5.3.

To understand the value of bad news disclosure, we first consider how it affects the manager's beliefs and the monitor's incentives. We rewrite the evolution of beliefs for $t < \tau$

After that, the reports of non-compliant firms are released publicly on the PCAOB website, at `https://pcaobus.org/Inspections/Reports/Pages/FirmsFailedToAddressQCSatisfactorily.aspx`.

[37] An analog argument holds for the case with fixed time delay. The proof of Proposition 4 is exactly the argument we have just outlined. We hence skip the formal proof for the sake of brevity.

[38] Note that, whenever $\bar{\gamma}$ is sufficiently small, the equilibrium remains a shirk-work-shirk equilibrium for any disclosure policy. This, in particular, guarantees existence of an equilibrium for any disclosure policy.

as

$$\dot{x} = \lambda x \hat{m}(x)^2 + \gamma(x) x (1 - x). \tag{15}$$

Equation (15) follows from Bayes' rule and shows that when the rate of bad news is positive, the reputation drift is steeper: in the absence of arrivals the monitor reputation increases faster (relative to the case without bad news, $\gamma = 0$) because no news becomes a more favorable signal of monitor quality.

However, this is not true from the strategic monitor's perspective. The strategic monitor knows his own type and therefore anticipates that the information the regulator reveals will, on average, lower his reputation. Formally, if the regulator reveals bad news at rate $\gamma(x)$, she expects the impact on the reputation to be zero, i.e.

$$\gamma(x_{t-})x_{t-}(1-x_{t-})dt - x_{t-}E^R(dN_t) = \gamma(x_{t-})x_{t-}(1-x_{t-})dt - \gamma(x_{t-})x_{t-}(1-x_{t-})dt = 0, \tag{16}$$

where $E^R[\cdot]$ denotes the expectation with respect to the regulator's ex-ante information. The strategic monitor instead expects the impact to be negative, i.e.

$$\gamma(x_{t-})x_{t-}(1-x_{t-})dt - x_{t-}E^M(dN_t) = \gamma(x_{t-})x_{t-}(1-x_{t-})dt - \gamma(x_{t-})x_{t-}dt < 0, \tag{17}$$

where $E^M[\cdot]$ denotes the expectation from the strategic monitor's perspective.

Thus, disclosing bad news diminishes the reputational incentives of the strategic monitor, because it makes his reputation accumulate more slowly. As a result, the monitor exerts less effort and the manager manipulates more. We study this effect formally in the proposition below.

**Proposition 5.** *Disclosure of bad news reduces the strategic monitor's incentive to exert effort for reputation sufficiently close to $x_h$ and on $[x_h, 1]$.*

Even though disclosure may reduce the monitor's incentives, it can be valuable for the regulator. Suppose the regulator provides delayed bad news when the reputation is just below the work region, i.e. when $x$ is just below $x_l$. In that case, the monitor shirks anyway,[39] so there are no adverse incentive effects. If the bad news does not realize, the reputation will increase faster (see Equation (15)), so the monitor will reach the work region sooner. This is valuable for the regulator, because manipulation is lower in the work region.

As the proposition below shows, disclosing bad news remains valuable even inside the work region, where it has a direct impact on the monitor's effort.

[39] Recall from Proposition 2 that without disclosure the monitor shirks for all $x < x_l$.

**Proposition 6.** *For $\bar{\gamma}$ sufficiently small, disclosure of bad news is valuable to the regulator when $x \leq x_d$, where $x_l \leq x_d < x_h$. It is not valuable for $x$ close to (or above) $x_h$.*

Intuitively, the regulator uses disclosure to gamble on reaching the work region faster (if $N_t$ does not realize) or destroying the strategic monitor's reputation (if $N_t$ realizes). This gamble is valuable, precisely because reaching the work region is valuable.

We can see the impact on the regulator's value given her HJB equation:

$$rW(x) = \alpha(1 - m(x)) + \left(\lambda x m(x)^2 + \gamma x(1-x)\right) W'(x) - \lambda m^2(x) W(x) - \gamma(1-x) W(x) \tag{18}$$

for some given $\gamma$. Now, the effect of bad news on the regulator's value is captured by

$$\gamma x(1-x) W'(x) - \gamma(1-x) W(x). \tag{19}$$

The first term captures the positive effect of bad news on the reputation drift; the second term is the loss caused by the expected realization of bad news. Below $x_l$, $W'(x)$ is relatively high because the regulator anticipates reaching the work region. Hence, for low reputations, disclosure of bad news has a favorable effect: the benefit of increasing the reputation drift outweighs the risk of a negative disclosure.

Consistent with this intuition, disclosing bad news is not valuable for sufficiently high reputations. Increasing the drift of reputation only makes the monitor leave the work region more quickly, so the value of having higher reputation is relatively low for the regulator. Then, the increase in reputation is not enough to compensate for the potential loss when bad news is revealed or for the effects on the monitor's incentives.[40]

Finally, for reputations above $x_h$ the monitor shirks, but disclosing bad news cannot be valuable. It does not induce the monitor to work on that region, but instead lowers his value from reaching a high reputation. This leads him to exert less effort at lower reputation levels.

### 5.2.2 Disclosure of Good News

We now show that delayed good news are valuable when the reputation is sufficiently large. Suppose that the regulator can disclose good news at rate $\gamma_t \in [0, \bar{\gamma}]$. That is, when the monitor is good, his type is revealed at rate $\gamma_t$, but the regulator discloses nothing if the monitor is strategic. Formally, we introduce a second publicly observable Poisson process $N_t^G$, which has arrival rate $\{\gamma_t\}_{t\geq 0}$ if the monitor is good and zero arrival rate if he is

[40] Formally, $W'(x)$ in Equation 19 is relatively small, so the expression is negative for $x$ close to $x_h$.

strategic. As in the previous section, we only consider admissible Markovian disclosure policies $\gamma_t = \gamma(x_{t-})$.[41]

When only good news are expected to arrive, then the absence of arrivals reduces the monitor's reputation. The change in reputation for $t < \tau$ is now given by

$$\dot{x} = \lambda x \hat{m}(x)^2 - \gamma(x)x(1-x). \tag{20}$$

Disclosure of good news hence reduces the drift from the regulator's perspective.

Just as in the bad news case, disclosing good news does not change the average reputation growth from the regulator's perspective, i.e.

$$-\gamma(x_{t-})x_{t-}(1-x_{t-})dt + (1-x_{t-})E^R(dN_t) = 0,$$

since $E^R(dN_t) = \gamma(x_{t-})x_{t-}$. For the strategic monitor however, the good news never realize. For him, disclosure reduces reputation growth on average and the impact on the drift is given by

$$-\gamma(x_{t-})x_{t-}(1-x_{t-})dt.$$

This reduces the monitor's value from building reputation and leads him to exert less effort. We show this result formally in the proposition below.

**Proposition 7.** *Disclosure of good news increases manipulation and decreases the value of the strategic monitor on the work region* $[x_l, x_h]$.

Thus, good news decreases the speed at which the monitor reputation improves in the absence of arrivals. This deteriorates the monitor's incentives and, consequently, exacerbates the manager's manipulation. Yet, such a policy may still benefit the regulator. If the reputation is close to the upper bound of the work region, the monitor stays in the work region longer, because the drift of reputation is lower. Effectively, the regulator can use good news to delay reaching a high reputation at which the monitor shirks.

**Proposition 8.** *For* $\gamma$ *sufficiently small and* $x < x_h$ *sufficiently close to* $x_h$*, the regulator benefits from disclosing good news. Disclosing good news is not valuable for* $x \geq x_h$.

Formally, when the good news disclosure rate is $\gamma$ the regulator's value follows the HJB

[41] Note that with good news, our definition with admissibility may not be sufficient to ensure that the ODE for reputation, in Equation (20), has a unique solution for all $x$ and all policies, because the ODE may become degenerate. However, for $\bar{\gamma}$ sufficiently small, this is not an issue, since the optimal policy involves no disclosure for $x > x_h$. Then, Equation (20) admits a unique solution on the relevant path of play.

equation:

$$rW(x) = \alpha(1-m(x)) + \left(\lambda x m(x)^2 - \gamma x(1-x)\right) W'(x) - \lambda m^2(x) W(x) + \gamma x (W(1) - W(x)) \tag{21}$$

with boundary condition $W(1) = \frac{\alpha}{r}$. The value of disclosing good news is given by

$$-\gamma x(1-x) W'(x) + \gamma x \left(\frac{\alpha}{r} - W(x)\right).$$

The first term is the negative effect of disclosure on reputation: the presence of disclosure reduces the drift, absent arrivals. The second term captures the effect of an arrival, which naturally benefits the regulator because it eliminates manager's manipulation going forward.

The good news policy is valuable when the slope of the regulator value $W'(x)$ is small, given that the effect of disclosure on the drift is negative. On the upper part of the work region (toward $x_h$) the slope of the regulator value $W'(x)$ is relatively low because a higher reputation means exiting the work region. This is why good news is valuable toward $x_h$: if no good news arrive, then the monitor reputation deteriorates, thereby delaying the process by which the reputation exits the work region. By contrast, if good news arrive, then the regulator obtains a large gain.

**Good and Bad News Combined** To summarize our results: *i*) disclosing bad news is valuable for low reputations, *ii*) Good news is valuable toward $x_h$ and *iii*) No disclosure is valuable for top reputations, in the upper shirking region.

The intuition is as follows. The value of disclosure is to bring reputations into the region where reputational incentives "do their job": for low values, this means disclosing bad news. This does not hurt the good type monitor, and allows the regulator to induce more effort from the strategic type if there is no disclosure realization. For relatively high values, this means disclosing good news, which lowers the reputation for the strategic type because for him the good news never materializes. This result is beneficial for the regulator, since it keeps the reputation inside the work region for a longer time. Of course, this is anticipated and may destroy some incentives for monitors with lower reputations. However, close to $x_h$, disclosing good news is beneficial overall. Finally, the optimal policy requires that the regulator stays silent for top reputations, above $x_h$. This feature is key for incentives: the regulator commits to opacity once the monitor has achieved a relatively high reputation, as a reward for good past performance. Disclosure above $x_h$ does not have the ability to improve incentives locally, but it weakens the incentives at lower reputation levels.

Because the ODE for the dynamics of manipulation is nonlinear (Equation (13)), we cannot qualitatively characterize the regulator's optimal disclosure on the interior of the work region. The full problem of the regulator is therefore not analytically tractable. However, we can solve the regulator's problem numerically, which we do in the following section.

## 5.3 Full Problem

We now consider the full problem of the regulator, who maximizes her value by choosing a disclosure policy $\{\gamma_{Bt}, \gamma_{Gt}\}_{t\geq 0}$, which is predictable with respect to the filtration generated by the public history $h^t$, but which is not necessarily Markovian in reputation.[42] We maintain the assumptions of Section 5. That is, the regulator has commitment, she chooses the policy ex-ante, disclosure is verifiable, and the rates of good news and bad news disclosure are bounded, i.e., $\gamma_{Bt}, \gamma_{Gt} \leq \bar{\gamma}$ for some $\bar{\gamma}$ sufficiently small. We show numerically that the results match the ones we described qualitatively in Section 5.2. Specifically, on the lower shirking region, disclosure of bad news is optimal, but disclosure of good news is not. By contrast, disclosure of good news is optimal on the working region, while disclosure of bad news is not optimal whenever the monitor's value is sufficiently high.

When choosing the rates $\gamma_{Bt}$ and $\gamma_{Gt}$, the regulator must take the effect on the monitor's value into account, which affects the monitor's incentive to exert effort. As we have seen, the monitor and the regulator evaluate disclosure policies using different information sets, because the monitor's type is private information. Denote with $N_t^B$ the Poisson process for bad news, which includes a loss arriving and the regulator disclosing that the monitor is strategic, and with $N_t^G$ the process for good news. We have

$$\begin{aligned} E^R\left(dN_t^B\right) &= \lambda\left(1-x_{t-}\right)\hat{m}_t\left(1-\hat{a}_t\right)+\gamma_B\left(1-x_{t-}\right) \\ E^R\left(dN_t^G\right) &= \gamma_{Gt}x_{t-} \end{aligned}$$

and

$$\begin{aligned} E^M\left(dN_t^B\right) &= \lambda\hat{m}_t\left(1-a_t\right)+\gamma_{Bt} \\ E^M\left(dN_t^G\right) &= 0. \end{aligned}$$

That is, the regulator believes that bad news arrives with rate $\lambda\hat{m}_t\left(1-\hat{a}_t\right)+\gamma_{Bt}$ if the monitor is strategic, which is true with probability $1-x_{t-}$. The monitor, by contrast, knows that he is strategic, and that the arrival rate is $\lambda\hat{m}_t\left(1-a_t\right)+\gamma_{Bt}$. Similarly, the regulator believes that bad news arrives at rate $\gamma_{Gt}$ if the monitor is good, which is true with

[42]Similarly to the baseline model, we restrict attention to policies such that the laws for reputation and the monitor's continuation value, Equations (22) and (23) admit unique solutions.

probability $x_{t-}$, but the strategic monitor knows that good news will never arrive.

From the regulator and the firm's perspective, the belief $x_t$ follows the law of motion

$$dx_t = \left(\lambda x_{t-}\hat{m}_t^2 + x_{t-}\left(1 - x_{t-}\right)\left(\gamma_{Bt} - \gamma_{Gt}\right)\right) dt - x_{t-}\left(1 - x_{t-}\right) dN_t + \left(1 - x_{t-}\right) x_{t-} dN_t^G \quad (22)$$

with initial condition $x_0$.

We now derive the regulator's optimal control problem, which uses the belief $x_t$ and the monitor's continuation value $V_t$ as state variables. We first reformulate the monitor's value function using a martingale representation result. The proof is in Appendix C.3.

**Lemma 1.** *For any admissible disclosure policy* $\left(\gamma_{Bt}, \gamma_{Gt}\right)_{t\geq 0}$, *the monitor's continuation value follows*

$$dV_t = \left(\left(r + \gamma_{Bt} + \lambda m_t\left(1 - a_t\right)\right) V_t - \alpha\left(1 - m_t^2\right) + c a_t\right) dt - V_t dN_t^B$$

*in equilibrium, with initial value* $V_0$.

On path, i.e. for $0 \leq t < \tau$, this law of motion reduces to the ODE

$$dV_t = \left(\left(r + \gamma_{Bt} + \lambda m_t\left(1 - a_t\right)\right) V_t - \alpha\left(1 - m_t^2\right) + c a_t\right) dt. \quad (23)$$

Using the martingale representation, we next reformulate the necessary conditions for a Markov Perfect Equilibrium, which are analogous to the ones in Section 4, and which we write as

$$m_t = \left(1 - x_t\right)\left(1 - a_t\right)$$

for the manager and

$$\begin{aligned} a_t &= 0 \text{ if } \lambda m_t V < c \\ a_t &\in [0,1] \text{ if } \lambda m_t V = c \\ a_t &= 1 \text{ if } \lambda m_t V > c \end{aligned}$$

for the monitor. These conditions are analogous to the ones in the main model. Combining the conditions of the monitor and regulator implies that at any time $0 \leq t < \tau$, there exists a unique pair $(a_t, m_t)$, which satisfies

$$m_t = 1 - x_t \text{ and } a_t = 0$$

if $\lambda(1-x_t)V_t \leq c$ and

$$m_t = \frac{c}{\lambda V_t} \text{ and } a_t > 0$$

if $\lambda(1-x_t)V_t > c$. Importantly, effort $a_t$ and manipulation $m_t$ are Markovian in the pair $(x_t, V_t)$. That is, $m_t = m(x_t, V_t)$ and $a_t = a(x_t, V_t)$, where

$$m(x_t, V_t) = \begin{cases} 1 - x_t & \text{if} \quad \lambda(1-x_t)V_t \leq c \\ \frac{c}{\lambda V_t} & \text{if} \quad \lambda(1-x_t)V_t > c \end{cases} \tag{24}$$

and

$$a(x_t, V_t) = \begin{cases} 0 & \text{if} \quad \lambda(1-x_t)V_t \leq c \\ 1 - \frac{c}{\lambda V_t(1-x_t)} & \text{if} \quad \lambda(1-x_t)V_t > c. \end{cases} \tag{25}$$

Intuitively, we can take $x_t$ and $V_t$ as state variables in the regulator's problem and treat $m(x, V)$ and $a(x, V)$ as given functions affecting the laws of motion in Equations (22) and (23).[43]

We can now state the regulator's problem, conditional on an initial belief $x_0$ and an initial value for the monitor $V_0$. In this problem, the regulator chooses arrival rates $\gamma_{Bt}$ and $\gamma_{Gt}$, and additionally recommends an effort choice $a_t$ and a manipulation choice $m_t$, subject to the incentive compatibility conditions we just outlined:

$$\begin{aligned} W_0 \quad = & \max_{\{a_t, m_t, \gamma_{Bt}, \gamma_{Gt}\}_{0 \leq t < \tau}} E^R \left[ \int_0^\tau e^{-rt} \alpha (1 - m_t) \, dt \right] \\ & s.t. \ (22), (23), (24), (25) \end{aligned}$$

This problem admits the following HJB equation, which is now a nonlinear PDE in two

[43]Indeed, see Equation (26) below. Note that the baseline model, in an MPE the strategies are functions of $x$ only, while here, we have defined them as functions of $x$ and $V$. This is not an inconsistency. For $\bar{\gamma}$ sufficiently small, the drifts of $x_t$ and $V_t$ in Equations (22) and (23) are strictly positive on path for any given policy $(\gamma_{Bt}, \gamma_{Gt})_{0 \leq t < \tau}$. Thus, there exists a one-to-one correspondence between $V_t$ and $x_t$ and, therefore, there exists functions $\tilde{m}(x)$ and $\tilde{a}(x)$ so that $\tilde{m}(x_t) = m(x_t, V_t)$ and $\tilde{a}(x_t) = a(x_t, V_t)$ for all $t < \tau$.

dimensions,[44]

$$W(x,V) = \max_{\gamma_B,\gamma_G} \alpha(1-m(x,V)) \tag{26}$$
$$+\left((r+\gamma_B+\lambda m(x,V))V - \alpha\left(1-m(x,V)^2\right)\right)W_V(x,V)$$
$$+\left(\lambda x m(x,V)^2 + x(1-x)(\gamma_B-\gamma_G)\right)W_x(x,V)$$
$$-\left(\lambda m(x,V)^2 + \gamma_B(1-x)\right)W(x,V)$$
$$+\gamma_G x\left(\frac{\alpha}{r} - W(x,V)\right).$$

To derive the correct boundary conditions for Equation (26), first notice that not all pairs $(x,V)$ are implementable by a disclosure policy. For example, when $x_t = 0$, we know that $V_t = 0$ for any disclosure policy. Hence, any pair $(0,V)$ with $V > 0$ cannot be implementable. To characterize the regulator's HJB equation, we must first identify the set of implementable pairs $(x,V)$.

To this end, consider $\bar{V}(x)$, the highest possible monitor value which is implementable given initial belief $x$. This value is derived as follows

$$\bar{V}(x_0) = \max_{\{a_t,\gamma_{Bt},\gamma_{Gt}\}_{0\le t<\tau}} E^M\left[\int_0^\tau e^{-rt}\left(\alpha\left(1-m_t^2\right)-ca_t\right)dt\right]$$
$$s.t.\ (22),(24),(25).$$

In this problem, we choose the disclosure policy $\{\gamma_{Bt},\gamma_{Gt}\}_{0\le t<\tau}$ to maximize the monitor's value starting from $x_0$. This problem admits the HJB equation

$$r\bar{V}(x) = \max_{a,\gamma_B,\gamma_G} \alpha\left(1-m\left(x,\bar{V}(x)\right)^2\right) - ca \tag{27}$$
$$+\left(\lambda x m\left(x,\bar{V}(x)\right)^2 + x(1-x)(\gamma_B-\gamma_G)\right)\bar{V}'(x)$$
$$-\left(\lambda m\left(x,\bar{V}(x)\right)(1-a)+\gamma_B\right)\bar{V}(x)$$

with boundary condition $\bar{V}(1) = \alpha/r$, where $m\left(x,\bar{V}(x)\right)$ is given by Equation (24).[45] We denote the resulting optimal policy with $\{\bar{\gamma}_{Bt},\bar{\gamma}_{Gt}\}_{0\le t<\tau}$. This policy is Markovian in $x$, i.e.

[44] Here, we have simplified the martingale representation in Equation (23), since the incentive compatibility conditions imply that

$$(\lambda m_t V_t - c)\,a_t = 0.$$

We discuss existence and uniqueness of solutions to this equation below, once we have formulated the boundary conditions.

[45] Existence and uniqueness to this equation can be derived using similar arguments as for the main model (see Appendix B). That the solution to the Hamilton-Jacobi equation and the original problem coincide follows from standard verification arguments (see e.g. Davis (1993)).

there exist functions $\bar{\gamma}_B(x)$ and $\bar{\gamma}_G(x)$, such that $\bar{\gamma}_{Bt} = \bar{\gamma}_B(x_{t-})$ and $\bar{\gamma}_{Gt} = \bar{\gamma}_G(x_{t-})$.

Similarly, consider the min-max value $\underline{V}(x)$, which is the lowest implementable monitor value given initial belief $x$, i.e.,

$$\begin{aligned} \underline{V}(x_0) &= \min_{\{\gamma_{Bt},\gamma_{Gt}\}_{0\le t<\tau}} \max_{\{a_t\}_{0\le t<\tau}} E^M\left[\int_0^\tau e^{-rt}\left(\alpha\left(1-m_t^2\right)-ca_t\right)dt\right] \\ & \quad s.t.\ (22),(24),(25). \end{aligned}$$

This problem admits the HJB-Isaacs equation[46]

$$\begin{aligned} r\underline{V}(x) &= \min_{\gamma_B,\gamma_G}\max_a \alpha\left(1-m\left(x,\underline{V}(x)\right)^2\right)-ca \\ &\quad + \left(\lambda x m\left(x,\underline{V}(x)\right)^2 + x(1-x)(\gamma_B-\gamma_G)\right)\underline{V}'(x) \\ &\quad - \left(\lambda m\left(x,\underline{V}(x)\right)(1-a)+\gamma_B\right)\underline{V}(x), \end{aligned} \tag{28}$$

with boundary condition $\underline{V}(1) = \alpha/(r+\bar{\gamma})$.[47] We denote the resulting optimal policy with $\{\underline{\gamma}_{Bt},\underline{\gamma}_{Gt}\}_{0\le t<\tau}$ or equivalently with $(\underline{\gamma}_B(x),\underline{\gamma}_G(x))$, since the optimal policy is again Markovian in $x$.

Thus, the set of implementable pairs $(x,V)$ is given by

$$\left\{(x,V)\in[0,1]\times\left[0,\frac{\alpha}{r}\right] | \forall x : V\in\left[\underline{V}(x),\bar{V}(x)\right]\right\}.$$

We can now formulate the regulator's boundary conditions. At a pair $\left(x,\bar{V}(x)\right)$, the regulator has a unique feasible continuation policy, which is consistent with the policy $(\bar{\gamma}_B(x),\bar{\gamma}_G(x))$ defined above. We denote the regulator's value under this policy as $\bar{W}(x)$. This value satisfies the HJB equation

$$\begin{aligned} r\bar{W}(x) &= \alpha(1-\bar{m}(x)) + \left(\lambda x\bar{m}(x)^2 + x(1-x)(\bar{\gamma}_B(x)-\bar{\gamma}_G(x))\right)\bar{W}'(x) \\ &\quad - \left(\lambda\bar{m}(x)^2 - \bar{\gamma}_B(x)(1-x)\right)\bar{W}(x) + \bar{\gamma}_G(x)\,x\left(\frac{\alpha}{r}-\bar{W}(x)\right) \end{aligned} \tag{29}$$

with boundary condition $\bar{W}(1) = \alpha/r$, where we have used the abbreviation $\bar{m}(x) = m\left(x,\bar{V}(x)\right)$. Similarly, at a pair $(x,\underline{V}(x))$, the regulator's continuation policy must be consistent with $(\underline{\gamma}_B(x),\underline{\gamma}_G(x))$. We denote the regulator's value under that policy as $\underline{W}(x)$,

[46] The same comments regarding existence, uniqueness, and verification as for $\bar{V}(x)$ apply.

[47] Here is why the boundary conditions for $\bar{V}$ and $\underline{V}$ differ. In calculating $\bar{V}$, we choose the disclosure policy optimal for the monitor and hence disclosing that he is the strategic type for $x$ close to 1 cannot be optimal, since the monitor shirks for such beliefs anyway. By contrast, in calculating $\underline{V}$, we minimize the monitor's value. Disclosing bad news when beliefs are close to 1 achieves this purpose.

which satisfies the HJB Equation

$$r\underline{W}(x) = \alpha(1-\underline{m}(x)) + \left(\lambda x \underline{m}(x)^2 + x(1-x)\left(\underline{\gamma}_B(x) - \underline{\gamma}_G(x)\right)\right)\underline{W}'(x) \qquad (30)$$
$$- \left(\lambda \underline{m}(x)^2 - \underline{\gamma}_B(x)(1-x)\right)\underline{W}(x) + \underline{\gamma}_G(x)\, x\left(\frac{\alpha}{r} - \underline{W}(x)\right)$$

with boundary condition $\underline{W}(1) = \alpha/r$ and using $\underline{m}(x) = m(x, \underline{V}(x))$.

Since the regulator's continuation policy is uniquely pinned down at any pair $(x, \underline{V}(x))$ and $\left(x, \bar{V}(x)\right)$, the graphs of the functions $\underline{V}(x)$ and $\bar{V}(x)$ are absorbing boundaries in $(x, V)$-space. We can thus assign Dirichlet boundary conditions $\underline{W}(x)$ and $\bar{W}(x)$, respectively.

We now turn to boundary conditions at $x = 0$ and $x = 1$. At $x = 0$, we have $W(0, V) = 0$ for any feasible $V$, since the monitor continues to shirk forever independently of any disclosure policy by the regulator.[48] Finally, at $x = 1$, the set of implementable $V$ is $[\alpha/(r + \bar{\gamma}), \alpha/r]$. For any $V$ in the set, there exists a $\gamma_B(1) \in [0, \bar{\gamma}]$ which implements that $V$. The monitor value at any such $\gamma_B(1)$ is constant and given by $W(1, V) = \alpha/r$. This follows from two facts. First, the agent never manipulates at $x = 1$ (i.e. $m(1, V) = 0$ for any implementable $V$), so a different value $\gamma_B(1)$ does not change manipulation at $x = 1$. Second, the likelihood of bad news arriving is zero from the monitor's perspective (since $x = 1$).

We summarize our results in the following Lemma, the proof of which we just gave.

**Lemma 2.** *The regulator's HJB equation* (26) *is defined on the domain*

$$\left\{(x, V) \in [0, 1] \times \left[0, \frac{\alpha}{r}\right] | \forall x : V \in \left[\underline{V}(x), \bar{V}(x)\right]\right\}$$

*and admits the following boundary conditions:*

$$\begin{aligned} W(0, V) &= 0 && \text{for} \quad V \in \left[\underline{V}(0), \bar{V}(0)\right], \\ W(1, V) &= \tfrac{\alpha}{r} && \text{for} \quad V \in \left[\underline{V}(1), \bar{V}(1)\right], \\ W(x, \underline{V}(x)) &= \underline{W}(x) && \text{for} \quad x \in (0, 1), \\ W\left(x, \bar{V}(x)\right) &= \bar{W}(x) && \text{for} \quad x \in (0, 1). \end{aligned}$$

Figure 2 illustrates the set of implementable $(x, V)$-pairs and the boundary values for the regulator's problem, for given parameter values. The boundaries of the implementable set are highly nonlinear, become very steep at $x = 0$, and at $x = 0$, the implementable set reduces to a singleton. These features pose significant problems for any numerical procedure.

Nonetheless, we are able to compute a solution to the PDE (26) using a finite difference

[48] In our numerical analysis, it turns out that $\underline{V}(0) = \bar{V}(0) = 0$, so the only feasible $V$ is zero.

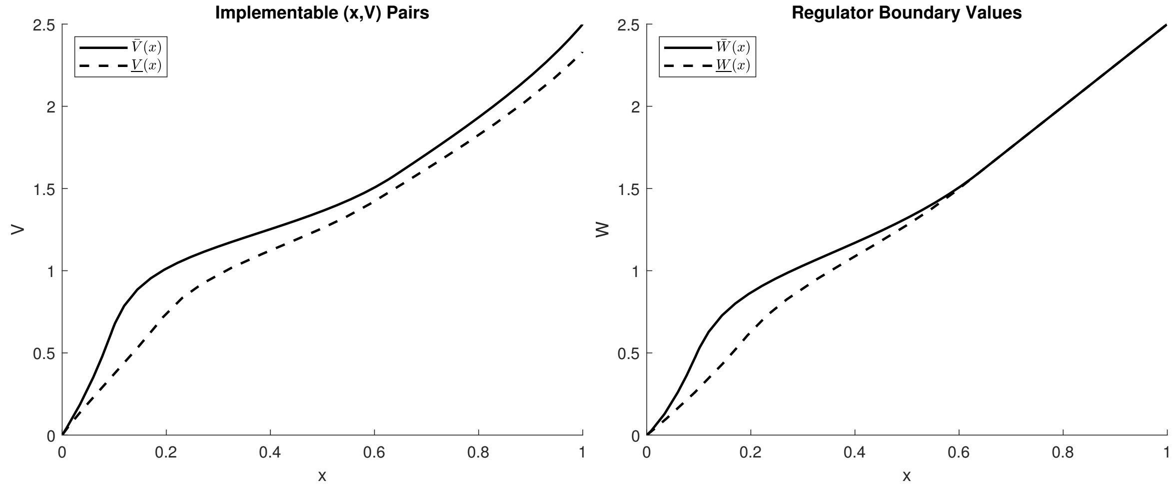
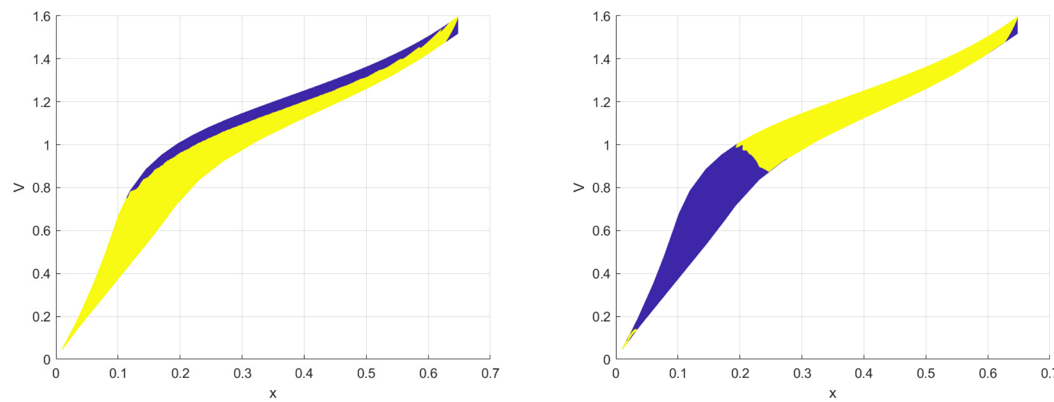



**Figure 2:** On the left is the set of implementable $(x, V)$ pairs for parameters $\alpha = 3.5$, $\lambda = 2.1$, $r = 1.4$, $c = 1.2$, and $\bar{\gamma} = 0.1$. The solid line is $\bar{V}(x)$ and the dashed line is $\underline{V}(x)$. On the right are the corresponding boundary values $\bar{W}(x)$ (solid) and $\underline{W}(x)$ (dashed).

scheme. We provide a complete description of the scheme, and the adjustments we made to overcome the apparent numerical difficulties, in Appendix E. The regulator's optimal policies are shown in Figure 3 and the regulator's optimal value is shown in Figure 4.




**(a)** $\gamma_B$ **(b)** $\gamma_G$

**Figure 3:** The regulator's optimal disclosure policy on the work region and lower shirking region, under the same parameters as in Figure 2. The yellow regions indicate when $\gamma_B = \bar{\gamma}$ (left) and $\gamma_G = \bar{\gamma}$ (right). The blue regions indicate when $\gamma_B = 0$ and $\gamma_G = 0$.

The results match our qualitative characterization in Section 5.2. For low beliefs, when the monitor shirks, revealing bad news is optimal for the regulator. For high beliefs, when the monitor works, revealing bad news is optimal only when the monitor's continuation value is relatively low. For high continuation values, not disclosing bad news is optimal. By contrast, revealing good news is not optimal for low beliefs, but optimal for high beliefs.

To close, we consider the case when the regulator can pick $V_0$ optimally, i.e. there is

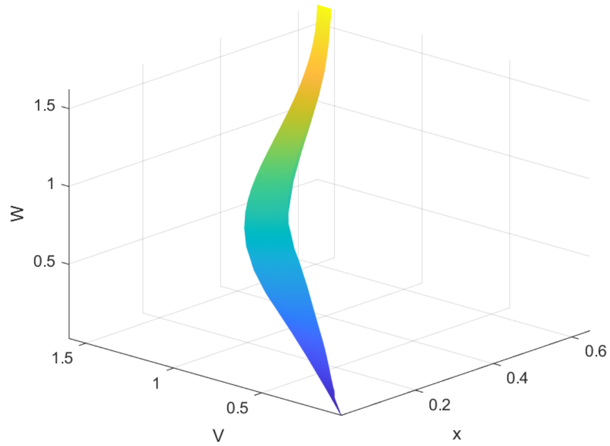
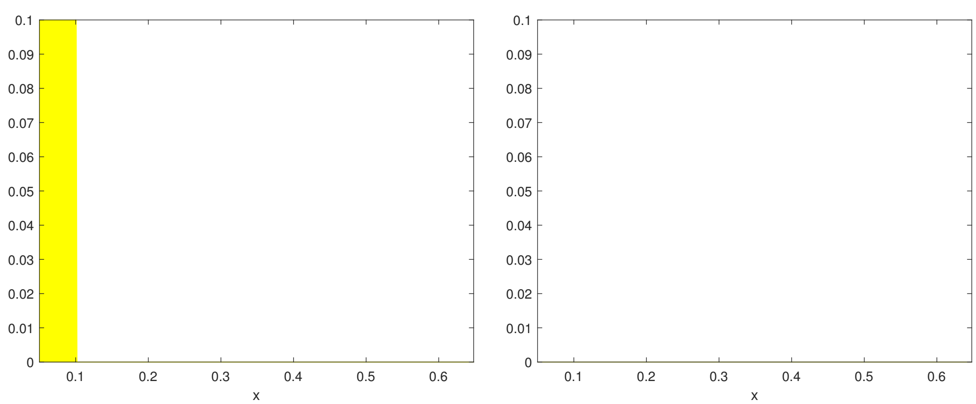



**Figure 4:** The regulator's optimal value function under the same parameters as in Figure 2.

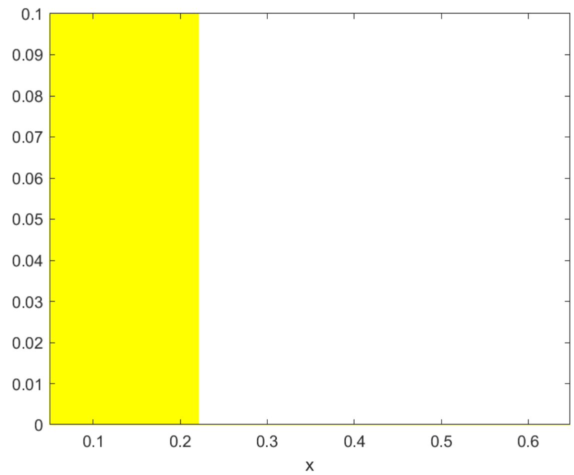
no constraint on the initial value of the monitor, except being implementable. Then, for any given $V_0$, there exists a unique trajectory $(\gamma_{Bt}, \gamma_{Gt}, x_t, V_t)_{0 \leq t < \tau}$. We can thus construct a mapping from (on-path) beliefs $x$ to optimal values $\gamma_B(x)$ and $\gamma_G(x)$, which facilitates comparison to our qualitative results in Section 5.2.1.

The results, which are in Figure 5, closely match our qualitative results. At the optimal $V_0$, the regulator discloses bad news for beliefs below the work region, but discloses no news once the work region is reached. By contrast, Figure 6 shows the optimal disclosure policies at each belief for an arbitrarily chosen $V_0$. The results are again consistent with our qualitative characterization. The regulator discloses bad news for beliefs below the work region and also on the lower of the work region. She discloses good news on the upper part of the work region.

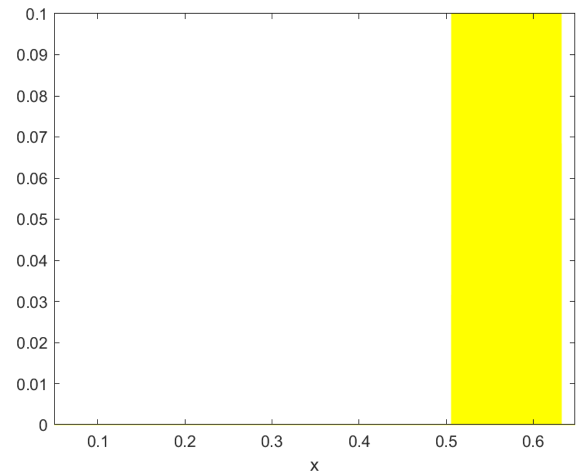



**(a)** $\gamma_B$ **(b)** $\gamma_G$

**Figure 5:** The regulator's optimal disclosure policy as a function of $x$, choosing $V_0$ optimally at initial belief $x_0 = 0.05$ and under the same parameters as in Figure 2. The yellow regions indicate when $\gamma_B = \bar{\gamma}$ (left) and $\gamma_G = \bar{\gamma}$ (right).




**(a)** $\gamma_B$




**(b)** $\gamma_G$

**Figure 6:** The regulator's optimal disclosure policy as a function of $x$, choosing $V_0 = (\bar{V}(x_0)+\underline{V}(x_0))/2$ at initial belief $x_0 = 0.05$ and under the same parameters as in Figure 2. The yellow regions indicate when $\gamma_B = \bar{\gamma}$ (left) and $\gamma_G = \bar{\gamma}$ (right).

# 6 Regulatory Implications

We have shown that a regulator optimally induces uncertainty about the monitor's type, even when this entails the risk of destroying the monitor's reputation altogether. One important question is how aggressive regulators should be in disclosing information. Specifically, when should the regulator disclose bad news and potentially destroy the monitor's reputation?

We investigate this next, by providing comparative statics on the threshold $x_d$, below which the regulator uses delayed bad news. We compute these results numerically, by solving the regulator's problem for given parameter values and then comparing the thresholds.

**Enforcement Tools** Suppose the regulator has access to better enforcement tools, which make it more costly for the manager to manipulate. Then, counterintuitively, the regulator becomes more aggressive. The threshold $x_d$ is higher.

We model better enforcement as increasing the agent's cost of manipulation, which now becomes $c_m \frac{1}{2} m^2$.[49] On the surface, one would think that stronger enforcement would crowd-out disclosure, given the potentially adverse incentive effect of disclosure. However, the

[49]Previously, the cost was $\frac{1}{2}m^2$. For example, the regulator may be able to impose fines or other penalties on the manager, which increase $c_m$. From the monitor's perspective, changing $c_m$ is equivalent to changing $\lambda$ since only the manipulation in response to monitor reputation matters for the monitor's incentives. For the monitor, an increase in $c_m$ is then equivalent to a decrease in $\lambda$.

opposite holds. Stronger enforcement leads to less manipulation by the manager, so that the monitor has stronger incentives to shirk. Hence, the work region shrinks and the lower shirking region expands. As a result, the regulator discloses bad news for higher reputation levels.

**Severity of Losses** Consider the effect of the loss $\alpha$ on the regulator's aggressiveness. When $\alpha$ is higher, a manipulation shock causes larger losses to the firm. Reputation becomes more valuable, because the monitor's fee $p(x_t)$ increases. This triggers more monitoring effort and the work region expands. This, in turn, crowds-out regulatory disclosure: the bad news region shrinks.

Empirically, this suggests that when firms are willing to pay more to the monitor, either because manipulation shocks are more costly to the firm, or because the monitor enjoys stronger monopoly power, the regulator should interfere less.

**Monitoring Costs** When the cost of monitoring $c$ increases, shirking becomes more attractive to the monitor. Accordingly, the working region shrinks and the bad news region expands. The regulator thus discloses more bad news.

Overall, our results suggest that regulatory disclosure and reputational incentives are substitutes. Whenever reputational incentives are stronger, the regulator should disclose less bad news.

# 7 Conclusion

In long-run relationships the desire to build a reputation can act as an incentive device when explicit penalties or contracting arrangements are not available. This is especially relevant for intermediaries such as banks, underwriters, rating agencies, or auditors. These intermediaries fulfill the role of monitors in the economy. Banks screen loans before they sell them off in a structured product, underwriters and venture capital firms monitor the quality of startups before their initial public offering, rating agencies monitor firms for behavior that may make default more likely, and auditors detect accounting fraud that may otherwise go unnoticed. If monitors neglect their duty, enforcement is often impractical. In the recent financial crisis, for example, mortgage underwriters have failed to properly screen applicants. Yet, it took many years before the problem became apparent, and few individual underwriters have been punished. In these situations, reputation may be the main incentive device.

In this paper, we characterize reputational incentives for monitors. In our model, the agent who is monitored is a rational player. He optimally chooses how much to misbehave in response to the monitor's reputation and the anticipated monitoring effort. This leads to a shirk-work-shirk equilibrium. When reputation is low, there is little value for the monitor to exert effort, so the monitor shirks. Likewise, when reputation is very high, the monitor shirks because the agent does not misbehave when he is faced with a high reputation monitor; in this situation, if the monitor shirks, the public is not likely to detect it. Instead, the monitor only exerts effort when reputation is in an intermediate interval. This finding has an important implication—uncertainty about the monitor is valuable.

In response to the financial crisis, regulators have started to rethink the transparency of financial intermediaries. The Sarbanes-Oxley Act (SOX) has brought with it a slew of disclosure requirements, a new regulatory authority has been formed to oversee auditors (the PCAOB), and many governments have designed stress tests for banks. If intermediaries anticipate that information about them will be revealed, how does this influence their desire to build a reputation in the first place? And how can regulators harness mandatory disclosure requirements to improve the functioning of markets for loans, equity, or auditing services?

We show that seemingly reasonable disclosures can have a detrimental effect. If the regulator provides verifiable disclosure about the monitor, any deterministic disclosure policy (i.e., a policy that reveals information with certainty at any given time) will at least partially destroy the incentive to acquire a reputation and lead the monitor to exert less effort. To improve the functioning of the underlying markets, regulators should therefore not aim to provide transparency about the monitor. Instead, they should aim to induce uncertainty about the monitor's type, since reputational incentives are strongest when reputation is in an intermediate region. This provides a rationale for disclosure policies that use delay.

# References

**Acito, Andrew A, Chris E Hogan, and Richard D Mergenthaler**, "The effects of PCAOB inspections on auditor-client relationships," *The Accounting Review*, 2017, *93* (2), 1–35.

**Alvarez, Fernando and Gadi Barlevy**, "Mandatory disclosure and financial contagion," Technical Report, Working Paper, Federal Reserve Bank of Chicago 2014.

**Aobdia, Daniel and Nemit Shroff**, "Regulatory oversight and auditor market share," *Journal of Accounting and Economics*, 2017, *63* (2-3), 262–287.

**Bailey, P.B., L.F. Shampine, and P.E. Waltman**, *Nonlinear two-point boundary value problems*, Academic Press, 1968.

**Banerjee, Snehal, Jesse Davis, and Naveen Gondhi**, "When transparency improves, must prices reflect fundamentals better?," *Review of Financial Studies*, 2018, *31* (6), 2377–2414.

**Bar-Isaac, Heski**, "Reputation and survival: learning in a dynamic signaling model," *Review of Economic Studies*, 2003, *70* (2), 231–251.

**Beatty, Randolph P**, "Auditor reputation and the pricing of initial public offerings," *Accounting Review*, 1989, pp. 693–709.

**Biglaiser, Gary**, "Middlemen as experts," *RAND Journal of Economics*, 1993, pp. 212–223.

**Board, Public Companies Accounting Oversight**, "The process for board determinations regarding firms' efforts to address quality control criticisms in inspection reports," *PCAOB Release*, 2006, (104-2006), 077.

**Board, Simon and Moritz Meyer ter Vehn**, "Reputation for quality," *Econometrica*, 2013, *81* (6), 2381–2462.

_ **and** _ , "A reputational theory of firm dynamics," *Unpublished paper, UCLA.[430]*, 2014.

**Bonatti, Alessandro and Johannes Hörner**, "Career concerns with exponential learning," *Theoretical Economics*, 2017, *12* (1), 425–475.

_ **and** _ , "Learning to Disagree in a Game of Experimentation," *Journal of Economic Theory*, 2017, *169*, 234–269.

**Boone, Jeff P, Inder K Khurana, and KK Raman**, "Did the 2007 PCAOB disciplinary order against Deloitte impose actual costs on the firm or improve its audit quality?," *The Accounting Review*, 2014, *90* (2), 405–441.

**Carter, Richard B., Frederick H. Dark, and Ajai K. Singh**, "Underwriter reputation, initial returns, and the long-run performance of IPO stocks," *Journal of Finance*, 1998, *53* (1), 285–311.

**Chemmanur, Thomas J. and Paolo Fulghieri**, "Investment bank reputation, information production, and financial intermediation," *Journal of Finance*, 1994, *49* (1), 57–79.

_ **and** _ , "Reputation, renegotiation, and the choice between bank loans and publicly traded debt," *Review of Financial Studies*, 1994, *7* (3), 475–506.

**Crandall, Michael G. and P.-L. Lions**, "Two approximations of solutions of Hamilton-Jacobi equations," *Mathematics of Computation*, 1984, *43* (167), 1–19.

**Cripps, Martin W, George J Mailath, and Larry Samuelson**, "Imperfect monitoring and impermanent reputations," *Econometrica*, 2004, *72* (2), 407–432.

**Davis, Mark HA**, *Markov Models and Optimization*, 2-6 Boundary Row, London: Chapman & Hall, 1993.

**Diamond, Douglas W.**, "Monitoring and reputation: The choice between bank loans and directly placed debt," *Journal of Political Economy*, 1991, *99* (4), 689–721.

**Dilmé, Francesc**, "Reputation Building through Costly Adjustment," 2014.

_ **and Daniel F Garrett**, "Residual Deterrence," 2015.

**Dye, Ronald A**, "Informationally motivated auditor replacement," *Journal of Accounting and Economics*, 1991, *14* (4), 347–374.

_ , "Auditing standards, legal liability, and auditor wealth," *Journal of Political Economy*, 1993, *101* (5), 887–914.

**Ekmekci, Mehmet**, "Sustainable reputations with rating systems," *Journal of Economic Theory*, 2011, *146* (2), 479 – 503.

**Firth, Michael**, "Auditor reputation: The impact of critical reports issued by government inspectors," *RAND Journal of Economics*, 1990, pp. 374–387.

**Frenkel, Sivan, Ilan Guttman, and Ilan Kremer**, "The effect of exogenous information on voluntary disclosure and market quality," *Journal of Financial Economics*, 2020.

**Geelen, Thomas**, "Information dynamics and debt maturity," *Swiss finance institute research paper*, 2019, (16-78).

**Goldstein, Itay and Yaron Leitner**, "Stress tests and information disclosure," *Journal of Economic Theory*, September 2018, *177*, 34–69.

**Gryglewicz, Sebastian and Simon Mayer**, "Delegated monitoring and contracting," *Available at SSRN 3175528*, 2019.

**Halac, Marina and Ilan Kremer**, "Experimenting with career concerns," *Working Paper*, 2018.

**Hansen, Robert S. and Paul Torregrosa**, "Underwriter compensation and corporate monitoring," *Journal of Finance*, 1992, *47* (4), 1537–1555.

**Holmström, Bengt**, "Managerial incentive problems: A dynamic perspective," *Review of Economic Studies*, 1999, *66* (1), 169–182.

**Horner, Johannes and Nicolas S. Lambert**, "Motivational ratings," 2016.

**Hu, Yunzhi and Felipe Varas**, "A Dynamic Theory of Learning and Relationship Lending," *Available at SSRN 3363685*, 2019.

**Jin, Ginger Zhe and Phillip Leslie**, "The Effect of Information on Product Quality: Evidence from Restaurant Hygiene Grade Cards," *Quarterly Journal of Economics*, 2003, *118* (2), 409–451.

**Jovanovic, Boyan**, "Truthful Disclosure of Information," *The Bell Journal of Economics*, 1982, *13* (1), pp. 36–44.

**Kamenica, Emir and Matthew Gentzkow**, "Bayesian persuasion," *American Economic Review*, 2011, *101* (6), 2590–2615.

**Keller, Godfrey and Sven Rady**, "Breakdowns," *Theoretical Economics*, 2015, *10* (1), 175–202.

**_ , _ , and Martin Cripps**, "Strategic experimentation with exponential bandits," *Econometrica*, 2005, *73* (1), 39–68.

**Marinovic, Iván and Felipe Varas**, "No news is good news: Voluntary disclosure in the face of litigation," *RAND Journal of Economics*, 2016, *47* (4), 822–856.

**Martel, Jordan, Kenneth S. Mirkin, and Brian Waters**, "Learning by owning in a lemons market," *Available at SSRN 2798088*, 2018.

**Mathis, Jerome, James McAndrews, and Jean-Charles Rochet**, "Rating the raters: are reputation concerns powerful enough to discipline rating agencies?," *Journal of Monetary Economics*, 2009, *56* (5), 657–674.

**Pei, Harry Di**, "Reputation with strategic information disclosure," *Available at SSRN*, 2016.

**Polyanin, A. D. and V. F. Zaitsev**, *Handbook of exact solutions for ordinary differential equations*, Chapman & Hall/CRC, 2002.

**Public Company Accounting Oversight Board**, *Report on observations of PCAOB inspectors related to audit risk areas affected by the economic crisis*, PCAOB Washington, DC, 2010.

_ , *Annual Report*, PCAOB Washington, DC, 2016.

**Royden, Halsey Lawrence**, *Real analysis*, 3rd ed., Prentice Hall, 1988.

**Sannikov, Yuliy**, "A continuous-time version of the principal-agent problem," *Review of Economic Studies*, 2008, *75* (3), 957–984.

**Stein, Jeremy C**, "Takeover threats and managerial myopia," *Journal of Political Economy*, 1988, *96* (1), 61–80.

**Varas, Felipe, Iván Marinovic, and Andrzej Skrzypacz**, "Random Inspections and Periodic Reviews: Optimal Dynamic Monitoring," *Review of Economic Studies*, 2020.

# A Preliminaries

We first provide a formal definition of Markov Perfect Equilibria.

**Definition 9.** *A Markov Perfect Equilibrium (MPE) consists of admissible*[50] *strategies $m(x) \in [0,1]$ and $a(x) \in [0,1]$, admissible conjectures $\hat{a}(x) \in [0,1]$, and $\hat{m}(x) \in [0,1]$, a price $p(x)$, and a reputation process $\{x_t\}_{t \geq 0}$, such that*

*(1) $a(x_{t-})$ is optimal for the monitor, i.e. it maximizes for all $0 \leq t < \infty$*

$$V(x_{t-}) = E^M \left[ \int_t^\infty e^{-r(s-t)} p(x_s) - ca(x_s) ds \right],$$

*(2) $m(x_{t-})$ is optimal for the manager all $0 \leq t < \infty$, i.e.*

$$m(x_{t-}) = \arg\max_m (1 - x_{t-})(1 - \hat{a}(x_{t-}))m - \frac{1}{2}m^2,$$

*(3) the conjectures $\hat{a}(x)$ and $\hat{m}(x)$ are correct, i.e., $\hat{a}(x_{t-}) = a(x_{t-})$ and $\hat{m}(x_{t-}) = m(x_{t-})$ for all $0 \leq t < \infty$,*

*(4) the reputation process follows Bayes' rule, i.e $x_t$ is given by Equation (10) for all $0 \leq t < \tau$ and we have $x_t = 0$ for all $t \geq \tau$, where $\tau$ is the first arrival time of the loss, and*

*(5) the price $p(x_{t-})$ satisfies Equation (7) given $x_{t-}$, $\hat{m}(x_{t-})$, and $\hat{a}(x_{t-})$ for all $0 \leq t < \infty$.*

Second, we provide a brief proof showing that the ODE for reputation in Equation (10) has a unique solution for any initial value $x_0 \in (0,1)$. Here, recall that in defining admissibility we have fixed a partition $\{x_1, ..., x_N\}$, such that strategies are Lipschitz continuous in $x$ for any $x \in (x_n, x_{n+1})$ for $n = 1, ..., N-1$.

**Lemma 3.** *For any admissible $\hat{m}(x)$ and $\hat{a}(x)$ and for any initial value $x_0 \in (0,1)$, the ODE (10) has a unique solution.*

*Proof.* First, pick some $x_0 \in (x_n, x_{n+1})$ for any $n \in \{1, ..., N-2\}$. Since the RHS of Equation (10) is Lipschitz continuous on $(x_n, x_{n+1})$, there exists a unique solution at least up until the time $T = \inf\{t : x_t = x_{n+1}\}$. This follows from the Picard-Lindelöf theorem. Now, let $x_{T-} = \lim_{t \uparrow T} x_t$ be the left limit of that solution at time $T$. We construct a solution $\hat{x}_t$ to Equation (10) on the interval $[x_{n+1}, x_{n+2})$ as follows. We pick $\hat{x}_0 = x_{T-}$ as its initial value. Then, we define $\hat{m}_-(x_{n+1}) \lim_{x \uparrow x_{n+1}} \hat{m}(x)$ and $\hat{m}_+(x_{n+1}) = \lim_{x \downarrow x_{n+1}} \hat{m}(x)$ and we define $\hat{a}_+(x_{n+1})$ and $\hat{a}_-(x_{n+1})$ analogously. Since $\hat{m}_-(x_{n+1}) > 0$ and $\hat{m}_+(x_{n+1}) > 0$, and $\hat{a}_-(x_{n+1}) < 1$ and $\hat{a}_+(x_{n+1}) < 1$, we can without loss of generality pick

$$\frac{dx_{n+1}}{dt} = \lambda \hat{m}_+(x_{n+1})(1 - x_{n+1})(1 - \hat{a}_+(x_{n+1})),$$

which, again by the Picard-Lindelöf theorem, yields a unique solution on the interval $[x_{n+1}, x_{n+2})$. Pasting the solution $\hat{x}_t$ to the solution $x_t$ starting at time $T$, i.e. $x_t = \hat{x}_{t-T}$ for $t \geq T$ yields a unique function of time which satisfies the ODE (10) on the interval $[x_0, x_{n+2})$. We can then proceed iteratively to cover the entire interval $(x_0, 1)$. It only remains to prove that there exists a unique solution when $x_0 = x_n$ for some $n \in \{1, ...N\}$. The proof argument is the same as the one we showed above, when $x_0 = x_{T-}$. ∎

[50] See Section 2 for our definition of admissibility.

Finally, we note that the admissibility requirements do not put a binding constraint on the equilibrium. Specifically, instead of restricting attention to admissible strategies, we could merely require that the strategies are such that the ODE (10) has a unique solution, without putting any explicit restrictions. Then, we would have to verify that the equilibrium strategies, i.e. in Propositions 1 and 2, are such that Equation (10) admits a unique solution.[51] As can be seen from Propositions 1 and 2, this is indeed the case.

# B Proofs

## B.1 Shirking Equilibrium (Proposition 1)

We first show that the shirking ODE (12) with the boundary condition $V_s(1) = \frac{\alpha}{r}$ has a unique continuously differentiable solution. This is technically involved, because the equation is singular at both $x = 0$ and $x = 1$. We express the solution as an initial value problem (IVP) at some value $x_0 \in (0, 1)$. Then, we characterize the solutions as $x$ approaches one and show that there can be at most one solution that satisfies the boundary condition. To prove existence, we use a rescaling of $V_s(x)$ together with the Arzelà-Ascoli Theorem (see e.g. Royden (1988), Ch. 7.10, p. 167ff) and an argument similar to the shooting method (see e.g. Bailey et al. (1968)).[52] That the solution to the ODE equals the strategic type's value then follows from a standard verification argument (see Davis (1993), Ch. 4).

To prove that this is indeed an equilibrium, we then use the firm's optimality condition for each $x$ and show that no deviation at that $x$ is optimal. Since the equilibrium is assumed to be Markovian, this is sufficient. We then establish uniqueness by showing that in any other potential equilibrium must have a discontinuous value function, which is impossible.

We start with recording some useful properties of solutions to the shirking ODE (12). The solutions can be indexed by an initial condition $v_0$ at a (common) initial point $x_0 \in (0, 1)$. To highlight this dependence, we denote them with $V_s(x, v_0)$. We continue writing the shirking value as $V_s(x)$.

**Lemma 4.** *Solutions to the initial value problem (IVP) in Equation (12) with initial condition $V_s(x_0, v_0) = v_0$ for some fixed $x_0 \in (0, 1)$ have the following properties:*

1. *For any interval $[\underline{x}, \bar{x}]$ with $0 < \underline{x} < x_0 < \bar{x} < 1$ and any $v_0$, the solution to the IVP exists and is unique.*

2. *$V_s(0, v_0) = 0$ for all $v_0$.*

3. *For any $x \in (0, 1)$, $V_s(x, v_0)$ is continuous and strictly increasing in $v_0$. In particular, two solutions $V_s(x, v_0')$ and $V_s(x, v_0)$ cannot cross on $(0, 1)$.*

4. *Larger solutions, i.e. $v_0' > v_0$, have larger slope: if $v_0' > v_0$, then for all $x \in (0, 1)$, $V_s'(x, v_0') > V_s'(x, v_0)$.*

5. *There exists at most one solution with $V_s(1, v_0) = \frac{\alpha}{r}$.*

[51] This guess-and-verify technique is common in the stochastic control literature.

[52] We defer the detailed proof to Section D, since it is purely technical.

*Proof.* 1. Existence and uniqueness for solutions to the IVP follows from the Picard-Lindelöf Theorem, which requires that the right hand side of

$$V_s'(x, v_0) = \frac{(r + \lambda(1-x)) V_s(x, v_0) - \alpha\left(1 - (1-x)^2\right)}{\lambda x (1-x)^2} \tag{31}$$

is Lipschitz in both $V_s$ and $x$. This is true as long as the interval $[\underline{x}, \bar{x}]$ is bounded away from 0 or 1, which we have assumed.

2. To show that for every $v_0$, we have $V_s(0, v_0) = 0$, we use the method of integrating factors (see Polyanin and Zaitsev (2002), p.4) to write

$$\begin{aligned} V_s(x, v_0) &= \exp\left(\int_{x_0}^{x} \frac{r + \lambda(1-s)}{\lambda s (1-s)^2} ds\right) \\ &\cdot \left(v_0 - \int_{x_0}^{x} \frac{\alpha\left(1 - (1-s)^2\right)}{\lambda s (1-s)^2} \exp\left(-\int_{x_0}^{s} \frac{r + \lambda(1-u)}{\lambda u (1-u)^2} du\right) ds\right). \end{aligned} \tag{32}$$

This equation can be written more explicitly as

$$\begin{aligned} V_s(x, v_0) &= \left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \\ &\cdot \left(\left(\frac{1-x_0}{x_0}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(-\frac{r}{\lambda}\frac{1}{1-x_0}\right) v_0 \right. \\ &\left. - \frac{\alpha}{\lambda}\int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} (1-s)^{\frac{r}{\lambda}-1} \left(1 - (1-s)^2\right) \exp\left(-\frac{r}{\lambda}\frac{1}{1-s}\right)\right) ds. \end{aligned}$$

We now show that this expression converges to zero as $x \to 0$. We can bound the value of the integral from above as follows:

$$\begin{aligned} &\int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} (1-s)^{\frac{r}{\lambda}-1} \left(1 - (1-s)^2\right) \exp\left(-\frac{r}{\lambda}\frac{1}{1-s}\right) ds \\ \leq\ & M \int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} \left(1 - (1-s)^2\right) ds \end{aligned}$$

for some $M > 0$, because all terms inside the integral except for $s^{-\left(\frac{r}{\lambda}+2\right)}$ are bounded as $x$ converges to zero. Computing this new integral, and multiplying it by $x^{\frac{r+\lambda}{\lambda}}$, we can show that $V_s(x, v_0)$ converges to zero if and only if

$$x^{\frac{r+\lambda}{\lambda}} \left(\frac{1}{-\left(\frac{r}{\lambda}+1\right)} x^{-\frac{r}{\lambda}} + \frac{\lambda}{r} x^{-\frac{r-\lambda}{\lambda}}\right)$$

converges to zero as $x \to 0$. Inspecting the exponents, we can confirm this is the case.

3. This follows directly from Equation (32). Using that equation we can write

$$\begin{aligned} V_s\left(x, v_0'\right) - V_s\left(x, v_0\right) &= \left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \\ &\quad \cdot \left(\frac{1-x_0}{x_0}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(-\frac{r}{\lambda}\frac{1}{1-x_0}\right) \cdot \left[v_0' - v_0\right] \\ &> 0 \end{aligned}$$

4. This follows from differentiating Equation (32) in $x$, which yields

$$V_s'\left(x, v_0'\right) - V_s'\left(x, v_0\right) = \frac{d}{dx}\left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \cdot \left[v_0' - v_0\right].$$

The derivative on the RHS is strictly positive for $x \in (0,1)$.

5. Suppose there exist two solutions $V_s(x, v_0)$ and $V_s(x, v_0')$ with $V_s(1, v_0) = V_s(1, v_0') = \frac{\alpha}{r}$. Without loss of generality, assume $v_0' > v_0$. By Point 3, for any $\varepsilon > 0$, we have $V_s(1-\varepsilon, v_0') > V_s(1-\varepsilon, v_0)$, so there must exist a $\delta < \varepsilon$ such that $V_s'(1-\delta, v_0') < V_s'(1-\delta, v_0)$, otherwise, the solutions cannot both hit $\frac{\alpha}{r}$ at $x = 1$. But by Point 4, such $\delta$ cannot exist.

∎

The last point of the Lemma shows that there is at most one solution to the IVP that satisfies $V_s(x) = \frac{\alpha}{r}$. In Section D, we use the Properties established in the Lemma to prove existence.

To show that shirking is indeed an equilibrium, note that shirking is optimal at $x$ whenever

$$\lambda(1-x) V_s(x) \leq c.$$

This is satisfied because of our assumption on $c$ in the statement of Proposition 1.

We now show that the shirking equilibrium is unique. To do this, we show that any equilibrium with working must feature a discontinuity in the firm's value function, which is not consistent with the firm being forward-looking.

**Lemma 5.** *In any equilibrium, the monitor must shirk whenever* $x \geq \frac{\alpha\lambda - rc}{\alpha\lambda}$.

*Proof.* Let $V(x)$ denote the value in an arbitrary equilibrium. Shirking is optimal whenever

$$\lambda(1-x) V(x) \leq c.$$

In any equilibrium, the value of the monitor is bounded, i.e. $V(x) \leq \frac{\alpha}{r}$. Combining these two inequalities and rearranging yields the result. ∎

If $\alpha\lambda \leq rc$, the Lemma implies that shirking is the unique equilibrium. We thus focus on the case $\alpha\lambda > rc$. In any equilibrium that features working, there must be an interval $[x_h, 1]$ when the firm shirks. On that interval, the value of the firm is simply given by $V_s(x)$. Importantly, this value is independent of anything that happens for $x' < x$. This is because we are in a "perfect bad news" case. In any equilibrium, the value of the firm is continuous at any $x > 0$, because the future evolution of reputations and prices is anticipated.

Now, assume that $[x_h, 1]$ is the largest interval where the firm shirks. If $x_h = 0$ we are done. Thus, assume by way of contradiction that $x_h > 0$. For any sufficiently small $\varepsilon > 0$, there exists an $x_\varepsilon \in (x_h - \varepsilon, x_h)$ such that working is optimal at $x_\varepsilon$. If this were not the case, then $[x_h, 1]$ would not be the largest interval where the firm shirks. Since working is optimal at $x_\varepsilon$, it must be the case that

$$V(x_\varepsilon) \geq \frac{c}{\lambda(1 - x_\varepsilon)}.$$

By the assumption in Proposition 1, we have

$$\frac{c}{\lambda(1-x)} - V_s(x) \geq K \; \forall x \in [0,1]$$

for some fixed $K > 0$. Therefore,

$$V(x_\varepsilon) \geq V_s(x_\varepsilon) + K.$$

At $x_h$, $V(x)$ must satisfy the value matching condition

$$V(x_h) = V_s(x_h).$$

However, letting $\varepsilon \to 0$ implies that $x_\varepsilon \to x_h$, which in turn implies that $V$ is discontinuous at $x_h$, which is impossible. Since this argument applies for any $x_h > 0$, it must be the case that $x_h = 0$. That is, the firm shirks for all $x$.

Finally, we provide sharp condition in terms of the model parameters for when shirking is the unique equilibrium. To facilitate the analysis, we introduce two new functions, $g$ and $l$. $g(x)$ is defined as $g(x) = (1-x)V_s(x)$. $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ whenever $g(x)$ crosses $\frac{c}{\lambda}$ and it is easier to study the crossing points of $g(x)$. It satisfies the ODE

$$\left(r + \lambda(1-x)^2\right) g(x) = \alpha\left((1-x) - (1-x)^3\right) + \lambda x (1-x)^2 g'(x) \tag{33}$$

with boundary conditions $g(0) = g(1) = 0$. It is continuously differentiable, because $V_s(x)$ is continuously differentiable.

The slope of $g(x)$ is determined by the function $l(x, v)$ for $x \in [0,1]$ and $v \geq 0$, which is given by

$$l(x, v) = \alpha\left((1-x) - (1-x)^3\right) - \left(r + \lambda(1-x)^2\right) v. \tag{34}$$

Specifically, we can write $g(x)$ as

$$0 = l(x, g(x)) + \lambda x (1-x)^2 g'(x),$$

so $g'(x)$ is positive whenever $l(x, g(x))$ is negative. The function $l(x, v)$ satisfies the following properties for all $v > 0$ and $x \in [0,1]$: $l(0, v) < l(1, v) < 0$, $l_{xx}(x, v) < 0$, $l_x(0, v) > 0$ and $l_x(1, v) < 0$.[53] Thus, for any fixed $v$, $l(x, v)$ is either always negative or hits zero exactly twice. It is also strictly decreasing in $v$ for all $x$ and has a unique interior maximum for all $v$.

**Proposition 10.** *Shirking is the unique equilibrium if and only if* $\max_x l\left(x, \frac{c}{\lambda}\right) \leq 0$. *The value* $\bar{c}$

[53] Here, $l_x(x, v)$ is the partial derivative with respect to $x$, etc.

*above which shirking is the unique equilibrium satisfies*

$$\max_x l\left(x, \frac{\bar{c}}{\lambda}\right) = 0.$$

*Equivalently, the equilibrium features working if and only if* $\max_x l\left(x, \frac{c}{\lambda}\right) > 0$.

*Proof.* Suppose that $\max_x l\left(x, \frac{c}{\lambda}\right) > 0$. We show that in this case, $g(x)$ defined in Equation (33) must cross $\frac{c}{\lambda}$, which implies that the shirking equilibrium cannot exist. To show this, we denote with $\bar{g}$ the maximum of $g$ on $[0,1]$, which is attained at $\bar{x}$, and we assume that $\bar{g} < \frac{c}{\lambda}$. The function $g$ is continuously differentiable, so we have $g'(\bar{x}) = 0$,[54] and therefore $l(\bar{x}, \bar{g}) = 0$. Since $l(x, \bar{g})$ has a unique interior maximum, we have

$$l(x, \bar{g}) \leq l(\bar{x}, \bar{g}) = 0$$

for all $x$. Because $l(x, v)$ is decreasing in $v$, we also have

$$l\left(x, \frac{c}{\lambda}\right) < l(x, \bar{g})$$

for all $x$. Let $\bar{x}_l$ be the maximizer of $l\left(x, \frac{c}{\lambda}\right)$. We then have

$$\max_x l\left(x, \frac{c}{\lambda}\right) = l\left(\bar{x}_l, \frac{c}{\lambda}\right) < l(\bar{x}_l, \bar{g}) \leq l(\bar{x}, \bar{g}) = 0,$$

which is a contradiction. This establishes that whenever $l\left(x, \frac{c}{\lambda}\right)$ exceeds zero, $g(x)$ crosses $\frac{c}{\lambda}$ so shirking cannot be an equilibrium.

We now show that whenever $\max_x l\left(x, \frac{c}{\lambda}\right) \leq 0$, the shirking equilibrium exists. Our previous arguments will then imply uniqueness and we do not repeat them here. If $\max_x l\left(x, \frac{c}{\lambda}\right) < 0$, then $g(x) = \frac{c}{\lambda}$ implies that $g'(x) > 0$. Thus, once $g(x)$ crosses $\frac{c}{\lambda}$ from below, it must always stay above it. But this is incompatible with the boundary condition $g(1) = 0$. Thus, we must have $g(x) < \frac{c}{\lambda}$ for all $x$. Shirking is then an equilibrium.

Finally, we study the remaining case $\max_x l\left(x, \frac{c}{\lambda}\right) = 0$. Suppose in that case $g(x)$ exceeds $\frac{c}{\lambda}$. If this is true, then $g(x)$ crosses $\frac{c}{\lambda}$ at at least two values $x_1 < x_2$. At both values we must have $g'(x_1) = g'(x_2) = 0$. But since at most a single value of $x$ attains $l\left(x, \frac{c}{\lambda}\right) = 0$, this is impossible. Thus, in that case, $g(x)$ is at most tangent to $\frac{c}{\lambda}$ at one point, but never crosses it. This means shirking is still an equilibrium. ∎

We have now concluded our characterization. Using the functions $g$ and $l$, we record some additional properties of $V_s(x)$ and $g(x)$ below. These will be useful when analyzing the shirk-work-shirk and work-shirk cases.

**Lemma 6.** *Either* $V_s(x) \leq \frac{c}{\lambda(1-x)}$ *for all* $x$, *or* $V_s(x)$ *crosses* $\frac{c}{\lambda(1-x)}$ *exactly twice.*

*Proof.* To prove the result, suppose by way of contradiction that $g(x)$ crosses $\frac{c}{\lambda}$ more than twice. Since $g(0) = g(1) = 0 < \frac{c}{\lambda}$, $g$ must cross $\frac{c}{\lambda}$ an even number of times. Thus, there must exist three values $x_1 < x_2 < x_3$ at which $g(x)$ equals $\frac{c}{\lambda}$ for which $g'(x_1) \leq 0, g'(x_2) \geq 0$,and $g'(x_3) \leq 0$. This

[54] $g(x)$ is positive and cannot be identically zero on $(0,1)$. Since $g(0) = g(1) = 0$ this implies that $g(x)$ indeed has an interior maximum.

implies that $l\left(x_1, \frac{c}{\lambda}\right) \geq 0$, $l\left(x_2, \frac{c}{\lambda}\right) \leq 0$, and $l\left(x_3, \frac{c}{\lambda}\right) \geq 0$. But this is impossible because $l\left(x, \frac{c}{\lambda}\right)$ is strictly concave in $x$. If $l\left(x_2, \frac{c}{\lambda}\right)$ is non-positive, then $l\left(x_3, \frac{c}{\lambda}\right)$ must be strictly negative. Thus we have our contradiction, which establishes the result. ∎

The result in the Lemma above extends to *all* solutions of the shirking ODE (12), for which $g(1) < \frac{c}{\lambda}$, not just the one that satisfies $V_s(1) = \frac{\alpha}{r}$. Intuitively, $l$ is independent of the particular solution we have used, all solutions satisfy $g(0) = 0$ (because $V_s(0) = 0$) and we only need $g(1) < \frac{c}{\lambda}$ to ensure that $g$ crosses $\frac{c}{\lambda}$ an even number of times.

As we have seen in Equation (32), all solutions to the shirking ODE (12) can be indexed by an initial value $v_0$ at some common initial point $x_0 \in (0,1)$. We write them as $V_s(x, v_0)$ again and we denote with $v_0^s$ the initial value that yields the shirking value. That is, $V_s(x, v_0^s) = V_s(x)$ is the solution that satisfies $V_s(1) = \frac{\alpha}{r}$.

**Lemma 7.** *All solutions $V_s(x, v_0)$ to Equation (12) satisfy the following properties:*

1. *The functions $g$ never cross: Let $g(x, v_0) = (1-x) V_s(x, v_0)$. For $v_0' > v_0$, we have $g(x, v_0') > g(x, v_0)$ for all $x \in (0,1)$. Moreover, $g'(x, v_0') > g'(x, v_0)$.*

2. *Any solution that satisfies $g(1, v_s) = 0$ has a single interior maximum. It is weakly increasing to the left and decreasing at the right of the maximum.*

3. *Any solution that is larger than the shirking value, i.e. $v_0 > v_0^s$, crosses $\frac{c}{\lambda(1-x)}$ exactly once below $x_h$. The crossing point is left of $x_l$.*

4. *Any solution that is smaller than the shirking value, i.e. $v_0 < v_0^s$, crosses $\frac{c}{\lambda(1-x)}$ at most twice.*

*Proof.* 1. We can rewrite Equation (12) as

$$\begin{aligned}\left(r + \lambda (1-x)^2\right) V_s(x) &= \alpha \left(1 - (1-x)^2\right) \\ &\quad + \lambda x (1-x) \left((1-x) V_s'(x) - V_s(x)\right).\end{aligned}$$

We have $g'(x) = (1-x) V_s'(x) - V_s(x)$. Thus, $V_s(x, v_0') > V_s(x, v_0)$ if and only if $g'(x, v_0') > g'(x, v_0)$. But since $g(0, v_0) = g(0, v_0')$, this means that $g(x, v_0') > g(x, v_0)$ for all $x \in (0,1)$.

2. Suppose that there is a local maximum $\tilde{g}$ at point $\tilde{x}$ and a global one at $\bar{g}$. Then, $l(\tilde{x}, \tilde{g}) = 0$. Since $\tilde{g} < \bar{g}$, $g(x)$ must cross $\tilde{g}$ for at least two values $x_1 < x_2$ right of $\tilde{x}$. Without loss of generality, we can choose them so that $g'(x_1) \geq 0$ and $g'(x_2) \leq 0$. But this implies that $l(x_1, \tilde{g}) \leq 0$ and $l(x_2, \tilde{g}) \geq 0$. Inspecting the shape of $l(x, v)$, we see this is impossible. Even if $\tilde{x}$ is the first point where $l(x, \tilde{g})$ intersects zero, $l(x_1 \tilde{g}) \leq 0$ must imply $l(x_2, \tilde{g}) < 0$.

3. From Lemma 4, Point 3, we know that $g(x, v_0) > g(x)$. Therefore, it must cross $\frac{c}{\lambda}$. Also, any points where $g(x, v_0)$ crosses $\frac{c}{\lambda}$ must lie left of $x_l$. For $x < x_l$, we have $g'(x) > 0$, which follows from the previous point. But if $g(x, v_0)$ crosses $\frac{c}{\lambda}$ multiple times on that region there must exist a value of $x$ where $g'\left(x, \frac{c}{\lambda}\right) < 0$. However, Lemma 4, Point 4 implies that $g'(x, v_0) > g'(x)$, so this is impossible.

4. Any solution with $v_0 < v_0^s$ must have $g\left(x, v^0\right) \leq g(x)$ for $x \in [0,1]$. Thus, $g(1, v_0) \leq 0$. We can now use exactly the argument we used to prove Lemma 6.

∎

The following result, which characterizes the derivative of $V_s(x)$ when the work region is nonempty, will be useful in further developments. We record it here to avoid repeating similar arguments later.

**Corollary 11.** *Whenever $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$, at two points $x_1 < x_2$, we must have*

$$V_s'(x_2) < \frac{d}{dx_2}\frac{c}{\lambda(1-x_2)}.$$

*Proof.* From Proposition 10 we know that $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ whenever $\max_x l\left(x, \frac{c}{\lambda}\right) > 0$. Its derivative is strictly below the derivative of $\frac{c}{\lambda(1-x)}$ whenever $g'(x)$ is negative. Since $g(x)$ is continuously differentiable, we have $g'(x_2) \leq 0$, so we only have to show that the inequality is strict. Suppose that $g'(x_2) = 0$.

To show the result, we study $g''(x)$, which we can express as

$$\lambda x (1-x)^2 g''(x) = 2\lambda(1-x) g(x) + 3\alpha(1-x)^3 - \alpha - g'(x)(r + 2\lambda x(1-x))$$

by differentiating Equation (33). We know that $g(x)$ attains its maximum $\bar{g}$ at some $\bar{x}$ between $x_1$ and $x_2$. At that value, we must have

$$g''(\bar{x}) \leq 0.$$

This implies that

$$2\lambda(1-\bar{x})\bar{g} + 3\alpha(1-\bar{x})^3 - \alpha \leq 0,$$

because $g'(\bar{x}) = 0$. Now, the expression above is strictly decreasing in $x$ and strictly increasing in $g$. Therefore, it must be negative at $x_2 > \bar{x}$ and $g(x_2) = \frac{c}{\lambda} < \bar{g}$. Thus, if $g'(x_2) = 0$, then $g''(x_2) < 0$. But this is impossible. If these two conditions hold then $x_2$ is a local maximum, whereas we constructed $x_2$ so that $g$ crosses $\frac{c}{\lambda}$ from above. ∎

The proof above does not rely on any particular boundary conditions for $g$. We can thus extend it to any arbitrary solution to the shirking ODE (12). That is, when a solution crosses $\frac{c}{\lambda(1-x)}$, its slope at the larger crossing point $x_h$ must be strictly below the slope of $\frac{c}{\lambda(1-x)}$. This observation will be useful in the shirk-work-shirk case.

## B.2 Shirk-Work-Shirk Equilibrium (Proposition 2)

We construct the equilibrium using a method similar to backward induction. For sufficiently high reputation, the firm will shirk because there is no manipulation and hence no news.[55] We construct this upper shirking interval by finding the point where $V_s(x)$ hits the function $\frac{c}{\lambda(1-x)}$. This point exists because of the parametric assumptions in Proposition 2. We call it $x_h$. The shirking value on $[x_h, 1]$ does not depend on the equilibrium played at $x < x_h$, so we can compute it independently.

When the strategic type's effort is interior, he must be indifferent between working and shirking. We combine this indifference condition with the ODE that describes the value function for arbitrary effort (Equation (11)) and use them to derive an ODE for the effort itself. In equilibrium, effort and manipulation are equivalent[56] and we express the ODE in terms of manipulation since it is easier to study. This is Equation (13).

[55]See Lemma 5.

[56]This is because we have $m(x) = (1-x)(1-a(x))$ in any equilibrium.

We then solve the initial value problem (IVP) for this ODE with the initial condition $m(x_h) = 1 - x_h$.[57] This condition is equivalent to $a(x_h) = 0$, which is consistent with the firm shirking for $x \geq x_h$. The solution to the IVP must cross the function $1 - x$ exactly once below $x_h$, which we show. We label this crossing point $x_l$. The interval $[x_l, x_h]$ is then our working interval.

Finally, we construct the lower shirking interval $[0, x_l]$. To do this, we solve the shirking ODE (12) on $[0, x_l]$ with the value matching condition $V_s(x_l) = \frac{c}{\lambda(1-x_l)}$. At $x_l$, the firm must be indifferent between working and shirking, which motivates this condition. Then, it only remains to verify that shirking is indeed optimal on $[0, x_l]$. That is, $V_s(x)$ cannot cross $\frac{c}{\lambda(1-x)}$ on the interior of that interval.

The resulting function $V(x)$ is continuous at all $x \in [0,1]$ and continuously differentiable at all $x$, except possibly $x_l$ and $x_h$. Thus, it solves the monitor's HJB Equation (11) almost everywhere. A standard verification argument (Davis (1993), Ch. 4) then implies that $V(x)$ is indeed the monitor's value.

We start with a the point $x_h \in (0,1)$ at which the shirking value satisfies

$$V_s(x_h) = \frac{c}{\lambda(1-x_n)}.$$

As we have just argued, such $x_h$ exists. We conjecture that on an interval left of $x_h$, the firm exerts effort $a(x) \in (0,1)$.[58] The strategic type's value (11) is linear in effort. Whenever $a(x) \in (0,1)$, he must therefore be indifferent between working and shirking. Equivalently, if $m(x)$ is the equilibrium manipulation, we must have

$$\lambda m(x) V(x) = c. \tag{35}$$

Differentiating this expression yields

$$m'(x) V(x) + V'(x) m(x) = 0.$$

In equilibrium, $m(x)$ must be such that the strategic type's value solves Equation (11) and the indifference condition, since he anticipates future effort and manipulation. Plugging in the two conditions above in the Equation (11), we can derive an expression for the equilibrium level of manipulation that satisfies this. After some algebra, we then arrive at Equation (13).

We pin down the working interval by showing that there exists a unique point $x_l \in (0, x_h)$ for which $m(x_l) = 1 - x_l$. To do this, we first characterize the solution to the ODE (13) with boundary condition $m(x_h) = 1 - x_h$ in the Lemmma below.

**Lemma 8.** *$m(x)$ has the following properties:*

1. $m'(x) < 0$ *and* $m(x) > 0$ *for* $x \in [0, x_h]$.
2. $\lim_{x \to 0} m(x) = \infty$.

[57]To be precise, we are solving Equation (13) backwards in $x$. In this sense we can still understand $m(x_h) = 1 - x_h$ as an initial condition. Equivalently, we can think of it as a boundary condition which has to be satisfied by the right solution to the ODE, with the initial condition being taken at some $x_0 \in (0, x_h)$.

[58]Recall that there can never be an equilibrium where the firm exerts an effort of one. This is because exerting effort is only optimal when

$$\lambda(1-x)(1-\hat{a}(x)) V(x) \geq c.$$

But if equilibrium effort is one, i.e. $\hat{a}(x) = 1$, this condition cannot hold.

*3.* $m'(x_h) > -1$.

*4.* $m(x)$ *crosses* $1-x$ *once for* $x < x_h$.

*Proof.* 1. To prove the first point, note that whenever $m'(x) \geq 0$, Equation (13) implies that

$$rc < \lambda(\alpha - c)\, m(x) - \alpha\lambda m(x)^3 .$$

Suppose that $m(x) \geq 0$. The right hand side of this inequality is strictly concave and it reaches its unique maximum at the point $m^* = \sqrt{\frac{\alpha-c}{3\alpha}}$. Its maximum value is

$$\lambda\left((\alpha-c)\sqrt{\frac{\alpha-c}{3\alpha}} - \alpha\left(\frac{\alpha-c}{3\alpha}\right)^{\frac{3}{2}}\right) = \frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}}\frac{2}{3\sqrt{3}}.$$

By our assumption in Proposition 2, this value is below $rc$. Therefore, $m'(x)$ cannot be positive whenever $m(x)$ is positive. Since we have started with the condition $m(x_h) = 1 - x_h \in (0,1)$, this implies that at $x_h$, $m(x)$ is decreasing. But then $m(x)$ must remain positive for all $x \leq x_h$ and therefore its derivative must remain negative.

2. To show that $m(x)$ must go to infinity as $x$ approaches zero, we rewrite Equation (13) as

$$\int_{m(x_h)}^{m(x)} \frac{\lambda cm}{\lambda\alpha(m - m^3) - \lambda cm - rc} dm = \log(x) - \log(x_h).$$

We obtain this expression by exploiting the fact that Equation (13) is a separable first-order ODE.[59] As $x \to 0$ the RHS diverges to minus infinity. The derivative of the LHS is

$$\frac{\lambda cm}{\lambda\alpha(m - m^3) - \lambda cm - rc},$$

which is negative for all $m \geq 0$. Thus, we must have $m(x) \to \infty$. Otherwise, the RHS cannot match the LHS.

3. We compare the ODE characterizing $V$ with the one characterizing $V_s$. The first equation is

$$(r + \lambda m(x))\, V(x) = \alpha\left(1 - m(x)^2\right) + \lambda x m(x)^2 V'(x)$$

while the second is the shirking ODE (12). At $x_h$, we have $m(x_h) = 1 - x_h$. Thus, the coefficients of both equations are the same at $x_h$. Since we also have $V(x_h) = V_s(x_h)$, it must be that $V'(x_h) = V_s'(x_h)$. From Corollary 11, we know that $V_s'(x_h) < \frac{c}{\lambda(1-x_h)^2}$. On the work region, we have $V'(x) = -m'(x)\frac{c}{\lambda m(x)^2}$. Combining these expressions and plugging in $m(x_h) = 1 - x_h$ then yields $m'(x_h) > 1$.

[59]See e.g. Polyanin and Zaitsev (2002), p. 3.

4. The proof proceeds similarly to the proof of Lemma 6. Whenever $m(x) = 1 - x$, Equation (13), which is the ODE characterizing $m(x)$, becomes

$$0 = \lambda (\alpha - c)(1 - x) - \lambda\alpha (1 - x)^3 - \lambda c x (1 - x) m'(x) - rc.$$

We can rewrite it as

$$0 = \lambda l \left(x, \frac{c}{\lambda}\right) - \lambda c x (1 - x) \left(m'(x) + 1\right),$$

where $l(x, v)$ is the function we have defined previously in Equation (34). Note that $m'(x) > -1$ if and only if $l\left(x, \frac{c}{\lambda}\right) > 0$. Likewise, $m'(x) < -1$ if and only if $l\left(x, \frac{c}{\lambda}\right) < 0$. Since we are in the case with a nonempty working region, we have $\max_x l\left(x, \frac{c}{\lambda}\right) > 0$, according to Proposition 10. Recall that $l\left(x, \frac{c}{\lambda}\right)$ is hump-shaped. That is, it is negative for $x$ small, positive for intermediate $x$, and negative again for large $x$.Also, by Point 3 we know that $m'(x_h) > -1$ and that $l\left(x_h, \frac{c}{\lambda}\right) > 0$. Since $m(x)$ diverges to infinity as $x$ becomes small, we already know that it must cross $1 - x$ at least once below $x_h$. It must also cross $1 - x$ an odd number of times left of $x_h$.[60] We thus only have to prove that it does not cross more than once. By way of contradiction, suppose there are three points $x_1 < x_2 < x_3 < x_h$ at which $m(x)$ crosses $1 - x$. We must have $m'(x_1) \leq -1$, $m'(x_2) \geq -1$, and $m'(x_3) \leq -1$ and thus $l\left(x_1, \frac{c}{\lambda}\right) \leq 0$, $l\left(x_2, \frac{c}{\lambda}\right) \geq 0$ and $l\left(x_3, \frac{c}{\lambda}\right) \leq 0$. Since $l$ is hump-shaped and strictly concave, the last condition is incompatible with $l\left(x_h, \frac{c}{\lambda}\right) > 0$. Once $l$ becomes negative after being positive, it must stay negative. This is our contradiction.

∎

The first point of the Lemma guarantees that the IVP for $m(x)$ with initial condition $m(x_h) = 1 - x_h$ has a *unique* solution on any interval $[\varepsilon, x_h]$ for some small $\varepsilon > 0$. This is because $m(x) \geq 1 - x_h > 0$, so we can apply the Picard-Lindelöf Theorem to Equation (13). The last point of the Lemma characterizes the working region.

We now conclude the proof by showing that shirking is indeed optimal on $[0, x_l]$. For any point $(v, x)$ such that $x \in (0, x_h)$ and $v = \frac{c}{\lambda(1-x)}$, there exists a solution to the shirking ODE (12) that hits that point. This is because any solution to Equation (12) is strictly increasing and continuous in its initial condition.[61] Take a solution which hits $\left(x_l, \frac{c}{\lambda(1-x_l)}\right)$, which we denote with $\tilde{V}_s(x)$.[62] For the shirking equilibrium to exist on $[0, x_l]$, we need that $\tilde{V}_s(x)$ stays below $\frac{c}{\lambda(1-x)}$ on that interval. Otherwise, the firm would strictly prefer to exert effort. At $x_l$, we have $m(x_l) = 1 - x_l$ and $V(x_l) = \tilde{V}_s(x_l)$. Comparing the ODEs for $V$ and the ODEs for $\tilde{V}_s$, just as we did in the proof of Lemma 8 above, we see that $V'(x_l) = \tilde{V}_s'(x_l)$. That is, $V$ and $\tilde{V}_s$ satisfy smooth pasting. Since at $x_l$, we have $m'(x_l) \leq -1$, we know that $\tilde{V}_s'(x_l) \geq \frac{c}{\lambda(1-x_l)^2}$.

Recall that solutions to the shirking ODE are ordered, so that one solution is always larger than the other on $(0, 1)$.[63] If $\tilde{V}_s(x) < V_s(x)$, then Corollary 11 applies. $\tilde{V}_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ at two points $\tilde{x}_l$ and $\tilde{x}_h$, first from below and then from above. Thus either $x_l = \tilde{x}_l$ or $x_l = \tilde{x}_h$. At $\tilde{x}_h$,

[60] If $m(x)$ would cross $1 - x$ an even number of times for $x < x_h$, then after it crosses the last time (at the lowest $x$), it must stay below $1 - x$. This can be seen graphically. But this is impossible, because we have shown that $m(x)$ goes to infinity as $x$ goes to zero.

[61] This can be seen by inspecting Equation (32) in the proof of Lemma 4.

[62] We label this equation differently from $V_s$ to avoid confusion. Both equations satisfy the Equation (12), but they are *different solutions*. In particular, $\tilde{V}_s$ cannot satisfy the boundary condition $\tilde{V}_s(1) = \frac{\alpha}{r}$ unless it is identical to $V_s$.

[63] We have shown this in Lemma 4, Point 3.

we have $\tilde{V}_s' (\tilde{x}_h) < \frac{c}{\lambda(1-\tilde{x}_h)^2}$, which is incompatible with the smooth pasting condition. Therefore, $x_l = \tilde{x}_l$, i.e. $x_l$ is the first time $\tilde{V}_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ and it cannot cross left of $x_l$. Shirking on the region $[0, x_l]$ is thus optimal. If $\tilde{V}_s(x) > V_s(x)$ ,by Lemma 7, Point 2, it crosses $\frac{c}{\lambda(1-x)}$ at exactly one point left of $x_h$. By construction, that point is $x_l$. Therefore, $\tilde{V}_s(x)$ for $x < x_l$.

We have established that a shirk-work-shirk equilibrium exists. We now show that it is unique. First, any equilibrium with working must have an upper shirking region. This region must equal $[x_h, 1]$ . We have shown this when we proved uniqueness for the shirking equilibrium in Proposition 1 and the same argument applies here. Thus, if there exists a working region, there must be one that has $x_h$ as its upper bound. On any nonempty working region, $m(x)$ must satisfy Equation (13), otherwise, it is not consistent with the indifference condition of the strategic type (35) or the value function. Lemma 8 guarantees that Equation (13) has a *unique* solution that satisfies $m(x_h) = 1 - x_h$. Thus, in any equilibrium with a working region bordering $x_h$, that working region must be $[x_l, x_h]$ . We now show that there cannot be any other working region left of $x_l$. Let $\hat{x}_h < x_l$ be the right boundary of such a region. We must have $m(\hat{x}_h) = 1 - \hat{x}_h$ and $V(\hat{x}_h) = \frac{c}{\lambda(1-\hat{x}_h)}$. For $x \in (\hat{x}_h, x_l)$, we must have $V(x) = \tilde{V}_s(x)$ . Here, $\tilde{V}_s(x)$ is the solution to the shirking ODE with boundary condition $\tilde{V}_s(x_l) = \frac{c}{\lambda(1-x_l)}$ which we have defined above.[64] But since $\tilde{V}_s(x) < \frac{c}{\lambda(1-x)}$ for $x < x_l$, we must have a discontinuity at $\hat{x}_h$. This is a contradiction, since the value $V$ in any equilibrium must be continuous for $x > 0$. Therefore, there can be no equilibrium with a working region left of $x_l$. This proves that our equilibrium is unique.

## B.3 Work-Shirk Equilibrium

We now consider the case when the effort cost is sufficiently low, so that the equilibrium is work-shirk.

**Proposition 12.** *If* $\frac{c}{\lambda} < \max_x (1-x) V_s(x)$ *and* $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}} \frac{2}{3\sqrt{3}} \geq rc$, *there is a unique equilibrium, which is a work-shirk equilibrium. That is, there is a* $x_h$ *such that* $a(x) \in (0,1)$ *for* $x < x_h$ *and* $a(x) = 0$ *for* $x \geq x_h$ *and at* $x = 0$. *On the work region* $(0, x_h)$, *the manager's manipulation satisfies ODE* (13) *with boundary condition* $m(x_h) = 1 - x_h$.

This proposition shows that for a low cost of effort, the monitor works even if his reputation is close to zero, but as long as it is positive. Monitor value and monitor effort are discontinuous in reputation, being zero at $x = 0$ but strictly positive for positive reputations. Intuitively, when the cost $c$ is sufficiently low, the monitor has a high continuation value even when the reputation is very low, and therefore slow to increase. This high continuation value is sufficient to motivate strictly positive effort.

The proof proceeds similarly to the proof of Proposition 2. We define $x_h$ as the right-most point where $V_s(x)$ hits $\frac{c}{\lambda(1-x)}$, which given our assumptions is guaranteed to exist. Then, we solve for the ODE for $m(x)$. Unlike in Proposition 2, the solution does not necessarily hit $1 - x$. Instead, it can converge to a finite value between zero and one as $x$ approaches zero. This will constitute a work-shirk equilibrium. The strategic type's value is discontinuous at $x = 0$. [65]

[64] Again, it is important to note here that any solution to the shirking ODE is "forward-looking". It only depends on higher values of $x$ and is independent of the equilibrium played for lower values.

[65] That is, $V(0) = 0$ but $V(x) \geq \varepsilon > 0$ for all $x > 0$ and some $\varepsilon > 0$. Previously, we have argued that in any equilibrium, $V(x)$ needs to be continuous for $x > 0$. We have *not* argued that it needs to be continuous at $x = 0$. A possible discontinuity at zero is not surprising. The state $x = 0$ is absorbing, while for any $x > 0$ all other states $x' > x$ can be reached with positive probability.

We start with recording some properties of $m(x)$. This Lemma is analogous to Lemma 8.[66]

**Lemma 9.** *$m(x)$ has the following properties.*

1. *There exist two values $0 < \underline{m} \leq \bar{m} < 1$ such that $m'(x) > 0$ if $m(x) \in (\underline{m}, \bar{m})$, $m'(x) < 0$ if $m(x) > \bar{m}$ or $m(x) < \underline{m}$, and $m'(x) = 0$ if $x \in \{\underline{m}, \bar{m}\}$.*
2. *$m(x)$ never crosses $\underline{m}$ or $\bar{m}$.*
3. *$m(x) > 0$.*
4. *If $1 - x_h < \bar{m}$, then $\lim_{x\to 0} m(x) = \underline{m}$.*

*Proof.* 1. We can write Equation (13) as

$$0 = \lambda(\alpha - c)\, m(x) - \lambda\alpha m(x)^3 - rc - \lambda c x m'(x)\, m(x). \tag{36}$$

We are interested in the properties of the function

$$h(m) = \lambda(\alpha - c)\, m - \lambda\alpha m^3 - rc.$$

This function determines whether $m'(x)$ is positive or negative, because

$$0 = h(m(x)) - \lambda c x m'(x)\, m(x).$$

Under the condition $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}} \frac{2}{3\sqrt{3}} \geq rc$, $h(m)$ has two roots $0 < \underline{m} \leq \bar{m} < 1$. Inspecting its shape, we have $h(m) < 0$ for $m < \underline{m}$ and for $m > \bar{m}$ and $h(m) > 0$ for $m \in (\underline{m}, \bar{m})$. If $m(x) < \underline{m}$ or $m(x) > \bar{m}$, we must have $m'(x) < 0$. If $m(x) \in (\underline{m}, \bar{m})$ we must have $m'(x) > 0$.

2. Whenever $m(x)$ equals $\underline{m}$ or $\bar{m}$, $m'(x)$ and all higher derivatives are zero. We can show this by differentiating Equation (13) and plugging in values. Specifically, if $m = \bar{m}$ we have $m'(x) = 0$. The second derivative satisfies

$$\lambda x m(x)\, m''(x) = m'(x) \left(\lambda(\alpha - c) - 3\lambda\alpha m(x)^2 - \lambda c m(x) - \lambda c x \left|m'(x)\right|\right),$$

which is zero. Successively differentiating this equation and plugging in lower order terms yields the result.

3. If $1 - x_h < \underline{m}$, then $m(x)$ is decreasing, which implies that $m(x)$ is bounded above zero on $[0, x_h]$. If $1 - x_h \geq \underline{m}$ then $m(x)$ can never cross $\underline{m}$, so it must be positive.

4. We use the integral representation for Equation (13),

$$\int_{m(x_h)}^{m(x)} \frac{\lambda c m}{h(m)} dm = \log(x) - \log(x_h),$$

[66] The behavior of $m(x)$ is now different because of the changed parameter assumptions, even though $m(x)$ still satisfies Equation (13) and the same boundary condition as before, albeit at a different $x_h$.

which we have used previously in the proof of Lemma 8. If $1-x_h<\underline{m}$, we have $m(x)\leq\underline{m}$, so the derivative of the LHS must be negative at the solution $m(x)$. If $m(0)<\underline{m}$, then as $x\to 0$, the LHS is bounded whereas the RHS diverges to $-\infty$. Thus, we must have $m(0)=\underline{m}$. If $1-x_h\in(\underline{m},\bar{m})$, $m(x)$ is increasing and $h(m(x))$ is positive. Then, we can rewrite the equation as

$$-\int_{m(x)}^{m(x_h)}\frac{\lambda cm}{h(m)}dm=\log(x)-\log(x_h).$$

The LHS goes to zero as $x\to 0$ whenever $m(0)=\underline{m}$.

∎

A work-shirk equilibrium exists whenever $1-x_h<\bar{m}$. In that case, $m(x)$ remains below $1-x$ for all $0<x<x_h$ and it converges to $\underline{m}$ as $x\to 0$. We now show this is the case given our assumptions in Proposition 12. To prove this result, we exploit the functions $l\left(x,\frac{c}{\lambda}\right)$ in Equation (34) and $h(m)$ in Equation (36). We construct a sequence of inequalities which will imply that $1-x_h<\bar{m}$. We are interested in $h(m)$ only when $m=1-x$, so with slight abuse of notation we write it as

$$h(x)=\lambda(\alpha-c)(1-x)-\lambda\alpha(1-x)^3-rc.$$

$\bar{m}$ is the largest root of $h(m)$, so $1-\bar{m}$ is the smallest root of $h(x)$. We are trying to prove that $1-\bar{m}<x_h$, i.e. that $x_h$ lies above the smallest root of $h(x)$.[67] The condition $c<\alpha$ will imply that the maximum of $h(x)$ must be left of the maximum of $l\left(x,\frac{c}{\lambda}\right)$ and we will argue that $x_h$ must lie to the right. Since $h$ is hump-shaped its maximum is right of its smallest root, so this implies $1-x_h<\bar{m}$.

Consider the value $c_0$ at which the shirking value $V_s(x)$ is tangent to $\frac{c_0}{\lambda(1-x)}$ at a single point, $x_0$. At that point, we must have $l\left(x_0,\frac{c_0}{\lambda}\right)=0$ and $x_0$ must achieve the maximum of $l\left(x,\frac{c_0}{\lambda}\right)$ For any $c<c_0$, we have $x_l<x_0<x_h$.[68] Inspecting $l\left(x,\frac{c}{\lambda}\right)$, we can see that $\frac{\partial^2 l\left(x,\frac{c}{\lambda}\right)}{\partial x\partial c}>0$, i.e. the value at which $l$ attains its maximum is increasing in $c$. We denote this value with $x_l^*$ and omit the dependence on $c$. For $c<c_0$, we have $x_l^*<x_0$. We can rewrite $h(x)$ as

$$h(x)=\lambda l\left(x,\frac{c}{\lambda}\right)-\lambda cx(1-x).$$

This implies

$$h'(x_l^*)=\lambda c(2x_l^*-1).$$

If $x_l^*<\frac{1}{2}$, then $h$ is decreasing at $x_l^*$. But then $x_h^*$, the value at which $h$ attains its maximum, must lie left of $x_l^*$. $1-\bar{m}$ is the first root of $h(x)$ and it must lie left of $x_h^*$. Taken together, our arguments yield the following chain of inequalities:

$$\text{if } x_l^*<\frac{1}{2},\ \text{then } 1-\bar{m}\leq x_h^*<x_l^*<x_0<x_h.$$

We now only have to show that $x_l^*<\frac{1}{2}$ We can compute $\frac{\partial}{\partial x}l\left(x,\frac{c}{\lambda}\right)=0$ and find the maximizer

[67] Recall that whenever $h(x)$ exceeds zero on $[0,1]$ it has exactly two roots on that interval.

[68] Note that the shirking value $V_s(x)$ is independent of $c$, so changing $c$ does not affect it. For lower $c$, $\frac{c}{\lambda(1-x)}$ is lower while $V_s(x)$ is the same, so the intersection points must move to the left and right of $x_0$, respectively.

that lies in $[0,1]$. It is given by

$$x_l^* = 1 + \frac{1}{3}\frac{c}{\alpha} - \sqrt{\frac{1}{9}\left(\frac{c}{\alpha}\right)^2 + \frac{1}{3}}.$$

Then, the result follows from the fact that we assumed $c < \alpha$ and simple algebra.

Whenever the work-shirk equilibrium exists, it is unique. The argument for this is analogous to the one in our proof of Proposition 2. As we have already seen, the shirking region $[x_h, 1]$ must be the same for any potential equilibrium. Suppose that there exists another equilibrium where on some region $[\hat{x}_l, \hat{x}_h] \subset [0, x_h]$, the agent shirks. Then, we need that $m(\hat{x}_h) = 1 - \hat{x}_h$, otherwise, the value function would not be continuous.[69] However, the ODE (13) with boundary condition $m(x_h) = 1 - x_h$ has a *unique* solution and in a work-shirk equilibrium, that solution never hits $1 - x$ for $x < x_h$. Thus, we cannot have such a region. This establishes uniqueness.

# C Disclosure

## C.1 Delayed Bad News

The value function of the regulator in Equation (14) is constructed similarly to the function of the firm. On the upper shirking region $[x_s, 1]$, the regulator's value is her shirking value. It solves Equation (14) with $m(x) = 1 - x$, i.e.

$$\left(r + \lambda (1-x)^2\right) W_s(x) = \alpha x + \lambda x (1-x)^2 W_s'(x), \tag{37}$$

with boundary condition $W(1) = \frac{\alpha}{r}$. The shirking value has the closed form solution $W_s(x) = \frac{\alpha}{r}x$. On the working region $[x_l, x_h]$, the regulator's value solves Equation (14) and $m(x)$ is the manipulation obtanined in Equation (13) of Proposition 2. Since the regualtor's value is continuous, it satisfies the boundary condition $W(x_h) = \frac{\alpha}{r}x_h$. On the lower shirking region $[0, x_l]$, the regulator's value is again the shirking value in Equation (37), but it has boundary condition $W_s(x_l) = W(x_l)$, where $W$ is the solution obtained on the working region.[70]

**Lemma 10.** *On the working region, we have $W(x) > W_s(x)$.*

*Proof.* On the working region, the regulator's value satisfies Equation (14) with boundary condition $W(x_h) = W_s(x_h) = \frac{\alpha}{r}x_h$. If $W(x) \le \frac{\alpha}{r}x$ for some $x \in (x_l, x_h)$,

$$\lambda x m(x)^2 \left(W'(x) - \frac{\alpha}{r}\right) \le \alpha (x - (1 - m(x))).$$

The last term is negative, because $m(x) < 1 - x$. Thus, $W'(x) < \frac{\alpha}{r}$ whenever $W(x) \le \frac{\alpha}{r}x$. Once $W(x)$ touches $\frac{\alpha}{r}x$ on the working region, it must stay strictly below to the right. But this is incompatible with the boundary condition at $x_h$. Thus, the solution to $W(x)$ must satisfy $W(x) > \frac{\alpha}{r}x$ on $(x_l, x_h)$. ∎

[69] Specifically, if $m(\hat{x}_h) < 1 - \hat{x}_h$, then $V(\hat{x}_h) = \frac{c}{\lambda m(\hat{x}_h)} > \frac{c}{\lambda(1-\hat{x}_h)}$. But since the firm is supposed to shirk left of $\hat{x}_h$, we must also have $V(x) \le \frac{c}{\lambda(1-x)}$ for all $x \in (\tilde{x}_n - \varepsilon, \hat{x}_h)$ for some small $\varepsilon > 0$. This means the value is discontinuous, which cannot be true.

[70] Because of this boundary condition, $W_s(x)$ does not equal $\frac{\alpha}{r}x$ on $[0, x_l]$.

We are now ready to prove Proposition 6, which we restate below for convenience.

**Proposition 6.** *For $\bar{\gamma}$ sufficiently small, disclosure of bad news is valuable to the regulator when $x \leq x_d$, where $x_l \leq x_d < x_h$. It is not valuable for $x$ close to (or above) $x_h$.*

*Proof.* Delayed disclosure is valuable to the regulator whenever $W'(x)\,x - W(x)$ is strictly positive. This is equivalent to

$$\phi(x) := rW(x) - \alpha(1 - m(x)) > 0,$$

which follows from rearranging Equation 14. From Lemma 10, we know that $W(x_l) > \frac{\alpha}{r} x_l = \frac{\alpha}{r}(1 - m(x_l))$. Thus, $\phi(x_l) > 0$. By continuity $\phi(x)$ remains positive on a neighborhood to the right of $x_l$. On $[0, x_l]$, the value of information is positive whenever[71]

$$\phi(x) = rW_s(x) - \alpha x > 0,$$

where $W_s(x)$ is the shirking value in Equation (37) with boundary condition $W_s(x_l) = W(x_l) > \frac{\alpha}{r} x_l$. Thus, we have to show that the shirking value exceeds $\frac{\alpha}{r} x$. This is true because two solutions to Equation (37) cannot cross. This follows from an argument analogous to the one in Lemma 4. We can express $W_s(x)$ using the method of integrating factors and then show that each solution can be indexed by the initial value at a (wlog) common initial point. A solution with a larger initial value must then always lie above a solution with a lower one. The function $\frac{\alpha}{r} x$, which is the solution that satisfies $W_s(x_l) = \frac{\alpha}{r} x_l$ must therefore lie below the solution that satisfies $W_s(x_l) = W(x_l)$. This implies the value of disclosure is positive on $(0, x_l)$.

To show that disclosure is detrimental when reputation becomes sufficiently large, we plug in $\phi(x)$ and $\phi'(x) = rW'(x) + \alpha m'(x)$ into Equation (14), to obtain an ODE for $\phi(x)$, which is

$$\left(r + \lambda m(x)^2\right)\phi(x) = \lambda m(x)^2 \left(x\phi'(x) - \alpha x m'(x) - \alpha(1 - m(x))\right).$$

$\phi(x)$ satisfies the boundary condition $\phi(x_h) = 0$, because $W(x_h) = \frac{\alpha}{r} x_h = \frac{\alpha}{r}(1 - m(x_h))$. As $x$ approaches $x_h$, we have

$$\phi'(x) \to \alpha\left(m'(x_h) + 1\right),$$

which is strictly positive because $m'(x_h) < -1$. But if $\phi'(x) > 0$ for $x$ close to $x_h$ and $\phi(x_h) = 0$, then $\phi(x)$ must be negative close to $x_h$. ∎

Finally, we prove Proposition 5, which is restated below.

**Proposition 5.** *Disclosure of bad news reduces the strategic monitor's incentive to exert effort for reputation sufficiently close to $x_h$ and on $[x_h, 1]$.*

*Proof.* To show the result for $x$ left of $x_h$, note that since $V(x) = \frac{c}{\lambda m(x)}$, it is enough to show that delayed disclosure reduces the value $V(x)$. This is true whenever $x(1-x)V'(x) - V(x)$ is negative. The value of delayed disclosure is strictly negative at $x_h$. The result then follows from continuity of $V(x)$, $V'(x)$, and $m(x)$ and the fact that $m(x_h) = 1 - x_h$. On $[x_h, 1]$, the result is immediate. Disclosure does not affect effort on that region, since the monitor shirks anyway, but it lowers the value, because is changes his HJB equation by a factor of $-\gamma V_s(x)$. This in turn lowers the monitor's value for all lower reputations and since on the work region $V(x) = \frac{c}{\lambda m(x)}$ it implies more manipulation in equilibrium. ∎

[71] Here we have just substituted $m(x) = 1 - x$ and the shirking value $W_s(x)$.

## C.2 Delayed Good News

We first prove Proposition 7.

**Proposition 7.** *Disclosure of good news increases manipulation and decreases the value of the strategic monitor on the work region* $[x_l, x_h]$.

*Proof.* The strategic type's value for any positive $\gamma$ is

$$(r + \lambda m(x)) V(x) = \alpha \left(1 - m(x)^2\right) + \lambda x m(x)^2 V'(x) - \gamma x (1 - x) V'(x).$$

Using the indifference condition in Equation (35) yields the analog of Equation (13), which describes $m(x)$ for any given $\gamma$.

$$rc = \lambda\alpha \left(m(x) - m(x)^3\right) - \lambda c m(x) - \lambda c x m(x) m'(x) + \gamma c m'(x) \frac{x(1-x)}{m(x)}.$$

The result follows from applying Grönwall's Lemma to this equation. We have to take additional care because the initial condition for that equation is at the right boundary of the work region, so the eqation is effectively solved backwards. We use the identity $m(x) = m(x_h - y)$ where $y = x_h - x$ to define $\tilde{e}(y) = m(x_h - y)$. Substituting this and the derivative $\tilde{e}'(y)$ into Equation (13) yields

$$\tilde{e}'(y) = \frac{-h(\tilde{e}(y))}{\lambda c (x_h - y) \tilde{e}(y) - \gamma c \frac{(x_h - y)(1 - x_h + y)}{\tilde{e}(y)}},$$

which for all $y$ and $\tilde{e}(y)$ is increasing in $\gamma$, because $h(\tilde{e}(y))$, which is the function we have defined in Equation (36), is negative. Grönwall's Lemma then implies that $\tilde{e}(y, \gamma') \geq \tilde{e}'(y, \gamma)$ for all $y$ and $\gamma' \geq \gamma$, or, equivalently, $m(x, \gamma') \geq m(x, \gamma)$. Higher $\gamma$ decreases the value because $V(x) = \frac{c}{\lambda m(x)}$ and $m$ increases with $\gamma$. ∎

Finally, we prove Proposition 8.

**Proposition 8.** *For $\gamma$ sufficiently small and $x < x_h$ sufficiently close to $x_h$, the regulator benefits from disclosing good news. Disclosing good news is not valuable for $x \geq x_h$.*

*Proof.* The regulator's value for a given $\gamma$ is

$$\left(r + \lambda m(x)^2\right) W(x) = \alpha (1 - m(x)) + \left(\lambda x m(x)^2 - \gamma x (1 - x)\right) W'(x) + \gamma x \left(\frac{\alpha}{r} - W(x)\right).$$

Taking derivatives, we can derive the following representation describing the sensitivity or the regulator's value with respect to $\gamma$

$$\begin{aligned} \frac{\partial W(x)}{\partial \gamma} &= E_x \left[ \int_0^{\tau_0 \wedge \tau_h} m^{-rt} \left( \frac{dm(x_t)}{d\gamma} \left(-\alpha - \lambda W(x_t) + \lambda x_t m(x_t) W'(x_t)\right) \right.\right. \\ &\quad \left.\left. + \frac{\alpha}{r} - W(x_t) - (1 - x_t) W'(x_t) \right) dt \right]. \end{aligned}$$

Delayed good news is beneficial at $x$ if the term in the inner brackets is positive for all $x' \in [x, x_h]$. For $x$ sufficiently close to $x_h$, the term in the second line is positive, because $W(x)$

approaches $\frac{\alpha}{r}x$ and $W'(x_h)$ is below $\frac{\alpha}{r}$. $m(x)$ is decreasing in $\gamma$, by Proposition 7. The term multiplying it is negative sufficiently close to $x_h$. Thus, the integral is positive. ∎

### C.3 Proof of Lemma 1

Following a similar argument as Sannikov (2008), we define

$$\hat{V}_t = \int_0^t e^{-rs}\left(\alpha\left(1-m_s^2\right) - ca_s\right)ds + e^{-rt}V_t$$

for any $t < \tau$. Under the monitor's information set (i.e. knowing his own type), bad news follows a Poisson process with arrival rate $\lambda \hat{m}_t(1-a_t) + \gamma_{Bt}$, while good news follows a Poisson process with zero arrival rate. Using the martingale representation theorem for Poisson processes (e.g. Davis (1993)), it follows that there exists an adapted process $(\varphi_t)_{0\le t\le\tau}$, such that

$$\hat{V}_t = \hat{V}_0 + \int_0^t e^{-rs}\varphi_s(dN_s^B - (\lambda\hat{m}_t(1-a_t) + \gamma_{Bt})dt).$$

Since the monitor's continuation value is zero after bad news arrives, we have $\varphi_t = -V_t$ for all $0 \le t \le \tau$. Equating the two expressions and differentiating then yields

$$dV_t = \left(\left(r + \gamma_{Bt} + \lambda\hat{m}_t(1-a_t)\right)V_t - \alpha\left(1-\hat{m}_t^2\right) + ca_t\right)dt - V_t dN_t^B.$$

Since in equilibrium, the monitor's conjecture about manipulation is correct, i.e. $\hat{m}_t = m_t$ for all $t < \tau$, this expression simplifies to

$$dV_t = \left(\left(r + \gamma_{Bt} + \lambda m_t(1-a_t)\right)V_t - \alpha\left(1-m_t^2\right) + ca_t\right)dt - V_t dN_t^B,$$

which is the expression in the statement of the lemma.

## D Existence of Shirking Equilibrium

We now show that a solution to ODE (12) with boundary condition $V_s(1) = \frac{\alpha}{r}$ exists, using an approximation argument. The proof uses a "bounding box" which has finite upper and lower boundaries and whose right boundary is fixed below one. For any point on the boundary of this box, we can find an initial value so that the unique solution to the IVP hits this point. We then construct a sequence of boxes so that the right boundary approaches 1 and a corresponding sequence of solutions so that the value at the right boundary of the box converges to $\frac{\alpha}{r}$. To show that the limit actually satisfies $V_s(1) = \frac{\alpha}{r}$, we need to show that the sequence of solutions converges uniformly. For this we use the Arzelà-Ascoli Theorem, which we apply to a rescaled version of $V_s(x)$ that has a finite derivative.

The bounding box is for all $n \in \mathbb{N}$ given by

$$\begin{aligned} B_n \quad = \quad & \left\{(x,v) \in \mathbb{R}^2 | x \in [x_0, x_n], v \in \{-M, M\} \text{ if } x \in (x_0, x_n)\right. \\ & \left.\text{and } v \in [-M, M] \text{ if } x \in \{x_0, x_n\}\right\} \end{aligned}$$

for some finite $M > \frac{\alpha}{r}$. Here, $x_n$ is the right boundary of the box. We assume $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence with $x_n \in (x_0, 1)$ for all $n$ which converges to one as $n \to \infty$. Point 3 of Lemma 4 then implies that each point on $B_n$ can be reached by some solution to the IVP, which we show below.

**Corollary 13.** *For each $(\hat{x}, \hat{v}) \in B_n$, there exists a $v_{0n}$ such that the solution to the IVP with initial condition $v_{0n}$ satisfies $V_s(\hat{x}, v_{0n}) = \hat{v}$.*

*Proof.* Picking $v_{0n} = -M$ ensures that $V_s(x_0) = -M$ and picking $v_{0n} = M$ ensures that $V_s(x_0) = M$. For any $v_{0n} \in (-M, M)$, either hits the upper or lower bounds or it hits the right boundary at $x_n$. Since $V_s(x)$ is continuous and monotone in $v_{0n}$ by Point 3 of Lemma 4, the continuous mapping theorem implies that for any point $(\hat{x}, \hat{v}) \in B_n$, there exists an initial condition $v_{0n}$ such that $V_s(\hat{x}) = \hat{v}$. ∎

We use this result to construct a sequence of solutions which satisfy a boundary condition at $x_n$. That condition will converge to $\frac{\alpha}{r}$. Since we are only interested in the properties of these solutions as $x$ becomes large, we omit any dependence on the initial condition $v_{0n}$ to save notation. We denote with $V_{sn}(x)$ the solution to Equation (12) which satisfies the boundary condition

$$V_{sn}(x_n) = \frac{\alpha}{r} - \kappa(1 - x_n) \tag{38}$$

for some fixed $\kappa > 0$. As $n \to \infty$, the derivative $V'_{sn}(x_n)$ becomes potentially unbounded, because $x_n$ approaches one and the shirking ODE (12) has a singularity at $x = 1$. Therefore, we cannot use the Arzelà-Ascoli Theorem on $V_{sn}$ directly. Instead, we study the transformation

$$g_n(x) = V_{sn}(x)(1 - x),$$

which we extend to the entire interval $[x_0, 1]$ as follows:

$$\bar{g}_n(x) = \begin{cases} V_{sn}(x)(1-x) & \text{if} \quad x_0 \le x \le x_n \\ \frac{\alpha}{r}(1 - x_n) - \kappa(1 - x_n)^2 & \text{if} \quad x_n < x \le 1. \end{cases}$$

**Lemma 11.** *For all $n \in \mathbb{N}$, $\bar{g}_n(x)$ is uniformly bounded. It is also differentiable at all $x \in [x_0, 1]$ except at $x_n$ and has a uniformly bounded derivative.*

*Proof.* $\bar{g}_n(x)$ is uniformly bounded because we have constructed the sequence $V_{sn}(x)$ so that for all $x \in [x_0, x_n]$, $V_{sn}(x)$ is inside the "bounding box", i.e. $V_{sn}(x) \in [-M, M]$. Since $g_n(x) = V_{sn}(x)(1-x)$, we must also have $g_n(x) \in [-M, M]$. From the definition of $\bar{g}_n(x)$ we can also see that it is uniformly bounded on $[x_n, 1]$ for all $n$.

To show the derivative is uniformly bounded whenever it exists, we only have to consider the derivatives on the intervals $[x_0, x_n]$.[72] We can substitute $g_n(x) = V_{sn}(x)(1-x)$ and $g'_n(x) = V'_{sn}(x)(1-x) - V_{sn}(x)$ into Equation (12) to obtain an ODE for $g_n(x)$. This ODE is

$$\left(r + \lambda(1-x)^2\right) g_n(x) = \alpha\left((1-x) - (1-x)^3\right) + \lambda x (1-x)^2 g'_n(x). \tag{39}$$

[72] On $[x_n, 1]$, the result follows from inspecting the definition of $\bar{g}_n(x)$ above.

For any $n$, the derivative at $x_n$ is bounded. To see this, we first solve for $V'_{sn}(x_n)$, using Equation (31) and the condition in Equation (38). This yields

$$V'_{sn}(x_n) = \frac{1}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right).$$

Therefore we have

$$g'_n(x_n) = \frac{1-x_n}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right) - \frac{\alpha}{r} - \kappa(1-x_n). \tag{40}$$

As $n \to \infty$, this expression converges to $-\frac{r}{\lambda}\kappa$. This means that there exists a $\bar{K} > 0$ so that for all $n$, $|g'_n(x_n)| \leq \bar{K}$. To see that $g'_n(x)$ must be bounded uniformly for all $n$ and $x \leq x_n$, we differentiate Equation (39) to obtain

$$\begin{aligned} 0 &= 2\lambda(1-x)\,g_n(x) + \alpha\left(3(1-x)^2 - 1\right) \\ &\quad + \lambda x(1-x)^2 g''_n(x) - (r + \lambda(1-x)x)\,g'_n(x). \end{aligned}$$

Suppose there exists an $n$ and an $x_0 \leq x < x_n$ so that $|g'_n(x)| > K$. We choose $K$ sufficiently large and larger than $\bar{K}$. Then, if $g'_n(x) > K$, the equation above immediately implies that $g''_n(x) > 0$, since $g_n(x)$ is uniformly bounded. But this means that $g'_n(x') > K$ for all $x' \geq x$. This is a contradiction, since we have just shown that $g'_n(x_n)$ is bounded by $\bar{K}$ for all $n$. Similarly, if $g'_n(x) < -K$, then $g''_n(x) < 0$, which again implies that $g'_n(x_n) < -\bar{K}$. ∎

We can now apply the Arzelà-Ascoli Theorem to the sequence of functions $\bar{g}_n(x)$. It establishes that there is a subsequence that converges to a continuous function $g^*(x)$. As we show below, we can take $g^*(x)$ to be continuously differentiable on $[x_0, 1]$ and to satisfy the ODE (39) on that interval without loss of generality.

**Lemma 12.** *There exists a subsequence of $\bar{g}_n(x)$ which converges uniformly to a function $g^*(x)$ which is continuously differentiable and satisfies Equation (39) on $[x_0, 1]$.*

*Proof.* From the previous Lemma and the Arzelà-Ascoli Theorem we know there exists a subsequence which converges to a continuous function $g^*(x)$. We now use a diagonalization procedure to show that there exists a subsequence such that $g^*(x)$ is continuously differentiable on $[x_0, 1)$. For a given $n$, the derivative $g'_n(x)$ satisfies

$$g'_n(x) = \frac{\left(r + \lambda(1-x)^2\right) g_n(x) - \alpha\left((1-x) - (1-x)^3\right)}{\lambda x(1-x)^2}$$

on some interval $[x_0, \bar{x}_1]$ for $\bar{x}_1 < x_n < 1$. Since the sequence $g_n$ is equicontinuous on that interval and the right hand side of the above equation is continuous in both $x$ and $g_n(x)$, $g'_n(x)$ is equicontinuous on that interval as well.[73] Thus, there exists a subsequence of $g_n$ which converges to a limit that is continuously differentiable on $[x_0, \bar{x}_1]$. Proceeding iteratively, we then take a sequence of boundaries $\bar{x}_k$ which converges to one as $k \to \infty$. For each such $k$ we can find a subsequence of

[73] Note we are holding $\bar{x}_1$ fixed here.

$g_n$ that converges to a continuously differentiable function. Thus, we can take the limit $g^*$ to be continuously differentiable on $[x_0, 1)$ without loss of generality. Because of this, it also satisfies the ODE (39) on $[x_0, 1)$.

It remains to establish that $g^*$ is continuously differentiable at $x = 1$. This follows from Equation (40) in the proof of the previous Lemma. We have

$$\lim_{n\to\infty} g^{*\prime}(x_n) = \lim_{n\to\infty} g_n'(x_n)$$

and Equation (40) shows that $\lim_{n\to\infty} g_n'(x_n) = -\frac{r\kappa}{\lambda}$. Thus, $g^{*\prime}(1)$ is finite. ∎

We now use the function $g^*$ to show that our initial sequence of solutions $V_{sn}(x)$ converges to a limit that is continuous, solves the shirking ODE (12), and satisfies the boundary condition $V_s(1) = \frac{\alpha}{r}$. To do this, we define the following function on the interval $[x_0, 1]$

$$V^*(x) = \frac{g^*(x)}{1-x}.$$

This function is continuously differentiable except perhaps at $x = 1$ and it satisfies the ODE (12), which can be seen by substituting it into Equation (39). We thus only have to show it satisfies the boundary condition at $x = 1$. If we let $n_k$ denote the subsequence of $n$ for which $g_n$ converges to $g^*$, we have

$$V^*(x) = \lim_{k\to\infty} \frac{g_{n_k}(x)}{(1-x)} = \lim_{k\to\infty} V_{nk}(x).$$

Since for any $n$, $V_n(1) = \frac{\alpha}{r}$, we have

$$\begin{aligned} \lim_{x\to 1} V^*(x) &= \lim_{x\to 1}\lim_{k\to\infty} V_{nk}(x) \\ &= \lim_{k\to\infty} V_{nk}(1) \\ &= \frac{\alpha}{r}. \end{aligned}$$

This concludes our proof. We have shown that there exists a solution to the shirking ODE (12) on the interval $[x_0, 1]$ which satisfies the boundary condition $V_s(1) = \frac{\alpha}{r}$. Since any solution to the equation must satisfy $V(0) = 0$ (by Lemma 4, Point 1), we can extend this solution to the entire interval $[0, 1]$.

# E Numerical Appendix

We use a finite difference approximation of the regulator value $W(x, V)$.[74] To improve speed and accuracy given the highly nonlinear domain, we use an unevenly spaced grid. Specifically, we start with evenly spaced grid in the $x$ dimension with $I$ elements, $X = (x_1, ..., x_I)$ and we denote a generic element $x_i$. Then, we compute a solution to the Hamilton-Jacobi equations defining the boundaries $\bar{V}(x)$ and $\underline{V}(x)$ (Equations (27) and (28)), using MATLAB's built in *bvp4c* function. Since using *bvp4c* does not guarantee that the grid is the same as the one we have defined, we

[74] See Crandall and Lions (1984) for a characterization of convergence properties.

linearly interpolate the solutions on the grid $X$. We denote the resulting values with $\widehat{\bar{V}}_i$ and $\widehat{\underline{V}}_i$, which are defined at each grid point $x_i \in X$.

Similarly, we use *bvp4c* to compute the ODEs defining the monitor's boundary conditions $\bar{W}(x)$ and $\underline{W}(x)$ (Equations (29) and (30)) and we denote with $\widehat{\bar{W}}_i$ and $\widehat{\underline{W}}_i$ the linear interpolation on $X$. We also obtain the derivatives $\bar{W}'(x)$ and $\underline{W}'(x)$ from *bvp4c*, which will be useful later, and we denote the linear interpolations on $X$ as $\widehat{\bar{W}}_i'$ and $\widehat{\underline{W}}_i'$.

Next, we construct the grid in the $V$ dimension. We fix a number $J$, and for each $i$, we define an evenly spaced grid with $J$ elements, $V_i = (v_{i1}, ..., v_{iJ})$, with $v_{i1} = \widehat{\underline{V}}_i$ and $v_{iJ} = \widehat{\bar{V}}_i$. This grid choice has the following desirable property. As can be seen in Figure 2, the boundaries $\bar{V}(x)$ and $\underline{V}(x)$ become simultaneously very steep and very close together as $x$ becomes small. Our grid features a smaller distance between elements in the $V$-dimension on that region, which improves accuracy where it is most needed.

To facilitate indexing, we define the $I \times J$ matrix of $x$-elements as

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_1 & ... & x_1 \\ x_2 & x_2 & ... & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_I & x_I & ... & x_I \end{bmatrix}$$

and the $I \times J$ matrix of $V$-elements as

$$\boldsymbol{V} = \begin{bmatrix} v_{11} & v_{12} & ... & v_{1J} \\ v_{21} & v_{22} & ... & v_{2J} \\ \vdots & & \ddots & \vdots \\ v_{I1} & v_{I2} & ... & v_{IJ} \end{bmatrix}.$$

Thus, a generic element $v_{i,j}$ of $\boldsymbol{V}$ corresponds to the $j$'th element of the vector $V_i$. From here on, we use the short-hand notation $W_{i,j} = W(x_i, v_{i,j})$, $m_{i,j} = m(x_i, v_{i,j})$, etc.

We approximate the partial derivatives $W_x(x_i, v_{i,j})$ and $W_V(x_i, v_{i,j})$ using forward differences. The derivative in the $V$ dimension is standard and given by

$$W_V(x_i, v_{i,j}) \approx \frac{W_{i,j+1} - W_{i,j}}{v_{i,j+1} - v_{i,j}} \equiv W_{V,i,j}. \tag{41}$$

Approximating the derivative in the $x$ dimension faces two challenges. (1) For a given $(x_i, v_{i,j})$, the pair $(x_{i+1}, v_{i,j})$ may lie outside the domain. For example, we may have $v_{i,j} > \underline{V}(x_i)$, but $v_{i,j} < \underline{V}(x_{i+1})$. (2) For a given $(x_i, v_{i,j})$, $v_{i,j}$ may not be an element of $V_{i+1}$, i.e. the node $(x_{i+1}, v_{i,j})$ does not exist. Thus, we generally cannot compute the "naive" forward difference

$$W_x(x_i, v_{i,j}) \approx \frac{W(x_{i+1}, v_{i,j}) - W(x_i, v_{i,j})}{x_{i+1} - x_i}.$$

To solve the issue of derivatives at the boundaries, we replace the forward difference with the interpolated derivatives $\widehat{\overline{W}}_i'$ and $\widehat{\underline{W}}_i'$, respectively. That is,

$$W_{x,i,j} \equiv \widehat{\underline{W}}_i' \tag{42}$$

whenever $v_{i,j} < \widehat{\underline{V}}_{i+1}$ and

$$W_{x,i,j} \equiv \widehat{\overline{W}}_i' \tag{43}$$

whenever $v_{i,j} > \widehat{\overline{V}}_{i+1}$.

To approximate derivatives inside the feasible domain, we compute the nearest neighbor to $v_{i,j}$ in the vector $V_{i+1}$. We denote with $n_+(i,j)$ its index, i.e. $v_{n_+(i,j)}$ is the nearest neighbor to $v_{i,j}$ in $V_{i+1}$. Then, we approximate the derivative as

$$W_x(x_i, v_{i,j}) \approx \frac{W_{i,n_+(i,j)} - W_{i,j}}{x_{i+1} - x_i} \equiv W_{x,i,j}. \tag{44}$$

The finite difference approximation to the regulator's Hamilton-Jacobi equation (26) is now defined as

$$\begin{aligned} rW_{i,j} &= \alpha(1 - m_{i,j}) + \left(\left(r + \gamma_{B,i,j} + \lambda m_{i,j}\right) v_{i,j} - \alpha\left(1 - m_{i,j}^2\right)\right) W_{V,i,j} \\ &\quad + \left(\lambda x_i m_{i,j}^2 + x_i(1 - x_i)\left(\gamma_{B,i,j} - \gamma_{G,i,j}\right)\right) W_{x,i,j} \\ &\quad - \left(\lambda m_{i,j}^2 + \gamma_{Bi,j}(1 - x_i)\right) W_{i,j} + \gamma_{G,i,j} x_i \left(\frac{\alpha}{r} - W_{i,j}\right). \end{aligned}$$

Here, $\gamma_{B,i,j}$ and $\gamma_{G,i,j}$ are the approximate optimal controls, which are given by

$$\gamma_{B,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad v_{i,j} W_{V,i,j} + x_i(1 - x_i) W_{x,i,j} - (1 - x_i) W_{i,j} \geq 0 \\ 0 & \text{if} \quad v_{i,j} W_{V,i,j} + x_i(1 - x_i) W_{x,i,j} - (1 - x_i) W_{i,j} < 0 \end{cases}$$

and

$$\gamma_{G,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad -x_i(1 - x_i) W_{x,i,j} + x_i\left(\frac{\alpha}{r} - W_{i,j}\right) \geq 0 \\ 0 & \text{if} \quad -x_i(1 - x_i) W_{x,i,j} + x_i\left(\frac{\alpha}{r} - W_{i,j}\right) < 0. \end{cases}$$

We use an explicit method to calculate a solution to the above equation. We start with an initial guess $W_{i,j}^0$, which is given by a linear interpolation between $\widehat{\overline{W}}_i$ and $\widehat{\underline{W}}_i$ at each $i$ and $v_{i,j}$.[75] Then, we update $W_{i,j}^n$, as

$$\begin{aligned} \frac{W_{i,j}^{n+1} - W_{i,j}^n}{\Delta} + rW_{i,j} &= \alpha(1 - m_{i,j}) \\ &\quad + \left(\left(r + \gamma_{B,i,j}^n + \lambda m_{i,j}\right) v_{i,j} - \alpha\left(1 - m_{i,j}^2\right)\right) W_{V,i,j}^n \\ &\quad + \left(\lambda x_i m_{i,j}^2 + x_i(1 - x_i)\left(\gamma_{B,i,j}^n - \gamma_{G,i,j}^n\right)\right) W_{x,i,j}^n \\ &\quad - \left(\lambda m_{i,j}^2 + \gamma_{Bi,j}^n(1 - x_i)\right) W_{i,j}^n + \gamma_{G,i,j}^n x_i \left(\frac{\alpha}{r} - W_{i,j}^n\right). \end{aligned} \tag{45}$$

[75] That is,

$$W_{i,j}^0 = \widehat{\underline{W}}_i + (v_{i,j} - v_{i,1}) \frac{\widehat{\overline{W}}_i - \widehat{\underline{W}}_i}{v_{i,J} - v_{i,1}}.$$

Here, $\Delta$ is the step size of the iteration and $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ are defined analogously as

$$\gamma^n_{B,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad v_{i,j} W^n_{V,i,j} + x_i (1 - x_i) W^n_{x,i,j} - (1 - x_i) W^n_{i,j} \geq 0 \\ 0 & \text{if} \quad v_{i,j} W^n_{V,i,j} + x_i (1 - x_i) W^n_{x,i,j} - (1 - x_i) W^n_{i,j} < 0 \end{cases} \tag{46}$$

and

$$\gamma^n_{G,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad -x_i (1 - x_i) W^n_{x,i,j} + x_i \left( \frac{\alpha}{r} - W^n_{i,j} \right) \geq 0 \\ 0 & \text{if} \quad -x_i (1 - x_i) W^n_{x,i,j} + x_i \left( \frac{\alpha}{r} - W^n_{i,j} \right) < 0. \end{cases} \tag{47}$$

The algorithm can be summarized as follows.

1. Start with guess $W^0_{i,j}$.

2. Compute $W_{x,i,j}$ and $W_{V,i,j}$ using Equations (41), (43), (42), and (44).

3. Compute $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ using Equations (46) and (47).

4. Compute $W^{n+1}_{i,j}$ using Equation (45).

5. Stop if the maximum distance
$$\max_{i,j} \left| W^{n+1}_{i,j} - W^n_{i,j} \right|$$
is below a specified tolerance, otherwise go to step 2.

Running this scheme on the entire $x$-domain $[0, 1]$ results in a number of problems. (1) As can be seen from Figure 2, the regulator is indifferent between any disclosure policy for any $x \geq x_h$.[76] This may lead the policies $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ to oscillate on the region $[x_h, 1]$, which may lead to convergence failures. (2) The law of motion for beliefs may have an endogenous interior singularity at high $x$ values. To see this, consider again the shirking region $[x_h, 1]$. On this region, $m(x, V) = 1 - x$, independently of $V$ or the regulator's disclosure policy. Then, we can calculate the law of $x_t$ as

$$\begin{aligned} \frac{dx_t}{dt} &= \lambda x_t (1 - x_t)^2 + (\gamma_{Bt} - \gamma_{Gt}) x_t (1 - x_t) \\ &= x_t (1 - x_t) (\lambda (1 - x_t) + \gamma_{Bt} - \gamma_{Gt}). \end{aligned}$$

When $x_t$ is sufficiently large and $\gamma_{Bt} = 0$, the $dx_t/dt$ may change sign depending on whether $\gamma_{Gt} = 0$ and $\gamma_{Gt} > 0$. (3) The law for $V_t$ may change sign depending on whether $\gamma_{Bt} > 0$ or $\gamma_{Bt} = 0$ for high values of $x$.

To avoid these issues, we use our theoretical characterization to restrict the problem as follows. For $x > x_h$, we know that the monitor shirks irrespective of the disclosure policy and that the regulator's value is linear (see also Figure 2). Thus, without loss of generality, the optimal disclosure policy at any $x_t > x_h$ is given by $\gamma_{Bt} = \gamma_{Gt} = 0$. The regulator's boundary values coincide at $x_h$, i.e. $\bar{W}(x_h) = \underline{W}(x_h)$. We can now restrict the grid of $x$-values to $[0, x_h]$ and use the boundary condition $W(x_h, V) = \bar{W}(x_h)$ for any $V \in \left[\underline{V}(x_h), \bar{V}(x_h)\right]$. With this modification, and sufficiently small step size $\Delta$, and a sufficiently fine grid $X$, the explicit scheme converges monotonically.

[76] Note that this is consistent with our previous qualitative results in Section 4.